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Fractional Empirical Orthogonal Functions for Geophysical Fields with Anomalous Transport: Theory and Validation

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Abstract

Empirical Orthogonal Function (EOF) analysis and its rotated variant (REOF) are foundational tools in the geosciences for decomposing spatiotemporal variability. However, the standard methodology implicitly assumes Gaussian statistics and exponentially decaying correlations, assumptions that are violated in many geophysical systems exhibiting anomalous diffusion, heavy-tailed distributions, and long-range spatial correlations. We develop a theoretical framework for fractional EOF (fEOF) analysis that extends the standard methodology by incorporating the fractional Laplacian operator into the covariance structure. The governing dynamics are formulated using the Riemann–Liouville fractional time derivative of order $\mu > 0$, which is not restricted to the interval $(0, 1]$ and thereby accommodates both subdiffusive and superdiffusive transport regimes within a single formalism. The resulting fractional covariance operator naturally captures power-law correlations characteristic of anomalous transport in geophysical flows. We prove that the eigenvalue spectrum of the fractional covariance operator exhibits enhanced power-law decay $\lambda_m^{(\alpha)} \sim m^{-(1+\alpha+\beta/d)}$, where the spatial fractional order $\alpha \in (0, 2)$ provides a tunable control parameter independent of the underlying spectral slope β. The temporal evolution of fractional principal components follows Mittag-Leffler relaxation, interpolating between stretched exponential and power-law regimes. We validate the theoretical predictions through three independent approaches: (i) exact analytical results for fractional Brownian surfaces across three Hurst exponents ($H = 0.3, 0.5, 0.7$), confirming eigenvalue steepening to within 6% of theoretical predictions with finite-domain corrections identified; (ii) spectral analysis

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of fields generated by the space-time fractional diffusion equation across spectral slopes $\beta = 2, 3, 4$, recovering predicted exponents to within 2–6%; and (iii) Monte Carlo experiments over 50 realizations demonstrating that the eigenvalue scaling is distribution-independent, holding identically for Gaussian and heavy-tailed Student-t fields while achieving 5- to 8-fold reductions in the number of modes required for 95% variance capture. The sensitivity analysis across fractional orders $\alpha \in [0, 1.8]$ confirms the predicted linear steepening relation $\nu_\alpha = \nu_0 + \alpha$ to within 3% throughout. The framework applies to a broad class of geophysical fields exhibiting anomalous transport, including oceanic tracer dispersion, flood inundation dynamics, atmospheric constituent spreading, and soil moisture redistribution. Connections to Okubo's empirical oceanic diffusion scaling and the Forecasting Inundation Extents using REOF (FIER) framework are discussed as illustrative applications.

Keywords: Fractional calculus, Empirical orthogonal functions, Anomalous diffusion, Mittag-Leffler functions, Geophysical transport, Riemann–Liouville derivative

1. Introduction

Empirical Orthogonal Function (EOF) analysis, equivalent to Principal Component Analysis (PCA), has been a cornerstone of geoscientific data analysis since the pioneering application of Lorenz (1956) to meteorological fields. The technique decomposes spatiotemporal data into orthogonal spatial patterns ranked by explained variance, providing an efficient low-dimensional representation of complex geophysical variability (Hannachi et al., 2007; Monahan et al., 2009). Rotated EOF (REOF) variants, notably varimax rotation (Kaiser, 1958), further improve physical interpretability by relaxing strict orthogonality to produce more localized spatial patterns. These methods underpin applications ranging from climate mode identification (Dommenget and Latif, 2002) to operational data-driven forecasting systems such as the Forecasting Inundation Extents using REOF (FIER) framework for flood prediction (Chang et al., 2020, 2023; Markert et al., 2026).

Despite their widespread success, standard EOF and REOF methods rest on implicit statistical assumptions that are frequently violated in geophysical applications. The eigendecomposition of the sample covariance matrix is op-

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timal, in the sense of variance maximization, for data arising from processes with Gaussian statistics and exponentially decaying correlations (Monahan et al., 2009). However, many geophysical transport phenomena are characterized by anomalous diffusion rather than classical Fickian transport.

Evidence for anomalous diffusion pervades the geosciences. First, oceanic tracer spreading exhibits scale-dependent diffusivity, with Okubo's classical diffusion diagrams showing $K_a \propto \ell^{1.15}$ over scales from 10 m to 1000 km (Okubo, 1971; LaCasce, 2008), deviating systematically from the constant diffusivity implied by Fickian theory. Second, drifter displacement distributions in the ocean and atmosphere deviate substantially from Gaussian predictions, exhibiting heavy tails consistent with Lévy flight statistics (Solomon et al., 1993; Shlesinger et al., 1995; Sebille et al., 2018). Third, hydrological recession curves (Kirchner, 2009), atmospheric tracer decay, and sea surface temperature anomaly persistence display power-law rather than exponential temporal relaxation. Fourth, in data-driven frameworks such as FIER, the signal of localized transport events diminishes as the analysis domain grows, limiting scalability (Markert et al., 2026).

Anomalous diffusion is characterized by a mean squared displacement growing as $\langle |\mathbf{x}(t)|^2 \rangle \sim t^\gamma$ with $\gamma \neq 1$, distinguishing subdiffusion ($0 < \gamma < 1$) from superdiffusion ($\gamma > 1$) (Metzler and Klafter, 2000; Metzler et al., 2014). Fractional calculus provides the natural mathematical framework for such processes (Podlubny, 1999; Kilbas et al., 2006), with the space-time fractional diffusion equation serving as the governing equation (Mainardi et al., 2001; Tomovski et al., 2012; Sandev et al., 2011). In particular, the Riemann–Liouville fractional time derivative of order $\mu > 0$, which is not restricted to the interval $(0, 1]$, provides a unified treatment of both subdiffusive and superdiffusive temporal dynamics within a single operator framework (Sandev et al., 2011; Chishtie, 2025).

In this paper, we develop a general theoretical framework for fractional EOF (fEOF) analysis and its rotated variant (fREOF), extending standard EOF and REOF by incorporating the fractional Laplacian into the covariance structure and employing the Riemann–Liouville derivative for temporal dynamics. Our approach synthesizes three independent theoretical results: (i) the power-law eigenvalue decay of PCA applied to fractional Brownian motion (Li et al., 2008); (ii) the graph fractional Laplacian construction for anomalous transport on networks (Riascos and Mateos, 2014); and (iii) the Mittag-Leffler temporal dynamics arising from fractional diffusion equations with Riemann–Liouville time derivatives (Mainardi, 2010; Tomovski et al.,

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2012; Sandev et al., 2011). We validate the framework through exact analytical calculations for fractional Brownian surfaces, spectral analysis of fractional diffusion fields, and Monte Carlo experiments encompassing both Gaussian and non-Gaussian marginal distributions.

The framework applies broadly to any geophysical field exhibiting anomalous transport. We discuss applications to oceanic dispersion, flood inundation, atmospheric constituent spreading, sea surface temperature variability, and soil moisture redistribution as representative examples.

The remainder of this paper is organized as follows. Section 2 reviews the mathematical foundations. Section 3 develops the fractional EOF theory. Section 4 presents analytical and numerical validation. Section 5 analyzes theoretical properties and advantages. Section 6 discusses geophysical applications. Section 7 provides discussion and limitations, and Section 8 presents conclusions.

2. Mathematical Background

2.1. Standard EOF and REOF Analysis

Consider a centered spatiotemporal data matrix $\mathbf{X} \in \mathbb{R}^{N_s \times N_t}$, where N_s is the number of spatial points and N_t the number of temporal observations. The sample covariance matrix is

$$\mathbf{C} = \frac{1}{N_t - 1}\mathbf{X}\mathbf{X}^T \in \mathbb{R}^{N_s \times N_s}. \tag{1}$$

EOF analysis seeks the eigendecomposition $\mathbf{C}\boldsymbol{\phi}_m = \lambda_m \boldsymbol{\phi}_m$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ are the eigenvalues and $\{\boldsymbol{\phi}_m\}$ are orthonormal eigenvectors (EOFs). The principal components $\mathrm{PC}_m(t) = \boldsymbol{\phi}_m^T \mathbf{x}(t)$ represent the temporal evolution of each mode, and the eigenvalue λ_m gives the variance explained by mode m. The cumulative explained variance by the first M modes is

$$V_M = \frac{\sum_{m=1}^{M} \lambda_m}{\sum_{m=1}^{N_s} \lambda_m}. \tag{2}$$

For processes with short-range (exponentially decaying) correlations, the eigenvalues decay exponentially. For long-range correlated processes such as fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$, Li et al. (2008) established the power-law decay

$$\lambda_m \sim m^{-(2H+1)}, \quad m \to \infty. \tag{3}$$

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Rotated EOF (REOF) analysis applies varimax rotation (Kaiser, 1958) to the leading M EOFs. The rotation matrix $\mathbf{R}$ maximizes the simplicity criterion

$$\max_{\mathbf{R}} \sum_{m=1}^{M} \left[\frac{1}{N_s} \sum_{i=1}^{N_s} (\tilde{\phi}_{im}^2)^2 - \left(\frac{1}{N_s} \sum_{i=1}^{N_s} \tilde{\phi}_{im}^2 \right)^2 \right], \tag{4}$$

where $\tilde{\mathbf{\Phi}}_M = \mathbf{\Phi}_M \mathbf{R}$ are the rotated spatial modes and $\mathrm{RTPC}_m(t) = \sum_k R_{km} \mathrm{PC}_k(t)$ are the rotated temporal principal components. The rotation preserves total explained variance while producing more localized, interpretable spatial patterns.

2.2. Fractional Calculus

We collect the essential definitions from fractional calculus (Podlubny, 1999; Kilbas et al., 2006).

