Supplementary material: Modulation of glacier surge cycles on decadal to centennial timescales by intrinsic thermal-structure variability

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S1 Model equations

S1.1 Mechanical governing equations

We use a two-dimensional Cartesian coordinate system, where x lies in the horizontal plane and z is positive upwards. All lateral (y) variations are ignored, resulting in a plane strain approximation. For the two-dimensional Cartesian coordinate system we define the differential operator ∇ as

$$\nabla = \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_z \frac{\partial}{\partial z}, \tag{S1}$$

where $\{\boldsymbol{e}_x, \boldsymbol{e}_z\}$ is the set of orthonormal unit vectors associated with the two-dimensional coordinate system (Greve and Blatter, 2009, Ch. 3).

To solve the *mechanical* component of thermomechanically coupled fluid flow, we must solve balance equations for mass and momentum. Treating ice as an incompressible fluid with a uniform and constant density, the balance equation for mass is stated as:

$$\nabla \cdot \boldsymbol{u} = 0, \tag{S2}$$

where $\boldsymbol{u} = (u, w)$ is the velocity $(\mathrm{m\,yr^{-1}})$ vector in the horizontal (x) and vertical (z) directions, respectively. Because we treat ice as an incompressible fluid, only deviations of the stress from an isotropic state can cause deformation. Therefore, we define the deviatoric stress tensor $\boldsymbol{\sigma'}$ (MPa) in terms of the stress tensor $\boldsymbol{\sigma}$ (MPa) and pressure P (MPa) as

$$\boldsymbol{\sigma'} = \boldsymbol{\sigma} + P\boldsymbol{I}, \tag{S3}$$

where $P = -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})$, tr is the trace and $\boldsymbol{I}$ is the identity matrix. Using the deviatoric stress tensor, the momentum balance equation for an incompressible fluid is

$$\nabla \cdot \boldsymbol{\sigma'} - \nabla P + \rho \boldsymbol{g} = 0, \tag{S4}$$

where ρ is density ($\mathrm{kg\,m^3}$) and $\boldsymbol{g} = (0, -g)$ is the gravitational acceleration ($\mathrm{m\,yr^{-2}}$) vector for our two dimensional Cartesian coordinate system. Note Equation S4 ignores both acceleration and inertial forces (e.g., Coriolis). See Greve and Blatter (2009, p. 63–64) for the relevant scaling arguments for the momentum balance.

Glacier ice is a non-Newtonian fluid with a stress-dependent viscosity (?). To close the system of equations (Equations S2 & S4) we need a constitutive law that relates the deviatoric stress $\boldsymbol{\sigma'}$ from the momentum balance (Equation S4) to velocity in the mass conservation (Equation S2). The most commonly used constitutive law in glaciology is

$$\dot{\boldsymbol{\epsilon}} = A\boldsymbol{\sigma}_{\mathrm{e}}^{n-1}\boldsymbol{\sigma}, \tag{S5}$$

with n $\approx$ 3, and where $\dot{\boldsymbol{\epsilon}}$ ($\mathrm{yr^{-1}}$) is the strain rate tensor, $A(T')$ is the flow-law coefficient ($\mathrm{MPa^{-3}\,yr^{-1}}$), T' is the temperature relative to the pressure melting point (K), and $\boldsymbol{\sigma}_{\mathrm{e}}$ is the effective stress (MPa). The strain rate tensor is related to velocity gradients as

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}}\right). \tag{S6}$$

The effective stress $\boldsymbol{\sigma}_{\mathrm{e}}$, which is equivalent to the second invariant of the deviatoric stress tensor, is defined as:

$$\boldsymbol{\sigma}_{\mathrm{e}} = \sqrt{\frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma'}\right)^2}, \tag{S7}$$

where tr is the trace. An invariant (in this case the second) of the stress tensor is used to make Equation S5 applicable to all scenarios, irrespective of the coordinate system used. Details on the temperature-dependent flow-law coefficient $A(T')$ follows below in Section S1.3.

