Author comment in reply to Referee 2

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I thank Referee 2 for a precise and technically careful review. The referee has understood the construction exactly, and I have used the review to strengthen the manuscript in a substantial way. I first state plainly the points on which I agree, and I then address the main comment, that the consideration of C_α for dimensionality reduction is unmotivated. All of the changes described below will be made available in the revised version.

Points I concede. I agree with the algebra. The operator $C_\alpha = L^{-\alpha/2} C L^{-\alpha/2}$ is the ordinary covariance of the preconditioned field $y = L^{-\alpha/2}x$, its eigenvalues are the variances of y and not of x, and on a periodic and spatially homogeneous grid C and $L^{-\alpha/2}$ commute, so that C_α and C share the Fourier eigenbasis. On the homogeneous synthetic fields used in the original validation the spatial modes therefore coincide, and the differences visible in Figure 5 are finite sample behavior rather than a genuine change of pattern. I accept the consequence that the original M_{95} comparison, read as a statement about representing the field x in the Euclidean norm, is not a valid claim, since it compares variance capture of two different fields. I also accept that standard EOF is the optimal linear reduction of x in the Euclidean, that is the mean square, sense, by the Eckart–Young–Mirsky theorem. The revised version removes the Euclidean efficiency framing and the M_{95} comparison in that form, and corrects the caption of Figure 5 so that it states that the leading patterns are preserved rather than altered.

The main comment. The referee asks me to motivate, with heuristic or formal arguments, why the eigenvalues of C_α, equivalently the eigenvalues associated with $L^{-\alpha/2}x$, are meaningful objects for reducing the dimension of an anomalous transport field. I have prepared a formal answer for the revised manuscript, in the form of an optimality theorem with proof. The result establishes that the fractional EOF reconstruction, that is equation (31), is the optimal rank M approximation of the field in the scale weighted norm $\|A\|_W^2 = \mathrm{tr}(A^\top W A)$ with weight $W = L^{-\alpha}$, and that standard EOF is the same statement in the special case $\alpha = 0$, where the weight is the identity and the norm is Euclidean. The proof is short, since the map $A \mapsto L^{-\alpha/2}A$ is an invertible and rank preserving change of variable that converts the weighted problem into an ordinary Frobenius problem whose solution by Eckart–Young–Mirsky is the fractional EOF truncation. The statement and its proof are ready and are included in the revised version.

This places the referee's optimality observation in its proper context. The Eckart–Young–Mirsky theorem establishes Euclidean optimality, and Euclidean optimality is a statement about the variance of the field measured uniformly across all scales. Anomalous transport is a property of the generator of the process and of its scale dependence, not of the uniformly weighted variance of snapshots. The two notions of optimality concern different metrics, and the metric is selected by the physics of the field rather than by convenience. The weight $L^{-\alpha}$ has eigenvalues decaying as $k^{-2\alpha}$, so the weighted norm apportions reconstruction error across scales in proportion to their physical weight in a field with spectrum $S(k) \sim k^{-\beta}$, whereas the Euclidean norm weights all scales equally and so penalizes the omission of small scale components that carry little of the transport signal and, in the Lévy limit where the second moment diverges, no finite variance. I make this physical justification explicit in the revised version, and I fix α to the measured transport exponent of the field, which removes the arbitrariness the referee correctly identified in the original presentation. The choice of metric is therefore not circular. Any positive weight defines an optimality, but only the scale weight $L^{-\alpha}$ with α set by the field's anomalous exponent is the one dictated by the transport physics.

The referee's observation that an unconstrained α can drive M_{95} to unity is thereby answered. Once α is fixed by the measured exponent it is no longer a compression parameter, and I have removed the sensitivity sweep to $\alpha = 1.8$ that suggested otherwise.

The temporal contribution. The spatial reweighting is only one part of the framework. The signature of anomalous transport in these fields resides in the temporal memory of the modal amplitudes, which I model through the Mittag-Leffler kernels of equations (27) to (30). Euclidean optimality of standard EOF concerns the spatial reconstruction at each instant and makes no statement about whether the temporal evolution of the components is exponential or fractional. This part of the contribution is independent of the covariance reweighting and of the optimality discussion above, and I bring it forward in the revised version so that it is not obscured.

On the shared eigenvectors and Figure 5. I accept that on the periodic homogeneous fields used originally the eigenvectors coincide. I emphasize that this coincidence is exact only under periodicity and homogeneity. On the bounded and inhomogeneous domains characteristic of real geophysical fields, C and L do not commute, the graph Laplacian injects domain geometry and connectivity that the sample covariance does not encode, and the fractional modes differ materially from the standard modes. The revised validation adds a real, inhomogeneous field for which this difference is exhibited, in direct response to the referee's request that the technique be applied to a real dataset.

On Kavvas et al. (2020). I agree the relation should be stated. Kavvas et al. derive governing equations for transient groundwater flow in fractional time and space and solve them as a forward model. My use of the space time fractional diffusion equation is different in purpose. I do not solve it as a prognostic model. I use its spectral structure to identify the operator and the temporal kernels that the empirical decomposition then estimates from data. I add a paragraph making this distinction explicit in the revised version.

On organization. I agree that Section 2.2 read as a textbook treatment. In the revised version I remove the equations not used downstream and introduce the fractional operators through working definitions, retaining equations (10) and (14) for the fractional Laplacian and adding a short statement of how the temporal operator acts on time gridded fields.

I am grateful to the referee, whose objection led me to state the optimality on which the method rests, which I had previously left implicit. All of the changes above will be made available in the revised version.