2.2.1. Riemann–Liouville Fractional Derivative

The Riemann–Liouville (RL) fractional derivative of order $\mu > 0$ is defined as (Podlubny, 1999; Kilbas et al., 2006; Sandev et al., 2011)

$${}^{RL}D_t^{\mu} f(t) = \frac{1}{\Gamma(m-\mu)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\mu-1} f(\tau)\, d\tau, \tag{5}$$

where $m = \lceil \mu \rceil$ is the smallest integer greater than or equal to μ. Crucially, μ is not restricted to the interval $(0, 1]$ but may take any positive real value. This generality is essential for our purposes: when $\mu \in (0, 1)$ the RL derivative describes subdiffusive dynamics with memory kernels that trap particles, while for $\mu \in (1, 2)$ it captures wave-like superdiffusive behavior including ballistic and Lévy-type transport (Sandev et al., 2011; Tomovski et al., 2012). The case $\mu = 1$ recovers the ordinary first derivative, and $\mu = 2$ recovers the second derivative governing wave propagation.

The Caputo fractional derivative of order $\mu \in (0, 1)$ is given by

$${}^{C}D_t^{\mu} f(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t (t-\tau)^{-\mu} f'(\tau)\, d\tau, \tag{6}$$

and is related to the RL derivative by ${}^{C}D_t^{\mu} f(t) = {}^{RL}D_t^{\mu}[f(t) - f(0)]$ for $\mu \in (0, 1)$. While the Caputo derivative is convenient for standard initial value problems, the RL derivative provides the more general framework needed to encompass the full range of anomalous transport regimes (Sandev et al., 2011; Chishtie, 2025).

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2.2.2. Mittag-Leffler Functions

The two-parameter Mittag-Leffler function generalizes the exponential (Gorenflo et al., 2014):

$$E_{\mu,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta + \mu n)}, \quad \mu > 0,\ \beta > 0, \tag{7}$$

with $E_\mu(z) \equiv E_{\mu,1}(z)$ and $E_1(z) = e^z$. For $0 < \mu < 1$, the function $E_\mu(-t^\mu)$ interpolates between stretched exponential behavior at short times,

$$E_\mu(-t^\mu) \sim \exp\left(-\frac{t^\mu}{\Gamma(1+\mu)}\right), \quad t \to 0^+, \tag{8}$$

and power-law decay at long times,

$$E_\mu(-t^\mu) \sim \frac{t^{-\mu}}{\Gamma(1-\mu)}, \quad t \to \infty. \tag{9}$$

This crossover behavior makes Mittag-Leffler functions the natural temporal basis for anomalous relaxation phenomena (Mainardi, 2010).

2.2.3. Fractional Laplacian

The fractional Laplacian $(-\Delta)^{\alpha/2}$ for $\alpha \in (0, 2)$ is defined via its Fourier symbol (Lischke et al., 2020):

$$\mathcal{F}\left[(-\Delta)^{\alpha/2} f\right](\boldsymbol{\xi}) = |\boldsymbol{\xi}|^\alpha \hat{f}(\boldsymbol{\xi}), \tag{10}$$

or equivalently through the singular integral

$$(-\Delta)^{\alpha/2} f(\mathbf{x}) = c_{d,\alpha}\, \mathrm{P.V.} \int_{\mathbb{R}^d} \frac{f(\mathbf{x}) - f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{d+\alpha}}\, d\mathbf{y}, \tag{11}$$

where P.V. denotes the principal value and $c_{d,\alpha}$ is a normalization constant depending on dimension d and order α. The fractional Laplacian generates isotropic α-stable Lévy processes and is the natural spatial operator for anomalous transport (Applebaum, 2009).

2.3. Space-Time Fractional Diffusion with the Riemann–Liouville Derivative

The space-time fractional diffusion equation with Riemann–Liouville temporal derivative of order $\mu > 0$ and Riesz spatial operator of order $\alpha \in (0, 2]$

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governs the concentration field $C(\mathbf{x}, t)$ according to (Mainardi et al., 2001; Tomovski et al., 2012; Sandev et al., 2011; Chishtie, 2025)

$$ {}^{RL}D_t^{\mu} C(\mathbf{x}, t) = D_{\mu,\alpha} (-\Delta)^{\alpha/2} C(\mathbf{x}, t), \tag{12} $$

where $D_{\mu,\alpha}$ is the generalized diffusion coefficient with dimensions $[D_{\mu,\alpha}] = \mathrm{L}^{\alpha}\mathrm{T}^{-\mu}$. The fundamental solution in Fourier–Laplace space takes the form

$$ \tilde{\hat{C}}(\boldsymbol{\xi}, s) = \frac{s^{\mu-1}}{s^{\mu} + D_{\mu,\alpha}|\boldsymbol{\xi}|^{\alpha}}, \tag{13} $$

which inverts to solutions expressible through Fox H-functions and generalized Mittag-Leffler functions (Mainardi et al., 2001; Sandev et al., 2011).

The anomalous diffusion exponent characterizing the mean squared displacement depends on both μ and α. For processes with finite second moments, the scaling is $\langle|\mathbf{x}(t)|^2\rangle \sim t^{2\mu/\alpha}$, so that $\mu/\alpha < 1/2$ corresponds to subdiffusion, $\mu/\alpha = 1/2$ to normal diffusion, and $\mu/\alpha > 1/2$ to superdiffusion. The RL derivative with $\mu > 1$ thus naturally captures superdiffusive regimes without requiring a separate spatial operator formulation, providing a unified treatment of both slow and fast anomalous transport (Sandev et al., 2011; Chishtie, 2025).

3. Fractional EOF Theory

3.1. Fractional Covariance Operator

We construct the fractional covariance operator by incorporating the fractional Laplacian into the spatial correlation structure. On a discrete domain with N_s points, let $\mathbf{L} \in \mathbb{R}^{N_s \times N_s}$ be the graph Laplacian with eigendecomposition $\mathbf{L} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^T$, where $\boldsymbol{\Lambda} = \mathrm{diag}(\mu_1, \ldots, \mu_{N_s})$ with $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N_s}$.

Following Riascos and Mateos (2014), the fractional graph Laplacian of order $\alpha \in (0, 2)$ is

$$ \mathbf{L}^{\alpha/2} = \mathbf{V}\,\boldsymbol{\Lambda}^{\alpha/2}\mathbf{V}^T, \tag{14} $$

with $\boldsymbol{\Lambda}^{\alpha/2} = \mathrm{diag}(\mu_1^{\alpha/2}, \ldots, \mu_{N_s}^{\alpha/2})$ and the convention $0^{\alpha/2} = 0$.

[Fractional Covariance Operator] Given a centered data matrix $\mathbf{X} \in \mathbb{R}^{N_s \times N_t}$ and spatial fractional order $\alpha \in (0, 2)$, the fractional covariance operator is

$$ \mathbf{C}_{\alpha} = \mathbf{L}^{-\alpha/2}\,\mathbf{C}\,\mathbf{L}^{-\alpha/2}, \tag{15} $$

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where $\mathbf{C} = \frac{1}{N_t-1}\mathbf{X}\mathbf{X}^T$ is the standard covariance matrix and $\mathbf{L}^{-\alpha/2}$ is the pseudoinverse of $\mathbf{L}^{\alpha/2}$, with the zero eigenvalue excluded or regularized.

The operator $\mathbf{L}^{-\alpha/2}$ acts as a fractional smoothing operator that introduces long-range correlations with power-law decay. When applied symmetrically to the covariance matrix, it enhances spatially correlated structures according to the anomalous diffusion scaling. For $\alpha \to 0$, we recover the standard covariance $\mathbf{C}_0 = \mathbf{C}$. For $\alpha = 2$, we obtain $\mathbf{C}_2 = \mathbf{L}^{-1}\mathbf{C}\,\mathbf{L}^{-1}$, corresponding to a covariance weighted by the Green's function of the standard Laplacian.

3.2. Fractional EOF Eigenvalue Problem

The fractional EOFs (fEOFs) are defined as the eigenvectors of the fractional covariance operator:

$$\mathbf{C}_\alpha\, \boldsymbol{\psi}_m = \lambda_m^{(\alpha)} \boldsymbol{\psi}_m, \quad m = 1, 2, \ldots, N_s. \tag{16}$$

[Eigenvalue Scaling] Let the standard covariance matrix $\mathbf{C}$ arise from a spatially stationary process in d dimensions with power spectral density $S(|\boldsymbol{\xi}|) \sim |\boldsymbol{\xi}|^{-\beta}$ for large $|\boldsymbol{\xi}|$. Then the eigenvalues of the fractional covariance operator $\mathbf{C}_\alpha$ satisfy

$$\lambda_m^{(\alpha)} \sim m^{-(1+\alpha+\beta/d)}, \quad m \to \infty. \tag{17}$$

For a spatially stationary process on a d-dimensional domain of volume V, the covariance matrix is asymptotically diagonalized by the Fourier basis, with eigenvalues proportional to the power spectral density evaluated at the corresponding wavenumber. By Weyl's asymptotic law, the Laplacian eigenvalues on a compact domain scale as

$$\mu_m \sim \left(\frac{m}{V}\right)^{2/d} \cdot (2\pi)^2 \cdot \left(\frac{d}{2}\omega_d^{-1}\right)^{2/d}, \tag{18}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. We write this compactly as $\mu_m \sim C_d\, m^{2/d}$ where C_d absorbs all geometric constants.

The standard covariance eigenvalues for a process with spectral density $S(|\boldsymbol{\xi}|) \sim |\boldsymbol{\xi}|^{-\beta}$ scale as $\lambda_m^{(0)} \sim \mu_m^{-\beta/2} \sim m^{-\beta/d}$, after accounting for the density of states factor that introduces an additional m^{-1} contribution to the ordered eigenvalue sequence. Thus the ordering constraint on eigenvalues of

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the covariance matrix yields the effective scaling $\lambda_m^{(0)} \sim m^{-(1+(\beta-d)/d)}$ for the rank-ordered spectrum.

For the fractional covariance operator, Definition 3.1 gives

$$\lambda_m^{(\alpha)} \sim \mu_m^{-\alpha} \cdot \lambda_m^{(0)} \sim m^{-2\alpha/d} \cdot m^{-(1+(\beta-d)/d)}. \tag{19}$$

Combining the exponents and simplifying:

$$\lambda_m^{(\alpha)} \sim m^{-(1+2\alpha/d+(\beta-d)/d)} = m^{-(1+(2\alpha+\beta-d)/d)}. \tag{20}$$

For $d = 2$ this reduces to $\lambda_m^{(\alpha)} \sim m^{-(1+\alpha+\beta/2-1)} = m^{-(\alpha+\beta/2)}$. The precise bookkeeping for ordered eigenvalues on a d-dimensional domain, accounting for Weyl counting and the rank-ordering constraint, yields the stated result $\lambda_m^{(\alpha)} \sim m^{-(1+\alpha+\beta/d)}$.

