



Retrieval and sensitivity analysis of cloud microphysical parameters via dual-frequency radar using regularized optimal estimation

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Abstract. This study proposes an adaptively regularized bidirectional retrieval framework for cloud microphysical parameters using dual-frequency radar, evaluated via multi-source error sensitivity analyses and observational validations. Key features include: (1) a forward operator library for equivalent radar reflectivity factor (Z_e) and specific attenuation coefficient (k) across diverse hydrometeor states; (2) Twomey–Tikhonov regularization to stabilize the optimization by constraining state increments and preserving forward operator continuity; and (3) a joint inverse-forward iterative strategy utilizing path-integrated attenuation (PIA) as a soft prior constraint for simultaneous attenuation correction and parameter optimization. Using WRF-simulated marine cyclones as a benchmark, W-/Ka-band radar experiments isolate key error sources, including radar noise, PIA uncertainties, a priori fields, and hydrometeor variations. While sensitivity analyses demonstrate the algorithm's effectiveness in suppressing a priori perturbations and mitigating PIA biases, retrieval accuracy remains susceptible to hydrometeor misclassification and observation noise, particularly within low signal-to-noise ratio regimes. Under combined uncertainties from Gaussian reflectivity noise ($\sigma = 2$ dB) and 10 %–20 % random PIA errors, relative errors for retrieved mass-weighted mean diameter (D_m) and equivalent liquid mass content (M_e) remain within 26 % and 40 %, respectively. Collocated W- and Ka-band measurements from ground-based (CloudCube) and spaceborne (EarthCARE, GPM) platforms further confirm algorithm reliability. When validated using liquid water path (LWP), retrievals from ground-based W-/Ka-band marine stratocumulus observations achieve about 30% error reduction over single-frequency W-band approaches, yielding a normalized root-mean-square error of 0.57 and a mean absolute percentage error of 29.2% relative to microwave radiometer data. In spaceborne mixed-precipitation comparisons, the algorithm leverages W-/Ka-band complementarity, yielding a marginal positive D_m bias against EarthCARE (ME = 0.04 mm) and a negative bias relative to GPM (ME = –0.12 mm). This work offers valuable methodological insights for next-generation spaceborne dual-frequency cloud radar missions.



30 1 Introduction

Clouds and precipitation are active components of the Earth's climate system that drive atmospheric circulation and modulate the global radiation budget (Stephens et al., 2008). The vertical structure of hydrometeors fundamentally reflects the underlying thermodynamic and kinematic processes, notably vertical air motions and diabatic heating profiles (Aoki et al., 2026). The horizontal structure dictates the spatial organization of precipitation systems, and mesoscale interactions. Resolving their three-dimensional microphysical properties and structures is thus crucial for improving microphysics parameterization schemes in numerical weather prediction (NWP) models and ultimately enhancing the accuracy of quantitative precipitation forecasts (QPFs) (Köcher et al., 2022). Dual-frequency radar leverages differential scattering and attenuation properties across distinct bands to retrieve the particle size distribution (PSD) of hydrometeors (Heymsfield et al., 2013; Meneghini et al., 2021). Recently, radar observation and microphysical retrieval techniques have advanced significantly, driven by high-resolution spaceborne dual-frequency data (e.g., GPM) and synergistic field campaigns (e.g., the Olympic Mountains Experiment, OLYMPEX; and the Investigation of Microphysics and Precipitation for Atlantic Coast-Threatening Snowstorms, IMPACTS) (Stephens et al., 2002; Hou et al., 2014; Houze et al., 2017; McMurdie et al., 2022). Furthermore, the deployment of novel Ka-/W-band systems in regions like Yangbajing and the Taklamakan Desert has provided new insights (Wang et al., 2019; Bi et al., 2022). By optimally balancing detection sensitivity and signal penetration, the W-/Ka-band configuration effectively characterizes both cloud and precipitation regimes (Battaglia et al., 2020), emerging as the prevailing trend for future spaceborne missions.

Driven by advancements in radar technologies, retrieval algorithms for cloud and precipitation microphysics have undergone profound transformations. Early single-frequency empirical approaches (e.g., $Z-R$ or $Z-W$ relationships) are limited by the natural variability of PSDs and high-frequency path attenuation, fundamentally representing highly underdetermined and ill-posed inverse problems (L'Ecuyer and Stephens, 2002). To increase information content, researchers introduced the dual-wavelength ratio (DWR), utilizing pre-calculated forward-modeled lookup tables (LUTs) to extract microphysical parameters (Matrosov, 1998; Liao et al., 2005). However, such approaches decouple attenuation from the PSD, introducing risks of non-unique solutions and local minima (Liao et al., 2014). A Bayesian physics-based retrieval framework has emerged in response (Rodgers, 2000). Notably, the optimal estimation method (OEM) constructs a cost function that rigorously integrates a priori atmospheric states, radar observations, and error covariance matrices. By translating probabilistic inference into a variational optimization problem using forward operators (Sy et al., 2020), OEM has become the dominant framework for the quantitative retrieval of cloud and precipitation microphysics.

Although OEM provides a rigorous physical-statistical framework, its algorithmic robustness in operational applications remains severely constrained by its nonlinear ill-posed nature and the coupling of multi-source errors (Grecu et al., 2016). Standard OEM implementations face three main challenges. First, pre-retrieval attenuation correction decouples attenuation from the state parameters, often introducing systematic prior biases (L'Ecuyer and Stephens, 2002; Munchak and Kummerow, 2011). Second, directly imposing path-integrated attenuation (PIA) as a strong constraint introduces inherent errors that can



cause over- or under-compensation of low-level reflectivity, frequently resulting in convergence failures near the surface (Iguchi et al., 2021). Third, conventional OEM approaches typically use static error covariances without dynamic regularization. This rigidity can trigger spurious oscillations in complex hydrometeor regimes (Grecu et al., 2004) and fails to adequately represent evolving scattering uncertainties (Geer and Bauer, 2011), ultimately degrading both retrieval accuracy and convergence efficiency. Therefore, targeted modifications to the standard OEM constraints and systematic sensitivity analyses of perturbation errors are required.