S1.2 Additional thermodynamic relationships

Following Aschwanden et al. (2012) we conserve energy through an enthalpy H ($\mathrm{J\,kg^{-1}}$) formulation. Because ice is treated as an incompressible fluid (Equation S2), enthalpy is equivalent to internal energy and therefore solely a function of temperature T (K) and water content ω ($\mathrm{m^3m^{-3}}$) (Aschwanden et al., 2012). Enthalpy ($\mathrm{J\,kg^{-1}}$) is defined as,

$$H(T,\omega) = \begin{cases} \int_{T_0}^{T} c(T)\,\mathrm{d}T, & H < H_{\mathrm{f}}(P) \\ \int_{T_0}^{T'(P)} c(T)\mathrm{d}T + \omega L, & H \geq H_{\mathrm{f}}(P) \end{cases}, \tag{S8}$$

where T_0 is a reference temperature (K), $c(T)$ is the specific heat capacity ($\mathrm{J\,kg^{-1}\,K^{-1}}$), $H_{\mathrm{f}}(P)$ is the enthalpy of fusion ($\mathrm{J\,kg^{-1}}$), $T'(P)$ is the temperature relative to the pressure melting point (K), and L is the latent heat of fusion ($\mathrm{J\,kg^{-1}}$). The specific heat capacity of pure ice is a function of temperature (Cuffey and Paterson, 2010):

$$c(T) = 152.5 + 7.122T. \tag{S9}$$

Temperature relative to the pressure melting point is defined as:

$$T'(P) = T_{\mathrm{ptr}} - \mathcal{B}'\left(P - P_{\mathrm{ptr}}\right), \tag{S10}$$

where T_{ptr} (K) and P_{ptr} (MPa) are the temperature and pressure at the triple point of water, respectively and $\mathcal{B}'$ (K MPa^{-1}) is the Clausius-Clapeyron constant (i.e., pressure melting slope). The enthalpy of fusion $H_{\mathrm{f}}(P)$ is the enthalpy of temperate ice with zero liquid water present such that:

$$H_{\mathrm{f}}(P) = \int_{T_0}^{T'(P)} c(T'(P))\mathrm{d}T, \tag{S11}$$

which is equivalent to

$$H_{\mathrm{f}}(P) = 152.5\,(T'(P) - T_0) + \frac{7.122}{2}\left(T'(P)^2 - T_0^2\right). \tag{S12}$$

Enthalpy from temperature and water content

To convert from temperature (T) and water content (ω) to enthalpy, Eqn. S8 becomes:

$$H(T,\omega) = \begin{cases} 152.5\,(T - T_0) + \frac{7.122}{2}\left(T^2 - T_0^2\right), & H < H_{\mathrm{f}}(P) \\ 152.5\,(T'(P) - T_0) + \frac{7.122}{2}\left(T'(P)^2 - T_0^2\right) + \omega L, & H \geq H_{\mathrm{f}}(P)\,, \end{cases} \tag{S13}$$

where T_0 is a reference temperature (K), P is pressure (MPa), $H_{\mathrm{f}}(P)$ is the enthalpy of fusion (J kg^{-1}), $T'(P)$ is the temperature relative to the pressure melting point (K), and L is the latent heat of fusion (J kg^{-1}).

Temperature and water content from enthalpy

To convert from enthalpy (J kg^{-1}) to temperature (K), let us first consider the case of cold ice with zero water content (i.e. $H < H_{\mathrm{f}}(P)$). We begin with a simplified version of Eqn. S13:

$$H(T) = 152.5\,(T - T_0) + \frac{7.122}{2}\left(T^2 - T_0^2\right). \tag{S14}$$

Equation S14 can be rearranged to

$$-\frac{7.122}{2}T^2 + 152.5T + \frac{7.122}{2}T_0^2 + 152.5T_0 + H = 0, \tag{S15}$$

such that it resembles a quadratic polynomial in T. Using the quadratic formula, and enforcing that temperature in K must be bounded by 0, we find the temperature T as a function of enthalpy to be

$$T(H) = \frac{-152.5 + \sqrt{152.5^2 + 2 \cdot 7.122 \cdot \left(\frac{7.122}{2}T_0^2 + 152.5T_0 + H\right)}}{7.122}. \tag{S16}$$