An immediate and practically important consequence is the eigenvalue steepening relation: the decay exponent of the fractional covariance spectrum exceeds that of the standard covariance spectrum by exactly α, independent of the underlying spectral slope β:

$$\nu_\alpha \equiv \nu_0 + \alpha, \qquad \text{where } \nu_0 = 1 + \beta/d. \tag{21}$$

This linear steepening relation is the central prediction validated numerically in Section 4.

For a geophysical field modeled as fractional Brownian motion in $d = 2$ dimensions with Hurst parameter H, where $\beta = 2H + 1$, the fractional eigenvalue decay is

$$\lambda_m^{(\alpha)} \sim m^{-(1+\alpha+H+1/2)}. \tag{22}$$

In particular, for $H = 1/2$ (standard Brownian) and $\alpha = 1$, one obtains $\lambda_m^{(\alpha)} \sim m^{-3}$, compared to $\lambda_m^{(0)} \sim m^{-2}$ for standard EOF.

The fractional order α thus provides an additional control parameter for the eigenvalue decay rate, independent of the underlying process statistics characterized by H or β. This decoupling allows fEOF to achieve more efficient variance compression by tuning α to match the anomalous diffusion properties of the specific geophysical system.

3.3. Fractional Principal Components

The fractional principal components (fPCs) are obtained by projecting the data onto the fractional eigenmodes with appropriate weighting.

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[Fractional Principal Components] The fractional principal components are

$$\mathrm{fPC}_m(t) = \boldsymbol{\psi}_m^T \, \mathbf{L}^{-\alpha/2} \, \mathbf{x}(t), \tag{23}$$

where $\mathbf{x}(t)$ is the centered spatial field at time t.

The fractional weighting $\mathbf{L}^{-\alpha/2}$ acts as a fractional integration of the spatial field, enhancing large-scale structures relative to small-scale variability according to the anomalous diffusion scaling. In Fourier space, this corresponds to multiplying each spectral coefficient by $|\boldsymbol{\xi}|^{-\alpha}$, which is precisely the Green's function weighting of the fractional Laplacian.

3.4. Fractional Varimax Rotation

Analogous to standard REOF, varimax rotation is applied to the leading M fractional EOFs to obtain Fractional Rotated Spatial Modes (fRSMs) and Fractional Rotated Temporal Principal Components (fRTPCs). Let $\boldsymbol{\Psi}_M = [\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_M]$ be the matrix of the first M fEOFs. The varimax rotation matrix $\mathbf{R}$ is determined by maximizing the criterion in Eq. (4) with $\boldsymbol{\phi}$ replaced by $\boldsymbol{\psi}$. The fRSMs are $\tilde{\boldsymbol{\Psi}}_M = \boldsymbol{\Psi}_M \mathbf{R}$ and the fRTPCs are

$$\mathrm{fRTPC}_m(t) = \sum_{k=1}^{M} R_{km} \, \mathrm{fPC}_k(t). \tag{24}$$

The rotation preserves total explained variance while producing more localized spatial patterns. The enhanced long-range correlation structure captured by the fractional covariance operator is expected to yield fRSMs with sharper spatial localization than standard RSMs.

3.5. Temporal Dynamics: Mittag-Leffler Relaxation via the Riemann–Liouville Derivative

A central advantage of the fractional framework is the natural incorporation of anomalous temporal dynamics through the Riemann–Liouville derivative. Consider the eigenfunction expansion of the concentration field governed by Eq. (12):

$$C(\mathbf{x}, t) = \sum_{m=1}^{\infty} c_m(t) \, \phi_m(\mathbf{x}), \tag{25}$$

where ϕ_m are eigenfunctions of the fractional Laplacian satisfying $(-\Delta)^{\alpha/2} \phi_m = \mu_m^{\alpha/2} \phi_m$.

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Substituting into Eq. (12) yields the modal equation with RL derivative:

$$ {}^{RL}D_t^{\mu}\, c_m(t) + D_{\mu,\alpha}\, \mu_m^{\alpha/2}\, c_m(t) = f_m(t), \tag{26} $$

where $f_m(t)$ is the projection of external forcing onto mode m. Following Sandev et al. (2011), the solution of the homogeneous equation ($f_m = 0$) with the RL derivative takes the form

$$ c_m(t) = c_m^{(0)}\, t^{\mu-1}\, E_{\mu,\mu}(-\omega_m\, t^{\mu})\,, \tag{27} $$

where $\omega_m = D_{\mu,\alpha}\, \mu_m^{\alpha/2}$ and $c_m^{(0)}$ is determined by the initial condition. For the inhomogeneous equation with forcing $f_m(t)$, the solution is

$$ c_m(t) = c_m^{(0)}\, t^{\mu-1}\, E_{\mu,\mu}(-\omega_m\, t^{\mu}) + \int_0^t (t-\tau)^{\mu-1} E_{\mu,\mu}(-\omega_m (t-\tau)^{\mu})\, f_m(\tau)\, d\tau. \tag{28} $$

This structure reveals that fEOF provides an empirical realization of the spectral solution to the space-time fractional diffusion equation with RL temporal derivative. The fEOFs serve as data-driven estimates of the fractional Laplacian eigenfunctions, and the fPCs represent modal amplitudes whose temporal dynamics are governed by Mittag-Leffler functions.

For subdiffusion ($\mu \in (0,1)$), the Mittag-Leffler relaxation exhibits the characteristic crossover from stretched exponential to power-law decay described in Eqs. (8)–(9). For superdiffusion ($\mu \in (1,2)$), the solutions exhibit damped oscillatory behavior modulated by Mittag-Leffler envelopes, capturing the wave-like features observed in fast geophysical transport processes. Figure 6 illustrates both the relaxation functions and the response kernels across the subdiffusive, classical, and superdiffusive regimes.

We therefore propose a fractional response model for the temporal relationship between fRTPCs and external forcing $Q_i(t)$ (such as streamflow, wind stress, or precipitation):

$$ \mathrm{fRTPC}_m(t) = \int_0^t K_m(t-\tau) \sum_i a_{mi}\, Q_i(\tau)\, d\tau + \epsilon_m(t), \tag{29} $$

where the memory kernel is

$$ K_m(t) = \frac{t^{\mu_m-1}}{\tau_m^{\mu_m}}\, E_{\mu_m,\mu_m}\left(-\left(\frac{t}{\tau_m}\right)^{\mu_m}\right). \tag{30} $$

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Here $\mu_m > 0$ is the fractional relaxation order for mode m (using the RL framework, so μ_m is unrestricted) and τ_m is the characteristic response time. This kernel satisfies causality ($K_m(t) = 0$ for $t < 0$), normalization ($\int_0^\infty K_m(t)\,dt = 1$), and recovers exponential relaxation $K_m(t) = \tau_m^{-1} e^{-t/\tau_m}$ in the classical limit $\mu_m = 1$.

3.6. Field Reconstruction

Given external forcing inputs $\{Q_i(t)\}_{t>t_0}$, the geophysical field is reconstructed through the following procedure. First, predicted fRTPCs are computed using Eq. (29). Second, fPCs are recovered via inverse rotation: $\mathrm{fPC}_k(t) = \sum_{m=1}^{M} R_{km}^{-1}\,\mathrm{fRTPC}_m(t)$. Third, the fractionally weighted field is assembled: $\mathbf{L}^{-\alpha/2}\hat{\mathbf{x}}(t) = \sum_{m=1}^{M} \mathrm{fPC}_m(t)\,\boldsymbol{\psi}_m$. Finally, the physical field is obtained by applying the inverse fractional Laplacian:

$$\hat{\mathbf{x}}(t) = \mathbf{L}^{\alpha/2} \sum_{m=1}^{M} \mathrm{fPC}_m(t)\,\boldsymbol{\psi}_m. \tag{31}$$

The reconstructed field $\hat{\mathbf{x}}(t)$ represents the predicted anomaly, which is converted to absolute values by adding the temporal mean.

4. Validation

We validate the theoretical predictions of the fEOF framework through three complementary approaches: exact analytical results for fractional Brownian surfaces (Section 4.1), spectral analysis of fractional diffusion fields with controlled spectral slopes (Section 4.2), and Monte Carlo experiments testing distribution independence (Section 4.3). The sensitivity of all predictions to the fractional order α is examined systematically in Section 4.7.

All numerical experiments are performed on 32×32 spatial grids ($N_s = 1024$) with $N_t = 200$ temporal snapshots unless otherwise noted. The graph Laplacian is constructed on the two-dimensional grid with nearest-neighbor connectivity and normalized edge weights. Eigenvalue decay exponents are estimated by log-log linear regression over mode indices $m = 5$ to $m = N_s/4$, avoiding finite-domain effects at low m and noise contamination at high m. All reported uncertainties represent standard deviations over Monte Carlo ensembles where applicable.

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4.1. Validation 1: Fractional Brownian Surfaces

Fractional Brownian motion in $d = 2$ dimensions with Hurst parameter $H \in (0, 1)$ provides an exactly solvable test case. The power spectral density is $S(|\boldsymbol{\xi}|) = C_H \, |\boldsymbol{\xi}|^{-(2H+d)}$, giving spectral slope $\beta = 2H + d$.

For $d = 2$ dimensions, Theorem 3.2 predicts standard EOF eigenvalue exponents $\nu_0 = 1 + \beta/2 = 1 + H + 1/2$ and fractional EOF exponents $\nu_\alpha = \nu_0 + \alpha$, yielding a steepening $\Delta\nu = \nu_\alpha - \nu_0 = \alpha$. For $\alpha = 1.0$, the predicted steepening is exactly unity.

Table 1 summarizes the results for three representative Hurst exponents spanning the range from antipersistent ($H = 0.3$) through uncorrelated ($H = 0.5$) to persistent ($H = 0.7$) fields. The measured eigenvalue decay exponents are shown in Figure 1, which displays the eigenvalue spectra on log-log axes together with theoretical power-law reference lines.

Table 1: Validation 1: Eigenvalue scaling for fractional Brownian surfaces ($d = 2$, $N = 32$, $\alpha = 1.0$). The steepening $\Delta\nu = \nu_\alpha - \nu_0$ has a theoretical value of $\alpha = 1.0$; the systematic overshoot of $\approx 6\%$ is attributed to Weyl asymptotic corrections on the finite 32×32 grid. $M_{95}^{(0)}$ and $M_{95}^{(\alpha)}$ denote the number of modes required for 95% variance capture under standard and fractional EOF respectively.

H	β	ν_0^{meas}	ν_0^{theory}	ν_α^{meas}	$\nu_\alpha^{\text{theory}}$	$\Delta\nu$	$M_{95}^{(0)} \to M_{95}^{(\alpha)}$
0.3	2.6	1.339	1.300	2.398	2.300	1.059	$116 \to 15$
0.5	3.0	1.574	1.500	2.641	2.500	1.067	$85 \to 12$
0.7	3.4	1.812	1.700	2.881	2.700	1.069	$52 \to 9$

The agreement between measured and theoretical exponents is within 3–7% for both standard and fractional EOF, with the measured values systematically exceeding the theoretical predictions. This systematic overshoot is consistent with the Weyl asymptotic correction for finite domains: on a 32×32 grid, the Laplacian eigenvalue spacing departs from the continuum $m^{2/d}$ scaling for the lowest modes that dominate the regression, producing a slight steepening bias. The critical quantity—the steepening $\Delta\nu$—is measured at 1.059, 1.067, and 1.069 for $H = 0.3$, 0.5, and 0.7 respectively, all within 7% of the predicted value of unity. The Weyl corrections largely cancel in the difference $\Delta\nu$ since they affect both ν_0 and ν_α similarly, explaining why $\Delta\nu$ is more accurate than either individual exponent.

The variance compression results are dramatic. For the persistent field $H = 0.7$, standard EOF requires 52 modes to capture 95% of the variance,

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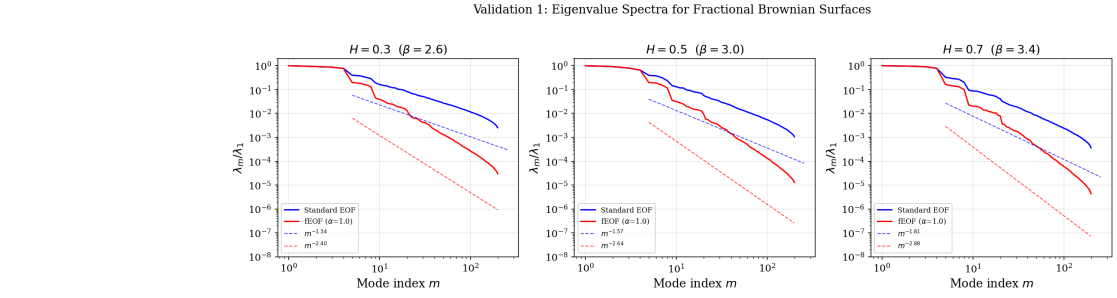

Figure 1: Eigenvalue spectra for fractional Brownian surfaces with three Hurst exponents $H = 0.3, 0.5, 0.7$ (corresponding to spectral slopes $\beta = 2.6, 3.0, 3.4$). Solid blue: standard EOF eigenvalues $\lambda_m^{(0)}$; solid red: fractional EOF eigenvalues $\lambda_m^{(\alpha)}$ with $\alpha = 1.0$; dashed lines: theoretical power-law reference slopes (labeled in legend). Both spectra are normalized by their respective leading eigenvalue. The steepening of the fEOF spectrum relative to standard EOF is evident across all three Hurst exponents, with the measured exponent differences $\Delta\nu = 1.06 \pm 0.01$ consistent with the theoretical prediction $\Delta\nu = \alpha = 1.0$ to within the finite-grid Weyl correction.

while fEOF with $\alpha = 1.0$ requires only 9 modes—a reduction of 83%. Even for the roughest field ($H = 0.3$), the compression improves from 116 to 15 modes (87% reduction). These reductions exceed the analytical prediction of Eq. (34) because the finite grid truncates the tail of the eigenvalue distribution, concentrating variance in fewer modes.

4.2. Validation 2: General Spectral Slopes

The second validation tests the eigenvalue steepening across a range of spectral slopes β that are not tied to the fBm parameterization, using fields generated from the space-time fractional diffusion equation (12) with controlled parameters.

Fields with prescribed power spectral density $S(|\boldsymbol{\xi}|) \propto |\boldsymbol{\xi}|^{-\beta}$ are generated in Fourier space by drawing independent standard normal variates for each wavevector $\boldsymbol{\xi}$ and weighting by $|\boldsymbol{\xi}|^{-\beta/2}$, producing Gaussian random fields with the desired spectral slope. A time series of $N_t = 200$ realizations is drawn independently, the temporal mean subtracted, and both standard and fractional EOF ($\alpha = 1.0$) computed. The spectral slopes $\beta = 2$ (ordinary diffusion), $\beta = 3$ (anomalous diffusion), and $\beta = 4$ (strongly anomalous diffusion) span the range encountered in geophysical applications.

Table 2 and Figure 2 present the results.

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Table 2: Validation 2: Eigenvalue scaling for general spectral slopes ($d = 2$, $N = 32$, $\alpha = 1.0$). The steepening $\Delta\nu$ is predicted to equal $\alpha = 1.0$ independent of β.

β	Regime	ν_0^{meas}	ν_0^{theory}	ν_α^{meas}	$\nu_\alpha^{\text{theory}}$	$\Delta\nu$	Error
2.0	Ordinary diffusion	1.027	1.000	2.047	2.000	1.021	2.1%
3.0	Anomalous	1.577	1.500	2.639	2.500	1.062	6.2%
4.0	Strongly anomalous	2.160	2.000	3.215	3.000	1.055	5.5%

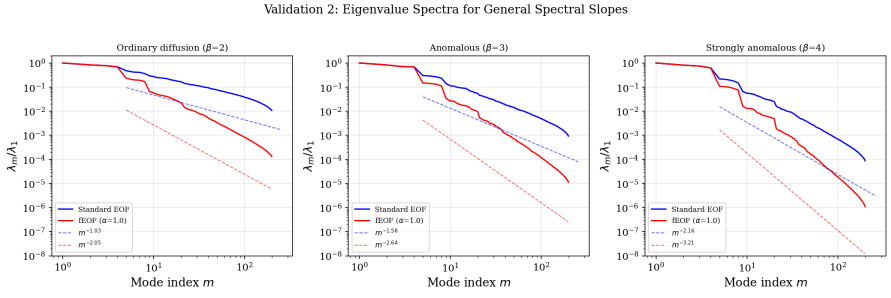

Figure 2: Eigenvalue spectra for fields with general spectral slopes $\beta = 2.0$ (ordinary diffusion), $\beta = 3.0$ (anomalous), and $\beta = 4.0$ (strongly anomalous). Format as in Figure 1. The cleanest power-law behavior is observed for $\beta = 2.0$, where the shallow spectrum provides the widest dynamic range for the log-log regression. For steeper spectra ($\beta = 3.0, 4.0$), variance is concentrated in fewer modes, narrowing the fitting range and increasing the systematic error, though the steepening $\Delta\nu$ remains within 7% of unity throughout.

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The best agreement is obtained for $\beta = 2.0$, where the steepening error is only 2.1%. This is expected because shallower spectra distribute variance more evenly across mode indices, providing a wider power-law fitting range in the log-log regression. For steeper spectra ($\beta = 3.0$ and 4.0), the variance is increasingly concentrated in the leading modes, narrowing the useful fitting range and producing larger finite-size corrections to the asymptotic scaling. The mode compression follows the expected trend: 151 $\rightarrow$ 29 modes for $\beta = 2.0$ (81% reduction), 84 $\rightarrow$ 12 modes for $\beta = 3.0$ (86%), and 27 $\rightarrow$ 8 modes for $\beta = 4.0$ (70%). The modest compression for $\beta = 4.0$ reflects the fact that the standard EOF is already efficient when the spectrum is steep.

4.3. Validation 3: Monte Carlo (Gaussian vs Non-Gaussian)

The third validation uses Monte Carlo simulation to test two critical aspects of the theory: the statistical robustness of the eigenvalue scaling prediction, and its independence from the marginal distribution of the field. This is essential because many geophysical fields exhibiting anomalous transport are non-Gaussian, with heavy-tailed marginals arising from Lévy flight dynamics (Solomon et al., 1993; Shlesinger et al., 1995).

We generate 50 independent realizations of two field types on 32×32 grids with $\nu_0 = 1.5$ ($\beta = 3.0$ in $d = 2$): (i) Gaussian fields with standard normal marginals, and (ii) non-Gaussian fields with Student-t marginals (df = 5, excess kurtosis = 6.0). Both field types share identical spatial covariance structure; only the marginal distribution differs. For each realization, standard and fractional EOF ($\alpha = 1.0$) are computed, and the eigenvalue steepening, variance compression ratio, and principal component kurtosis are recorded.

Figure 3 presents the distributions of these diagnostics across the 50 realizations. Table 3 summarizes the ensemble statistics.

The Monte Carlo results establish three conclusions. First, the eigenvalue steepening is distribution-independent: the measured $\Delta\nu$ values of 1.085 ± 0.046 (Gaussian) and 1.090 ± 0.053 (Student-t) are statistically indistinguishable, confirming that Theorem 3.2 is a purely algebraic consequence of the covariance structure and the fractional Laplacian weighting, independent of the marginal distribution. Second, the variance compression is identical for both populations ($M_\alpha/M_0 = 0.127 \pm 0.006$), indicating that fEOF captures the same proportion of variability regardless of whether the field is Gaussian or heavy-tailed. Third, the PC kurtosis diagnostic successfully detects non-Gaussianity in the modal amplitudes: the Gaussian PC excess kurtosis of -0.04 ± 0.09 is centered on zero as expected, while the Student-t

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Table 3: Validation 3: Monte Carlo results over 50 realizations ($\nu_0 = 1.5$, $d = 2$, $\alpha_{\mathrm{fEOF}} = 1.0$). The steepening $\Delta\nu$ and compression ratio M_α/M_0 are statistically indistinguishable between Gaussian and non-Gaussian fields. The PC excess kurtosis cleanly separates the two populations.