Finally, let us consider temperate ice with a finite water content. Within the temperate ice, the temperature will be equal to the pressure melting point, which can be found using Eqn. S16 with $H = H_{\mathrm{f}}$. All enthalpy in excess of the enthalpy of fusion H_{f}, represents water content, so water content as function of enthalpy reduces to:

$$\omega(H) = \max\left(0, \frac{H - H_{\mathrm{f}}}{L}\right). \tag{S17}$$

S1.3 Thermomechanical coupling

The coupling between the mechanical (Equation S2 & S4) and the thermodynamic (Equation 1) equations is accomplished via the flow-law coefficient A from the constitutive relation (Equation S5) which depends on temperature and water content:

$$A(H) = A_0(T'(H)) \exp\left(-\frac{Q_0(T(H))}{R}\frac{1}{T'(H)}\right) \times (1 + 181.25\,\omega(H)) \tag{S18}$$

$$A_0(T'(H)) = \begin{cases} A^+ & \text{if } T'(H) > T_* \\ A^- & \text{if } T'(H) < T_* \end{cases}, \tag{S19}$$

$$Q_0(T'(H)) = \begin{cases} Q^+ & \text{if } T'(H) > T_* \\ Q^- & \text{if } T'(H) < T_* \end{cases}, \tag{S20}$$

where A^+ and A^- are coefficients ($\mathrm{MPa}^{-3}\,\mathrm{yr}^{-1}$) specific to temperatures above and below the limit temperature $T^* = 263.15$ (K), Q^+ and Q^- are apparent activation energies ($\mathrm{J\,mol}^{-1}$) specific to temperatures above and below the limit temperature, and R is the gas constant ($\mathrm{J\,mol}^{-1}\,\mathrm{K}^{-1}$). Different values of A_0 and Q_0 above and below the limit temperature are used to account for different deformation mechanics in temperate versus cold ice (Cuffey and Paterson, 2010). The final term in Equation S18 accounts for the additional softening of ice due to water content (Greve and Blatter, 2009, pg. 239). In practice, enhancement is limited to $1\,\%$ due to our choosing a reference value of maximum englacial water content ω_{en}=$1\,\%$ (**?**).

S2 Convergence Metrics

S2.1 Mechanical metrics

For the discrete numerical model we calculate the glacier volume per unit width (m^2) as

$$V(t) = \sum_{j=1}^{N_\mathrm{x}} h(x_j, t)\,\Delta x, \tag{S21}$$

where h (m) is the ice thickness at position x_j and time t, N_x is the number of horizontal nodes in the model domain and Δx (m) is the horizontal gridcell spacing. For the sake of simplicity we often use the relative volume per unit width (-), calculated as:

$$V'(t) = \frac{V(t)}{V(0)}, \tag{S22}$$

to assess whether a simulation has reached steady state. We calculate the glacier length L (km) as:

$$L(t) = \sum_{j=1}^{N_\mathrm{x}} \begin{cases} 0 & \text{if } h(t, x_j, z_\mathrm{s}) = 0 \\ \Delta x/1000 & \text{if } h(t, x_j, z_\mathrm{s}) > 0. \end{cases} \tag{S23}$$

S2.2 Thermodynamic metrics

We use the fraction temperate FT (–), fraction of temperate bed FT_{b} (–), and mean enthalpy $\bar{H}$ (J kg−1) to assess whether a thermodynamic steady state has been achieved. Fraction temperate is defined as the fraction of the glacier volume that is at or above the enthalpy of fusion (H_{f}):

$$FT(t) = \sum_{k=1}^{\mathrm{M}} \begin{cases} 0 & \text{if } H_{\mathrm{k}}(t) < H_{\mathrm{f}}(t, P_{\mathrm{k}}) \\ V_{\mathrm{k}}(t)/V(t), & \text{if } H_{\mathrm{k}}(t) \geq H_{\mathrm{f}}(t, P_{\mathrm{k}}), \end{cases} \tag{S24}$$

where V_{k} (m^2), H_{k} (J kg−1), and P_{k} (MPa) are the volume, enthalpy, and pressure of the k^{th} element respectively, M is the total number of elements, and $V(t)$ (m^2) is the total glacier volume. Since we use quadrilateral elements, the total number of elements M is related to number of horizontal (N_{x}) and vertical (N_{z}) nodes by

$$M = (N_{\mathrm{x}} - 1)(N_{\mathrm{z}} - 1). \tag{S25}$$

The fraction of temperate bed FT_{b} is calculated as:

$$FT_{\mathrm{b}}(t) = \frac{1}{L(t)} \sum_{j=1}^{N_{\mathrm{x}}} \begin{cases} 0 & \text{if } H(t, x_j, z_{\mathrm{b}}) < H_{\mathrm{f}}(t, P(t, x_j, z_{\mathrm{b}})) \\ \Delta x/1000 & \text{if } H(t, x_j, z_{\mathrm{b}}) \geq H_{\mathrm{f}}(t, P(t, x_j, z_{\mathrm{b}})), \end{cases} \tag{S26}$$

where L (km) is the glacier length, H (J kg−1) and H_{f} (J kg−1) are the enthalpy and enthalpy of fusion along the ice base (z_{b}) at horizontal position x_j and time t. We use nodal values of enthalpy (index j vs. index k) to calculate FT_{b} since the ice/bed interface can be temperate with no overlying volumetric temperate ice. The weighted mean enthalpy is calculated as

$$\bar{H}(t) = \sum_{k=1}^{\mathrm{M}} H_{\mathrm{k}}(t) \frac{V_{\mathrm{k}}(t)}{V(t)}, \tag{S27}$$

where the elemental enthalpy $H_{\mathrm{k}}(t)$ is weighted by the variable element size $V_{\mathrm{k}}(t)$.

We also calculate the local Péclet number $\mathbf{Pe} = (\mathrm{Pe}_{\mathrm{x}}, \mathrm{Pe}_{\mathrm{z}})$, which is a non-dimensional vector value field with components

$$\begin{aligned} \mathrm{Pe}_{\mathrm{x}} &= \Delta x \frac{u}{\kappa/\rho} \\ \mathrm{Pe}_{\mathrm{z}} &= \Delta z \frac{w}{\kappa/\rho}, \end{aligned} \tag{S28}$$

where Δx (m) and Δz (m) are horizontal and vertical gridcell spacings, u (m yr^{-1}) and w (m yr^{-1}) are the x and z components of the velocity vector $\boldsymbol{u}$, ρ (kg m^{-3}) is density, and κ (kg m^{-1} yr^{-1}) is enthalpy diffusivity. We often use the magnitude of the Péclet vector Pe for visualization:

$$\mathrm{Pe} = |\mathbf{Pe}| = \left((\mathrm{Pe}_{\mathrm{x}})^2 + (\mathrm{Pe}_{\mathrm{z}})^2\right)^{1/2}. \tag{S29}$$

The Péclet number denotes the ratio of advection to diffusion, the two mechanisms of energy transport in Equation 1. $\mathrm{Pe} < 1$ indicates a diffusion-dominated regime, while $\mathrm{Pe} > 1$ indicates an advection-dominated regime.