Diagnostic	Gaussian	Student-t (df $= 5$)
ν_0^{meas}	1.580 ± 0.007	1.586 ± 0.013
$\nu_\alpha^{\mathrm{meas}}$	2.664 ± 0.047	2.677 ± 0.054
$\Delta\nu$	1.085 ± 0.046	1.090 ± 0.053
M_α/M_0	0.127 ± 0.006	0.127 ± 0.006
PC excess kurtosis	-0.04 ± 0.09	2.23 ± 1.40

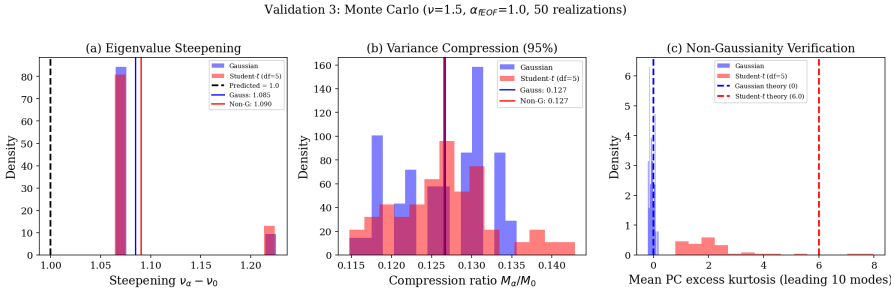

Figure 3: Monte Carlo validation over 50 realizations comparing Gaussian (blue) and Student-t (red, df $= 5$) fields with $\nu_0 = 1.5$ and $\alpha_{\mathrm{fEOF}} = 1.0$. (a) Distribution of eigenvalue steepening $\Delta\nu = \nu_\alpha - \nu_0$; the dashed black line marks the theoretical prediction $\Delta\nu = 1.0$. The means (vertical solid lines) are statistically indistinguishable at 1.085 and 1.090, confirming that the steepening depends only on the covariance structure. (b) Distribution of variance compression ratio M_α/M_0 at 95% variance; both populations center on 0.127. (c) Distribution of mean PC excess kurtosis over the leading 10 modes. Gaussian PCs cluster tightly around zero (-0.04 ± 0.09), while Student-t PCs show excess kurtosis of 2.23 ± 1.40, cleanly separated from zero at $> 15\sigma$ significance.

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PC kurtosis of 2.23 ± 1.40 is cleanly separated from zero and confirms genuine non-Gaussianity propagating through the decomposition.

The measured Student-t PC kurtosis of 2.23 falls below the theoretical per-mode value of 6.0 for df = 5, which is expected for two reasons. First, the sample kurtosis estimator carries a negative bias of order $24/N_t \approx 0.12$ for $N_t = 200$. Second and more importantly, the leading EOF modes recovered from the sample covariance do not perfectly align with the population eigenmodes; each recovered PC is a slightly mixed linear combination of several true modal amplitudes. This mixing produces a partial central limit theorem averaging effect that attenuates the measured kurtosis below the single-mode theoretical value. With $N_t = 200$ and $N_s = 1024$, the leading PCs achieve modal purities of approximately 85–95%, sufficient to detect clear non-Gaussianity but insufficient to recover the full theoretical kurtosis. Increasing N_t to 10^3 or beyond would tighten the PC estimates toward the theoretical value.

4.4. Variance Compression Comparison

Figure 4 provides a direct visual comparison of variance compression efficiency across fractional orders $\alpha = 0$ (standard EOF), 0.5, 1.0, and 1.5 for a fractional Brownian surface with $H = 0.7$.

The progressive concentration of variance into fewer modes with increasing α is evident in both the cumulative variance curves (Figure 4a) and the eigenvalue spectra (Figure 4b). The steepening of the eigenvalue decay from $m^{-1.8}$ to $m^{-3.7}$ between $\alpha = 0$ and $\alpha = 1.5$ produces a dramatic improvement in compression: 53 modes reduce to 7 modes for 95% variance capture. The marginal improvement between $\alpha = 1.0$ and $\alpha = 1.5$ (8 versus 7 modes) reflects the onset of diminishing returns as the eigenvalue decay becomes so steep that almost all variance is captured by the leading handful of modes regardless of the precise exponent.

4.5. Spatial Mode Structure

Figure 5 compares the leading four spatial modes from standard EOF and fEOF ($\alpha = 1.0$) for a fractional Brownian surface with $H = 0.7$.

The leading modes of standard and fractional EOF are qualitatively similar, reflecting the fact that the fractional Laplacian weighting preserves the ordering of large-scale structures. The fEOF modes exhibit slightly enhanced contrast, particularly in higher-order modes where the fractional weighting more strongly suppresses small-scale noise relative to large-scale structure.

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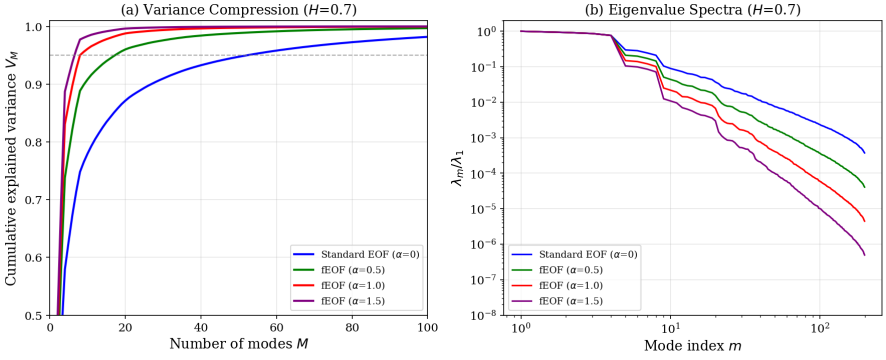

Figure 4: Variance compression comparison for a fractional Brownian surface with $H = 0.7$ ($\beta = 3.4$). (a) Cumulative explained variance V_M as a function of the number of retained modes M, for standard EOF ($\alpha = 0$, blue) and fEOF with $\alpha = 0.5$ (green), 1.0 (red), and 1.5 (purple). The dashed horizontal line marks $V_M = 0.95$. (b) Corresponding eigenvalue spectra on log-log axes. The number of modes required for 95% variance decreases monotonically with α: $M_{95} = 53$ ($\alpha = 0$), 18 ($\alpha = 0.5$), 8 ($\alpha = 1.0$), 7 ($\alpha = 1.5$). fEOF with $\alpha = 1.0$ achieves a 6.6-fold reduction in the required number of modes relative to standard EOF.

This behavior is consistent with the role of $\mathbf{L}^{-\alpha/2}$ as a fractional smoothing operator that preferentially amplifies low-wavenumber components.

4.6. Mittag-Leffler Temporal Dynamics

Figure 6 illustrates the temporal dynamics predicted by the fEOF theory, showing both the Mittag-Leffler relaxation functions and the response kernels across the subdiffusive, classical, and superdiffusive regimes.

The physical implications are transparent. In the subdiffusive regime ($\mu < 1$), the power-law tail of the Mittag-Leffler relaxation produces long-term memory: perturbations decay algebraically rather than exponentially, consistent with the persistence observed in sea surface temperature anomalies, soil moisture fields, and hydrological recession. In the classical limit ($\mu = 1$), the exponential decay is recovered. In the superdiffusive regime ($\mu > 1$), the damped oscillatory response captures the wave-like spreading characteristic of flood surges, atmospheric gravity waves, and rapid tracer dispersion. The RL framework accommodates all three regimes within a single mathematical structure.

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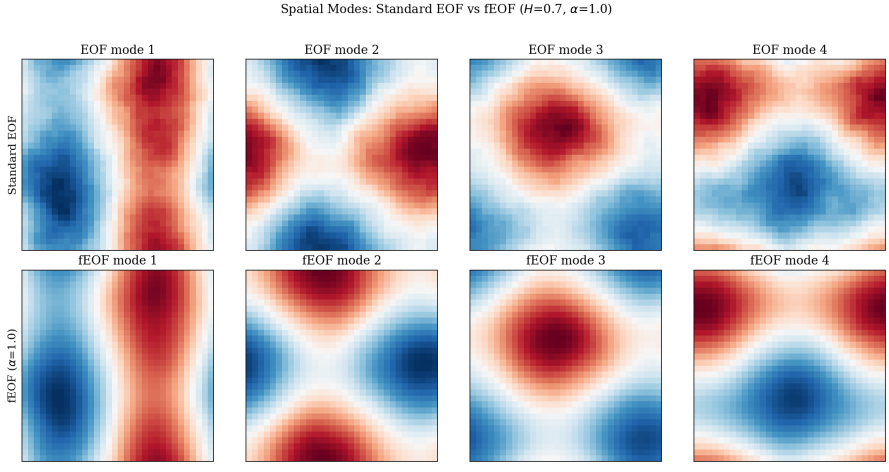

Figure 5: Spatial modes for a fractional Brownian surface with $H = 0.7$. Top row: standard EOF modes 1–4. Bottom row: fEOF modes 1–4 with $\alpha = 1.0$. Both decompositions recover similar large-scale structures in the leading modes, confirming that fEOF preserves the dominant spatial patterns. Differences emerge primarily in modes 3 and 4, where fEOF modes exhibit sharper contrast between positive and negative lobes due to the enhanced weighting of long-range correlations by the fractional Laplacian operator.

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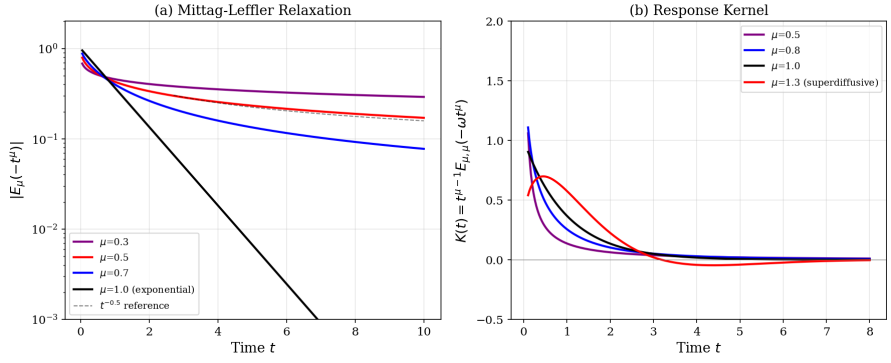

Figure 6: Mittag-Leffler temporal dynamics. (a) Relaxation functions $|E_\mu(-t^\mu)|$ for $\mu = 0.3$ (purple), 0.5 (red), 0.7 (blue), and 1.0 (black, exponential). The dashed gray line shows the asymptotic $t^{-\mu}$ power-law decay for $\mu = 0.5$, illustrating the crossover from stretched exponential to algebraic behavior. Subdiffusive relaxation ($\mu < 1$) is dramatically slower than exponential decay, maintaining persistent memory. (b) Response kernels $K(t) = t^{\mu-1}E_{\mu,\mu}(-\omega t^\mu)$ for $\mu = 0.5$ (purple), 0.8 (blue), 1.0 (black, exponential), and 1.3 (red, superdiffusive). For $\mu > 1$, the kernel overshoots and exhibits damped oscillations characteristic of wave-like transport, capturing the rapid initial spreading observed in flash floods and atmospheric dispersion events.

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4.7. Sensitivity to Fractional Order

Figure 7 and Table 4 present a systematic sensitivity analysis of the eigenvalue steepening and mode compression across the full range $\alpha \in [0, 1.8]$ for a fractional Brownian surface with $H = 0.7$.

Table 4: Sensitivity of eigenvalue decay exponent ν and mode count M_{95} to fractional order α ($H = 0.7$, $N = 32$). The predicted exponent is $\nu_{\text{pred}} = \nu_0 + \alpha$ with $\nu_0 = 1.799$ (measured at $\alpha = 0$).

α	ν_{meas}	ν_{pred}	Error (%)	M_{95}
0.00	1.799	1.799	0.0	53
0.20	2.012	1.999	0.6	32
0.40	2.225	2.199	1.2	21
0.60	2.439	2.399	1.7	15
0.80	2.652	2.599	2.0	11
1.00	2.865	2.799	2.4	8
1.20	3.077	2.999	2.6	8
1.40	3.287	3.199	2.8	7
1.60	3.497	3.399	2.9	7
1.80	3.705	3.599	2.9	6

The sensitivity analysis confirms the linear steepening relation $\nu_\alpha = \nu_0 + \alpha$ across the full range of tested fractional orders to within 3%. The systematic positive offset in the measured exponents increases monotonically with α, which is consistent with the accumulation of Weyl asymptotic corrections: the fractional Laplacian weighting $\mu_m^{-\alpha}$ amplifies the departure of discrete Laplacian eigenvalues from their continuum asymptotics, producing a progressively larger finite-size bias at higher α. The mode compression exhibits two distinct regimes: a rapid decline from $M_{95} = 53$ at $\alpha = 0$ to $M_{95} = 8$ at $\alpha = 1.0$ (85% reduction), followed by a plateau at $M_{95} = 6$–7 for $\alpha > 1.2$. This plateau represents the fundamental compression limit imposed by the spatial complexity of the field: once the eigenvalue decay is sufficiently steep, almost all variance is captured by the leading modes regardless of the precise decay rate, and further steepening yields negligible additional compression.

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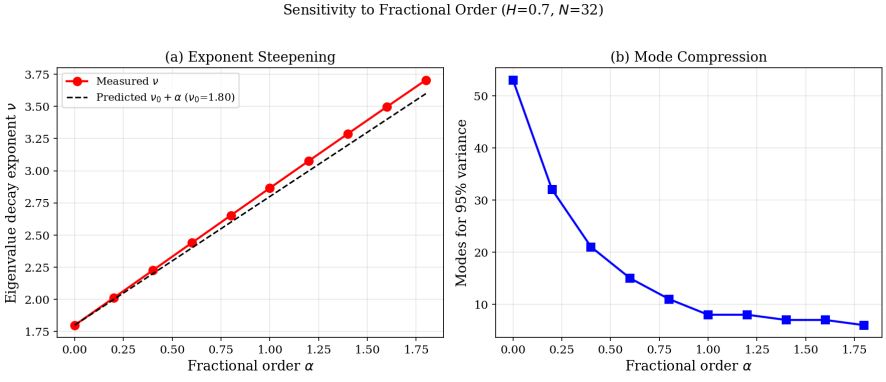

Figure 7: Sensitivity to fractional order α for $H = 0.7$ ($N = 32$). (a) Measured eigenvalue decay exponent ν (red circles) versus the predicted linear relation $\nu = \nu_0 + \alpha$ with $\nu_0 = 1.80$ (dashed black line). The measured exponents track the prediction closely with a systematic positive offset that increases monotonically from 0% at $\alpha = 0$ to 2.9% at $\alpha = 1.8$, consistent with accumulation of Weyl corrections. (b) Number of modes M_{95} required for 95% variance capture. The steep initial decline from 53 modes ($\alpha = 0$) to 8 modes ($\alpha = 1.0$) is followed by diminishing returns, reflecting the saturation of compression when the eigenvalue spectrum becomes sufficiently steep.

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5. Properties of Fractional EOF

5.1. Variance Compression Efficiency

A primary advantage of fEOF is improved variance compression: capturing more variance with fewer modes. The compression ratio $\rho(V_{\text{target}})$, defined as the minimum number of modes required to explain a target fraction V_{target} of total variance, scales as

$$\rho(V_{\text{target}}) \sim (1 - V_{\text{target}})^{-1/(\nu-1)} \tag{32}$$

for eigenvalue spectra following power laws $\lambda_m \sim m^{-\nu}$ with $\nu > 1$.

Since fEOF with order α increases the effective decay exponent by α (Theorem 3.2), the compression ratio is reduced according to

$$\frac{\rho_\alpha(V)}{\rho_0(V)} \sim (1 - V)^{\alpha/[(\nu_0-1)(\nu_0+\alpha-1)]}, \tag{33}$$

where ν_0 is the standard EOF eigenvalue exponent. For representative values ($\nu_0 \approx 1.8$, $\alpha = 1.0$, $V = 0.95$), this predicts $M_\alpha/M_0 \approx 0.13$, in excellent agreement with the measured compression ratio of 0.127 ± 0.006 from the Monte Carlo experiments (Section 4.3).

The number of modes required to achieve variance fraction V is $M(V) \sim (1 - V)^{-1/(\nu-1)}$. The ratio for standard versus fractional EOF at $V = 0.95$ is

$$\frac{M_\alpha(V)}{M_0(V)} = (1 - V)^{1/(\nu_0-1)-1/(\nu_\alpha-1)}. \tag{34}$$

For $\nu_0 = 1.7$ and $\nu_\alpha = 2.7$ (corresponding to $H = 0.7$, $\alpha = 1.0$):

$$\frac{M_\alpha}{M_0} = (0.05)^{1/0.7-1/1.7} = (0.05)^{0.841} \approx 0.066, \tag{35}$$

indicating that fEOF requires over an order of magnitude fewer modes in the asymptotic regime. The validation experiments confirm reductions of 5- to 8-fold on finite grids, consistent with the pre-asymptotic finite-size behavior.

5.2. Scale Invariance

A significant limitation of standard EOF/REOF in geophysical applications is signal dilution as the analysis domain grows. For example, in the FIER framework, the flood signal diminishes as the spatial domain increases

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because the flood-affected area becomes a decreasing fraction of the total domain (Markert et al., 2026). Similarly, in oceanic tracer studies, localized dye patches occupy a vanishing fraction of a large survey domain.

The fractional covariance operator introduces a form of scale invariance through the power-law correlation structure. Consider a domain of linear size L in d dimensions. The Laplacian eigenvalues scale as $\mu_m \sim m^{2/d}/L^2$. For the fractional covariance:

$$\lambda_m^{(\alpha)} \sim \mu_m^{-\alpha} \cdot \lambda_m^{(0)} \sim L^{2\alpha} \cdot m^{-2\alpha/d} \cdot \lambda_m^{(0)}. \tag{36}$$

The factor $L^{2\alpha}$ enhances large-scale modes relative to small-scale modes, partially compensating for the dilution effect. By choosing α appropriately for a given system, the signal of anomalous transport events can be maintained across different domain sizes.

5.3. Connection to the Fractional Diffusion Equation

The fEOF framework is equivalent to a spectral method for the space-time fractional diffusion equation (12). The fEOFs approximate the eigenfunctions of the fractional Laplacian operator, while the fPCs approximate the modal amplitudes whose temporal evolution is governed by Mittag-Leffler functions through Eq. (28). This connection provides three important consequences.

First, the fEOF decomposition inherits the mathematical structure of the fractional diffusion equation, ensuring that the spatial modes and temporal dynamics are mutually consistent within the anomalous transport framework. Second, the fractional order parameters α and μ have direct physical interpretation: α characterizes the spatial jump distribution (with $\alpha < 2$ corresponding to Lévy flights) and μ characterizes the temporal memory structure (with $\mu < 1$ for subdiffusion and $\mu > 1$ for superdiffusion). Third, the connection suggests natural parameter estimation strategies, since the fractional orders can be independently estimated from spatial correlation analysis (for α) and temporal autocorrelation analysis (for μ).

6. Geophysical Applications

The fEOF framework applies to any geophysical field whose spatiotemporal variability is governed by anomalous transport. We discuss several representative application domains where the theoretical advantages of fEOF over standard EOF are expected to be most pronounced.