S3 Additional figures

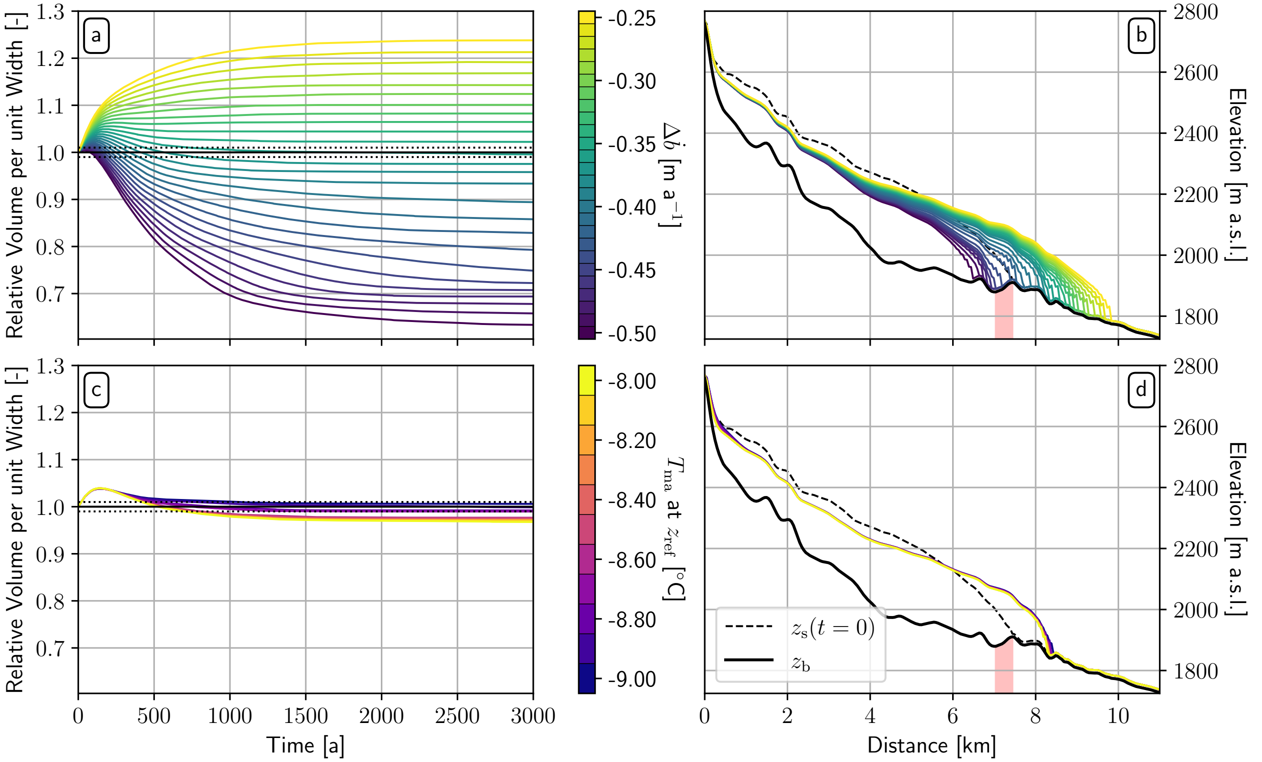

Figure S1. Relative volume per unit width as a function of time (a, c) and steady-state surface elevation (b,d) for different values of $\Delta\dot{b}$ (a,b) and $\hat{T}_{\mathrm{a}}$ (c,d). (a) $V'(t)$ and (b) $z_{\mathrm{s}}(t=3000)$ for various $\Delta\dot{b}$ values (color) with $\hat{T}_{\mathrm{a}} = -8.5°\mathrm{C}$. (c) $V'(t)$ and (d) $z_{\mathrm{s}}(t=3000)$ for various $\hat{T}_{\mathrm{a}}$ values (color) with $\Delta\dot{b} = -0.37\,\mathrm{m\,yr^{-1}}$. (a,c) Dotted black lines represent $V' = 1 \pm 0.01$. (b,d) Shaded red area highlights retrograde bed slope, which affects convergence to steady state.