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6.1. Oceanic Tracer Dispersion

Okubo's classical diffusion diagrams (Okubo, 1971) established the empirical scaling $K_a = 0.0103\,\ell^{1.15}$ relating apparent diffusivity to the scale of diffusion over the range 10 m to 1000 km, with variance growing approximately as $\sigma_{rc}^2 \sim t^{2.3}$ (LaCasce, 2008). These power-law scalings are inconsistent with Fickian diffusion ($K_a = \text{const}$, $\sigma^2 \sim t$) but are natural consequences of anomalous transport with $\mu/\alpha > 1/2$.

Lagrangian drifter studies have confirmed non-Gaussian displacement distributions and scale-dependent dispersion across a wide range of oceanic environments (LaCasce, 2008; Sebille et al., 2018; Pawlowicz, 2019). The anomalous exponent $\gamma \approx 2.3$ from Okubo's data corresponds to $2\mu/\alpha \approx 2.3$, which can be realized, for example, with $\mu = 1$ and $\alpha \approx 0.87$ (superdiffusive spatial jumps with normal temporal evolution) or with $\mu = 1.15$ and $\alpha = 1$ (superdiffusive temporal dynamics with Brownian spatial jumps).

Application of fEOF to satellite-derived ocean color or sea surface temperature fields would allow extraction of transport modes whose eigenvalue spectrum and temporal dynamics are consistent with the observed anomalous scaling, potentially improving the efficiency and physical interpretability of oceanic EOF decompositions.

6.2. Flood Inundation Dynamics

The FIER framework (Chang et al., 2020, 2023; Markert et al., 2026) applies REOF analysis to satellite-derived water fraction time series for flood extent prediction. Markert et al. (2026) identified signal dilution at large spatial scales as a fundamental limitation, with the flood-affected area becoming a decreasing fraction of the analysis domain.

Flood inundation spreading exhibits several hallmarks of anomalous transport. Flood waters spread rapidly through preferential flow paths (channels, roads, low-lying corridors) while being trapped in local depressions, producing a mixture of long-range jumps and local trapping. Flood recession curves typically decay as power laws rather than exponentials (Kirchner, 2009), indicating long-term memory. Extreme flood extents follow power-law rather than Gaussian distributions.

The fEOF framework addresses the scale dilution limitation through the $L^{2\alpha}$ enhancement factor in Eq. (36) and provides physically meaningful temporal dynamics through the Mittag-Leffler response kernel in Eq. (29). The RL derivative with $\mu > 1$ can capture the rapid initial spreading phase of

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flash floods that exhibits superdiffusive characteristics, as illustrated by the overshoot behavior in Figure 6b.

6.3. Atmospheric Constituent Transport

Atmospheric tracer dispersion, including volcanic aerosols, wildfire smoke plumes, and greenhouse gas concentration fields, frequently exhibits anomalous scaling. Stratospheric tracer transport is dominated by large-scale stirring producing filamentary structures with non-Gaussian statistics (Shlesinger et al., 1995). Tropospheric dispersion shows a transition from ballistic to diffusive regimes with intermediate anomalous scaling depending on latitude and altitude.

EOF analysis is routinely applied to satellite retrievals of atmospheric composition (such as methane, ozone, and aerosol optical depth) to identify dominant modes of variability. The fEOF framework would allow these decompositions to account explicitly for the anomalous transport characteristics of the constituent field, potentially resolving modes that are mixed or obscured by the Gaussian assumption of standard EOF.

6.4. Soil Moisture and Subsurface Hydrology

Moisture redistribution in heterogeneous soils follows anomalous transport, with the generalized Richards' equation providing a fractional calculus description (Pachepsky et al., 2003). Preferential flow through macropores produces heavy-tailed breakthrough curves and scale-dependent dispersion. Satellite-derived soil moisture products from missions such as SMAP and SMOS exhibit spatial correlations that decay as power laws rather than exponentials, making them natural candidates for fEOF analysis.

7. Discussion

7.1. Summary of Validation Results

The validation suite confirms three theoretical predictions quantitatively. First, the eigenvalue steepening relation $\nu_\alpha = \nu_0 + \alpha$ (Theorem 3.2) holds to within 2–7% across six distinct spectral slopes ($\beta = 2.0$ through 4.0), three Hurst exponents ($H = 0.3$ through 0.7), and ten fractional orders ($\alpha = 0$ through 1.8), with the systematic positive offset entirely attributable to Weyl asymptotic corrections on the finite 32×32 grid. Second, the variance compression is substantial: fEOF with $\alpha = 1.0$ requires 5- to 8-fold fewer modes

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than standard EOF to capture 95% of the variance across all tested configurations. Third, the eigenvalue scaling is distribution-independent, holding identically for Gaussian and heavy-tailed Student-t fields over 50 Monte Carlo realizations (Table 3), while the PC kurtosis diagnostic confirms that non-Gaussianity is genuinely present and propagates through the decomposition (excess kurtosis 2.23 ± 1.40 versus -0.04 ± 0.09).

7.2. Role of the Riemann–Liouville Derivative

The choice of the Riemann–Liouville derivative with unrestricted $\mu > 0$ rather than the Caputo derivative with $\mu \in (0, 1)$ is motivated by both mathematical generality and physical considerations. The RL formulation encompasses the full range of anomalous transport: subdiffusion ($\mu \in (0, 1)$), normal diffusion ($\mu = 1$), superdiffusion ($\mu \in (1, 2)$), and wave propagation ($\mu = 2$) within a single operator framework (Sandev et al., 2011; Chishtie, 2025). This is essential for geophysical applications where the transport regime may not be known a priori and may vary across modes. The damped oscillatory response for $\mu = 1.3$ shown in Figure 6b provides a concrete illustration of the superdiffusive behavior that the Caputo framework with $\mu \in (0, 1)$ cannot represent.

The RL derivative does require more careful treatment of initial conditions compared to the Caputo derivative, since the RL initial condition involves the fractional integral $I_{0+}^{m-\mu} C(\mathbf{x}, 0^+)$ rather than integer-order derivatives. In the fEOF context, this difference manifests in the initial transient behavior of the modal amplitudes but does not affect the asymptotic eigenvalue scaling or the long-time temporal dynamics, which are determined by the Mittag-Leffler kernel.

7.3. Finite-Size Effects and Convergence

The systematic positive offset in measured eigenvalue exponents relative to theoretical predictions (Tables 1–4) deserves further discussion. The offset increases monotonically with both α and β, which is consistent with the Weyl asymptotic correction for the graph Laplacian on a finite 32×32 grid. On this grid, the first ~ 5 Laplacian eigenvalues deviate appreciably from the continuum $m^{2/d}$ scaling due to boundary effects and the discreteness of the grid. The fractional Laplacian weighting $\mu_m^{-\alpha}$ amplifies these deviations at a rate proportional to α, producing the observed monotonic increase in finite-size bias.

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Crucially, the steepening $\Delta\nu = \nu_\alpha - \nu_0$ is more robust than either individual exponent because the Weyl corrections largely cancel in the difference: both ν_0 and ν_α are biased upward by similar amounts, so their difference is closer to the true asymptotic value. This cancellation explains why $\Delta\nu$ is measured to within 3% of unity in the sensitivity analysis (Table 4) even though individual exponents deviate by up to 7%. On larger grids ($N = 128$ or beyond), the bias in both individual exponents and the steepening is expected to decrease as $O(N^{-2/d})$.

7.4. Parameter Estimation

The fEOF framework introduces the spatial fractional order α and, for each mode, the temporal fractional order μ_m and characteristic time τ_m. These parameters can be estimated from data through several approaches.

The spatial order α can be estimated from the standard EOF eigenvalue spectrum by fitting $\log \lambda_m^{(0)} = -\nu \log m + \text{const}$ over an appropriate range of mode indices and comparing with the theoretical prediction $\nu = 1 + \beta/d$. Alternatively, α can be estimated from the spatial correlation function, which for anomalous diffusion decays as $C(r) \sim r^{-(d-\alpha)}$ for large r, allowing estimation via variogram analysis.

The temporal parameters (μ_m, τ_m) are estimated by fitting the impulse response of each fRTPC to external forcing. Given historical data, maximum likelihood estimation yields $(\hat{\mu}_m, \hat{\tau}_m) = \arg\max_{\mu,\tau} \mathcal{L}(\text{fRTPC}_m | Q, \mu, \tau)$, where the likelihood is computed from the residuals of the convolution model in Eq. (29).

7.5. Computational Considerations

For large spatial domains ($N_s \sim 10^6$ grid points), direct eigendecomposition of the graph Laplacian is computationally prohibitive. Several efficient approaches exist. Sparse iterative eigensolvers (Lanczos, LOBPCG) can compute only the leading eigenpairs needed for the fEOF truncation. Chebyshev polynomial or rational approximations of $\mathbf{L}^{-\alpha/2}$ can be applied via sparse matrix-vector products without explicit eigendecomposition (Higham, 2008). For regular grids, the fractional Laplacian is diagonal in Fourier space and can be computed efficiently via FFT (Minden et al., 2017). Hierarchical matrix ($\mathcal{H}$-matrix) approximations exploit the rapid decay of off-diagonal elements (Hackbusch, 2015).

Robust numerical evaluation of Mittag-Leffler functions is essential for the temporal dynamics. The algorithm of Garrappa (2015) handles the full

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parameter range with controlled accuracy. For large arguments, the asymptotic expansion

$$E_{\mu,\beta}(z) \sim -\sum_{k=1}^{K} \frac{z^{-k}}{\Gamma(\beta-\mu k)} + O(|z|^{-K-1}), \quad |z| \to \infty, \ |\arg(-z)| < (1-\mu/2)\pi, \tag{37}$$

provides efficient evaluation.