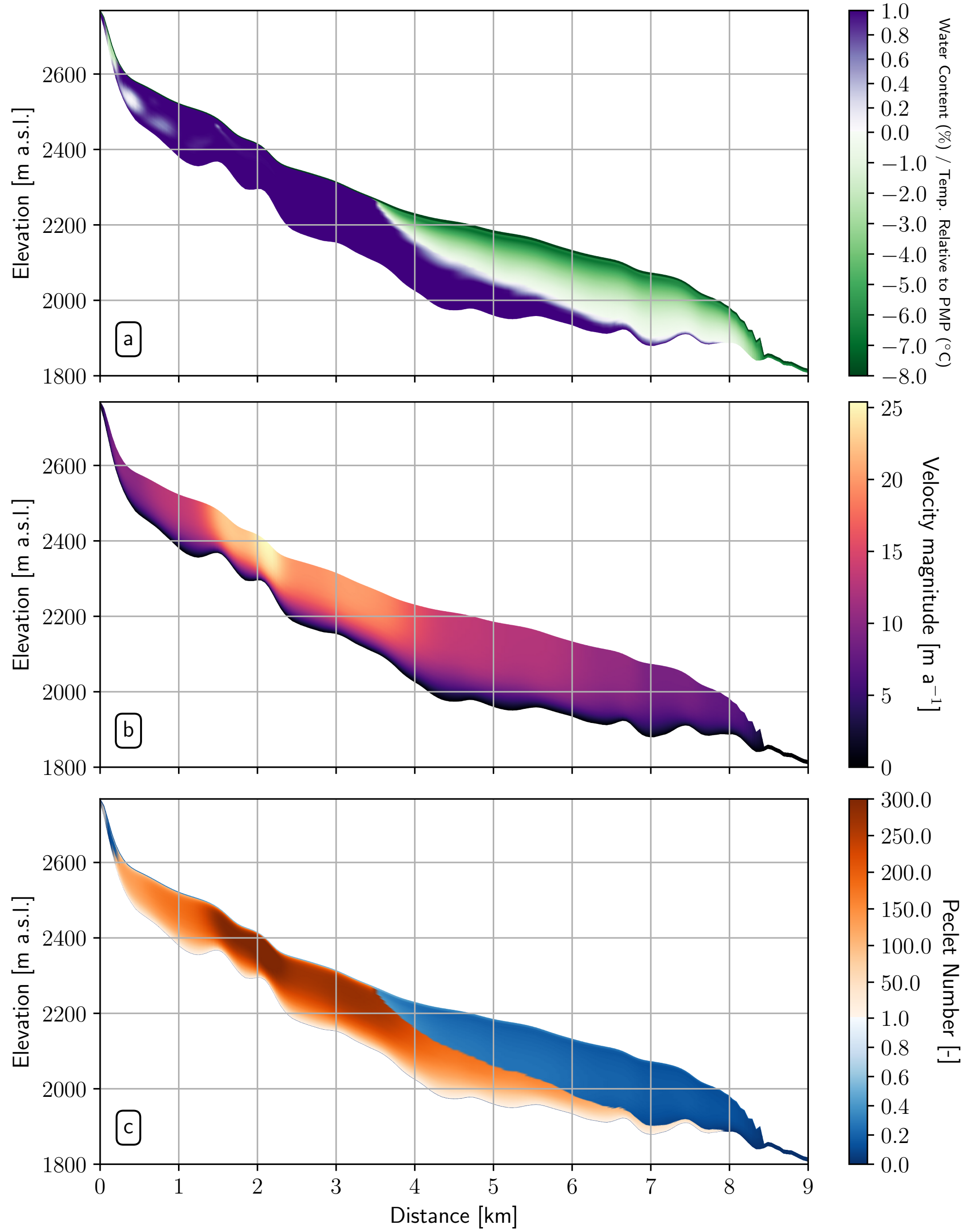

Figure S2. Steady-state reference glacier fields ($T_{\mathrm{ma}} = -8.5\,^{\circ}\mathrm{C}$, $\Delta \dot{b} = -0.36\,\mathrm{m\,y^{-1}}$). (a) Water content (%; purple) and temperature relative to the pressure melting point ($^{\circ}$C; green). (b) Velocity magnitude v_{m} ($\mathrm{m\,y^{-1}}$). (c) Péclet number magnitude Pe.