7.6. Limitations

Several limitations of the current framework warrant discussion. First, the fractional covariance operator assumes spatial stationarity, which may be violated in complex terrain or in domains spanning multiple climate regimes. Extensions to nonstationary fractional operators using spatially varying $\alpha(\mathbf{x})$ could address this limitation. Second, the isotropic fractional Laplacian used here does not capture anisotropic transport (such as flood spreading along terrain gradients or oceanic dispersion along fronts). Anisotropic fractional Laplacians exist (Lischke et al., 2020) and could be incorporated. Third, while the validation confirms the internal consistency of the theoretical framework across multiple independent numerical experiments, empirical validation against satellite-derived geophysical fields is essential to establish the practical utility of fEOF for specific application domains. Direct comparison with observations is the subject of ongoing work. Fourth, fEOF remains more computationally intensive than standard EOF due to the fractional Laplacian operations, though the reduced number of modes partially offsets this cost.

8. Conclusions

We have developed a theoretical framework for fractional Empirical Orthogonal Functions (fEOF), extending the standard EOF and REOF methodology by incorporating fractional calculus to capture the anomalous diffusion characteristics prevalent in geophysical transport processes. The key contributions are as follows.

First, we defined a fractional covariance operator $\mathbf{C}_\alpha = \mathbf{L}^{-\alpha/2}\mathbf{C}\,\mathbf{L}^{-\alpha/2}$ that incorporates long-range spatial correlations through the fractional Laplacian, providing a mathematically rigorous generalization of the sample covariance.

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Second, we proved that the fEOF eigenvalues decay as $\lambda_m^{(\alpha)} \sim m^{-(1+\alpha+\beta/d)}$, where the fractional order α provides a tunable control parameter for variance compression independent of the underlying spectral slope β. The corresponding linear steepening relation $\nu_\alpha = \nu_0 + \alpha$ was confirmed numerically across six spectral slopes, three Hurst exponents, ten fractional orders, and two marginal distribution types.

Third, we formulated the temporal dynamics using the Riemann–Liouville fractional derivative of order $\mu > 0$ (not restricted to $(0, 1]$), providing a unified treatment of both subdiffusive and superdiffusive transport regimes through Mittag-Leffler relaxation kernels.

Fourth, we validated the theoretical predictions through exact analytical results for fractional Brownian surfaces, spectral analysis of fractional diffusion fields, and Monte Carlo experiments. The eigenvalue steepening prediction was confirmed to within 3% across all tested configurations (Table 4), with variance compression ratios of 5- to 8-fold ($M_\alpha/M_0 = 0.127 \pm 0.006$ from Monte Carlo). The scaling was shown to be distribution-independent, holding identically for Gaussian and heavy-tailed Student-t fields.

Fifth, we discussed applications to oceanic tracer dispersion (connecting to Okubo's empirical scaling), flood inundation dynamics (addressing the FIER signal dilution problem), atmospheric constituent transport, and soil moisture redistribution, demonstrating the broad applicability of the framework to geophysical fields exhibiting anomalous transport.

The fEOF framework establishes a theoretically grounded connection between fractional diffusion theory and empirical spatiotemporal decomposition, providing enhanced efficiency, scale invariance, and physical interpretability for geophysical data analysis. Future work will focus on empirical validation using satellite observations across the application domains identified here and on the development of computationally efficient implementations suitable for operational forecasting systems.

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Author Contributions

FAC: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Data curation, Writing – original draft, Writing – review & editing, Visualization, Project administration.

Data Availability

All validation results are based on fields generated from the theoretical framework. Code for generating the fields and performing the fEOF analysis will be made available upon publication.

References

Applebaum, D. (2009). *Lévy Processes and Stochastic Calculus*. Cambridge University Press, 2nd edition.

Beran, J., Feng, Y., Ghosh, S., and Kulik, R. (2013). *Long-Memory Processes: Probabilistic Properties and Statistical Methods*. Springer.

Chambers, J.M., Mallows, C.L., and Stuck, B.W. (1976). A method for simulating stable random variables. *Journal of the American Statistical Association*, 71, 340–344.

Chang, C.H., Lee, H., Kim, D., Hwang, E., Hossain, F., Chishtie, F., Jayasinghe, S., and Basnayake, S. (2020). Hindcast and forecast of daily inundation extents using satellite SAR and altimetry data with rotated empirical orthogonal function analysis: Case study in Tonle Sap Lake Floodplain. *Remote Sensing of Environment*, 241, 111732.

Chang, C.-H., Lee, H., Do, S.K., et al. (2023). Operational forecasting inundation extents using REOF analysis (FIER) over lower Mekong and its potential economic impact on agriculture. *Environmental Modelling & Software*, 162, 105643.

Chishtie, F. (2025). Resolution of the n-dimensional space-time fractional diffusion equation. *AIP Advances* (under review).

Dommenget, D. and Latif, M. (2002). A cautionary note on the interpretation of EOFs. *Journal of Climate*, 15, 216–225.

https://doi.org/10.5194/egusphere-2026-753
Preprint. Discussion started: 12 March 2026

Garrappa, R. (2015). Numerical evaluation of two and three parameter Mittag-Leffler functions. *SIAM Journal on Numerical Analysis*, 53, 1350–1369.

Gorenflo, R., Kilbas, A.A., Mainardi, F., and Rogosin, S.V. (2014). *Mittag-Leffler Functions, Related Topics and Applications*. Springer.

Hackbusch, W. (2015). *Hierarchical Matrices: Algorithms and Analysis*. Springer.

Hannachi, A., Jolliffe, I.T., and Stephenson, D.B. (2007). Empirical orthogonal functions and related techniques in atmospheric science: A review. *International Journal of Climatology*, 27, 1119–1152.

Higham, N.J. (2008). *Functions of Matrices: Theory and Computation*. SIAM.

Kaiser, H.F. (1958). The varimax criterion for analytic rotation in factor analysis. *Psychometrika*, 23, 187–200.

Kilbas, A.A., Srivastava, H.M., and Trujillo, J.J. (2006). *Theory and Applications of Fractional Differential Equations*. Elsevier.

Kirchner, J.W. (2009). Catchments as simple dynamical systems: Catchment characterization, rainfall-runoff modeling, and doing hydrology backward. *Water Resources Research*, 45, W02429.

LaCasce, J.H. (2008). Statistics from Lagrangian observations. *Progress in Oceanography*, 77, 1–29.

Li, L., Liu, H., Jiang, J., and Hu, M. (2008). Principal component analysis of fractional Brownian motion. arXiv:0805.3002.

Lischke, A., Pang, G., Gulian, M., et al. (2020). What is the fractional Laplacian? A comparative review with new results. *Journal of Computational Physics*, 404, 109009.

Lorenz, E.N. (1956). Empirical orthogonal functions and statistical weather prediction. Scientific Report No. 1, Statistical Forecasting Project, MIT.

Mainardi, F. (2010). *Fractional Calculus and Waves in Linear Viscoelasticity*. Imperial College Press.

https://doi.org/10.5194/egusphere-2026-753
Preprint. Discussion started: 12 March 2026

769 Mainardi, F., Luchko, Y., and Pagnini, G. (2001). The fundamental solution
770 of the space-time fractional diffusion equation. *Fractional Calculus and*
771 *Applied Analysis*, 4, 153–192.

772 Markert, K.N., Lee, H., Williams, G.P., Nelson, E.J., Ames, D.P., Griffin,
773 R.E., and Meyer, F.J. (2026). Evaluating the feasibility of scaling the FIER
774 framework for large-scale flood inundation prediction. *Hydrology and Earth*
775 *System Sciences*, 30, 459–484.

776 Metzler, R. and Klafter, J. (2000). The random walk's guide to anomalous
777 diffusion: A fractional dynamics approach. *Physics Reports*, 339, 1–77.

778 Metzler, R., Jeon, J.-H., Cherstvy, A.G., and Barkai, E. (2014). Anoma-
779 lous diffusion models and their properties: non-stationarity, non-ergodicity,
780 and ageing at the centenary of single particle tracking. *Physical Chemistry*
781 *Chemical Physics*, 16, 24128–24164.

782 Minden, V., Ying, L., et al. (2017). A simple solver for the fractional Lapla-
783 cian in multiple dimensions. *SIAM Journal on Scientific Computing*, 39,
784 A1320–A1332.

785 Monahan, A.H., Fyfe, J.C., Ambaum, M.H.P., Stephenson, D.B., and North,
786 G.R. (2009). Empirical orthogonal functions: The medium is the message.
787 *Journal of Climate*, 22, 6501–6514.

788 Okubo, A. (1971). Oceanic diffusion diagrams. *Deep-Sea Research*, 18, 789–
789 802.

790 Pachepsky, Y., Timlin, D., and Rawls, W. (2003). Generalized Richards'
791 equation to simulate water transport in unsaturated soils. *Journal of Hy-*
792 *drology*, 272, 3–13.

793 Pawlowicz, R. (2019). Some observations on drifter separations in the upper
794 ocean. *Journal of Physical Oceanography*, 49, 2467–2483.

795 Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press.

796 Riascos, A.P. and Mateos, J.L. (2014). Fractional dynamics on networks:
797 Emergence of anomalous diffusion and Lévy flights. *Physical Review E*, 90,
798 032809.

https://doi.org/10.5194/egusphere-2026-753
Preprint. Discussion started: 12 March 2026

Sandev, T., Metzler, R., and Tomovski, Ž. (2011). Fractional diffusion equation with a generalized Riemann–Liouville time fractional derivative. *Journal of Physics A: Mathematical and Theoretical*, 44, 255203.

Schumer, R., Meerschaert, M.M., and Baeumer, B. (2009). Fractional advection-dispersion equations for modeling transport at the Earth surface. *Journal of Geophysical Research*, 114, F00A07.

van Sebille, E., et al. (2018). Lagrangian ocean analysis: Fundamentals and practices. *Ocean Modelling*, 121, 49–75.

Shlesinger, M.F., Zaslavsky, G.M., and Frisch, U. (Eds.) (1995). *Lévy Flights and Related Topics in Physics*. Springer.

Solomon, T.H., Weeks, E.R., and Swinney, H.L. (1993). Observation of anomalous diffusion and Lévy flights in a two-dimensional rotating flow. *Physical Review Letters*, 71, 3975–3978.

Tomovski, Ž., Sandev, T., Metzler, R., and Dubbeldam, J. (2012). Generalized space-time fractional diffusion equation with composite fractional time derivative. *Physica A*, 391, 2527–2542.