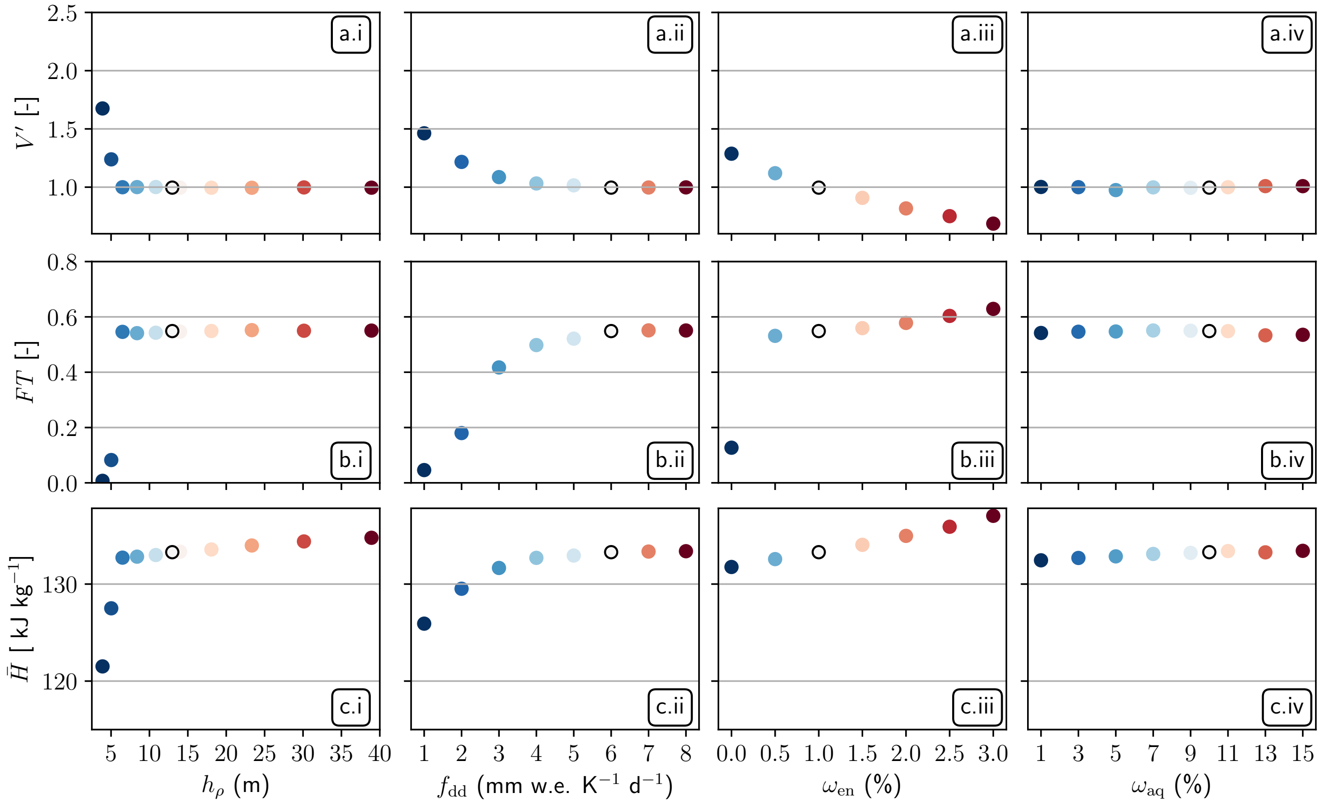

Figure S3. Sensitivity test results after 3kyr of initialization. (a) Relative volume per unit width V', (b) fraction temperate FT, and (c) mean enthalpy $\bar{H}$ ($\mathrm{kJ\,kg^{-1}}$) for variations in: (i) firn column thickness h_ρ (m), (ii) degree-day factor for snow $f_{\rm dd}$ ($\mathrm{mm\,w.e.\,K^{-1}\,d^{-1}}$), (iii) maximum englacial water content $\omega_{\rm en}$ (%), and (iv) maximum firn aquifer water content $\omega_{\rm aq}$ (%). Reference value for each parameter is denoted by the open circle. Blue markers denote values less than the reference, while red markers denote values greater.

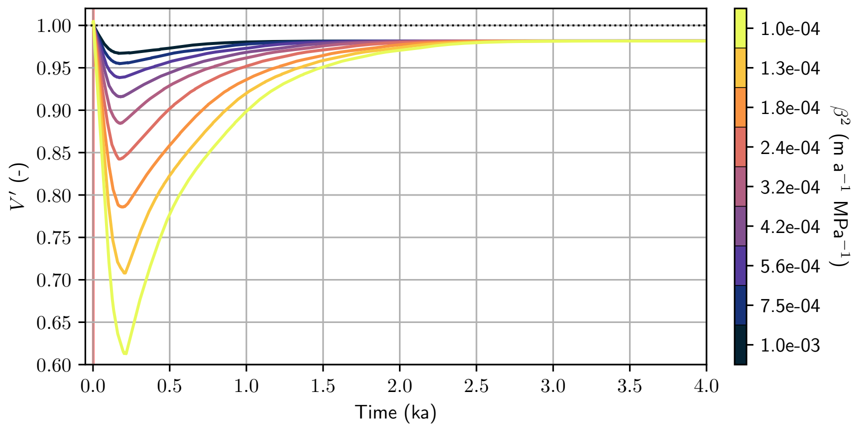

Figure S4. Recovery following a two-year surge starting at time 0 (vertical pink line) for an isothermal control case. Relative volume per unit width V' is shown as a function of time and the basal drag coefficient β^2. Dotted black line denotes the steady-state (i.e., pre-surge) value $V' = 1$ for the reference glacier.