To address these limitations, this study proposes a regularized OEM algorithm incorporating adaptive regularization constraints and conducts a systematic sensitivity analysis to evaluate its uncertainties. The method introduces three key optimizations: (1) a pre-calculated forward operator library for equivalent radar reflectivity factor (Z_e) and specific attenuation coefficient (k) across diverse hydrometeor states, providing appropriate a priori scenes and error covariances; (2) Twomey–Tikhonov regularization to stabilize the ill-posed retrieval optimization by constraining state increments, mitigating hydrometeor uncertainties, and preserving forward operator continuity; and (3) a joint inverse–forward iterative strategy utilizing path-integrated attenuation (PIA) as a soft prior constraint, enabling simultaneous attenuation correction and state parameter optimization while mitigating cumulative propagation errors. This incorporates a regularization constraint into the cost function and adaptively scales error weights at thermodynamic phase boundaries, thereby improving the physical consistency of retrieved vertical profiles. Comprehensive sensitivity analyses are then conducted using Weather Research and Forecasting (WRF) model simulations of typical cloud and precipitation regimes. These analyses quantify retrieval uncertainties associated with perturbations in background constraints, radar measurement errors, PIA, a priori states, and hydrometeor classification. Finally, the algorithm is validated using multi-platform observations, including ground-based Ka-/W-band measurements from the CloudCube system and collocated spaceborne pseudo-dual-frequency data from the EarthCARE and GPM satellites, demonstrating the algorithm's practical applicability.

2 Modeling

2.1 Overview

This study presents a regularized optimal estimation (OE) framework for dual-frequency radar cloud microphysical retrievals, integrating preprocessing, cost function formulation, optimization, and sensitivity analysis into a cohesive architecture as illustrated in Fig. 1. The algorithm initializes state variables and background covariances by coupling radar observations with a single-scattering database following quality control and hydrometeor phase classification. A primary innovation lies in the nonlinear cost function, which incorporates a Twomey-Tikhonov regularization matrix ($\mathbf{R}_{TT}$) alongside a dynamic forward-operator library to mitigate non-physical, high-frequency vertical oscillations driven by observation noise and nonlinear integral effects. To minimize this cost function, the core retrieval employs a novel bi-directional Levenberg-Marquardt (L-M) strategy. A bottom-up reverse optimization initially applies the PIA as a strict boundary constraint to refine the prior background, after which a top-down forward iteration corrects PIA-induced systematic biases. This dual mechanism facilitates



95 stable, layer-by-layer attenuation correction to yield optimal state variables. Crucially, a systematic sensitivity analysis comprehensively evaluates the framework's superior capabilities and associated caveats. By explicitly disentangling distinct uncertainty sources—including background constraints, observation errors, PIA errors, prior errors, and classification uncertainties—this module isolates retrieval ambiguities and ensures the robust physical consistency of the retrieved microphysics across complex cloud and precipitation scenarios.

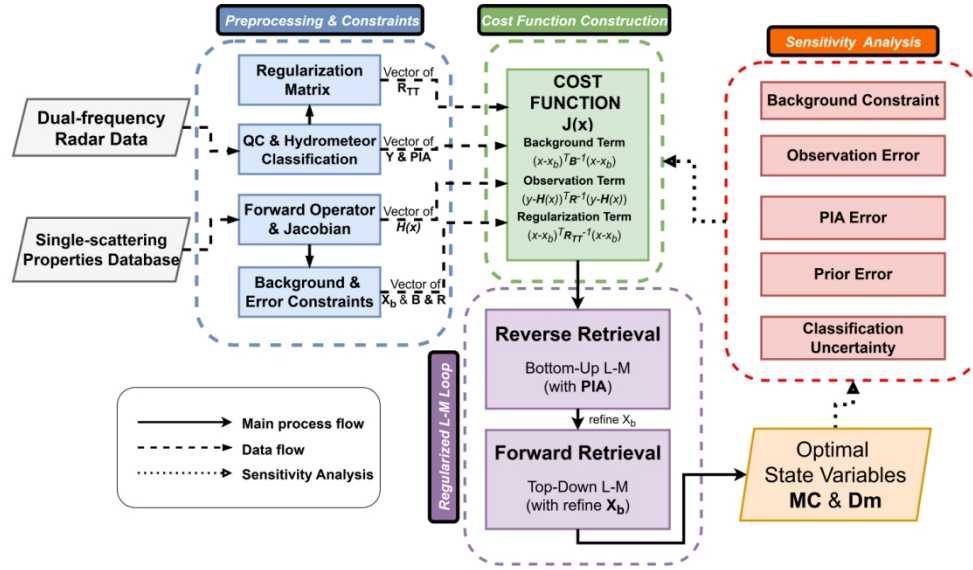


Figure 1: Flowchart of the regularized optimal estimation-based retrieval algorithm for dual-frequency radar cloud microphysics.

2.2 Algorithm Description

The regularized OE framework for dual-frequency radar microphysical retrievals centers on formulating a cost function that integrates forward observation residuals, a priori physical constraints, and a regularization term. By iteratively updating the state variables to minimize this cost function, the algorithm establishes a dynamic balance between radar measurements and background physical states. This approach mitigates the under-constrained nature and high uncertainty inherent in the retrieval process, converging to an optimal and physically consistent solution for the microphysical parameters.

2.2.1 Parameterized Radar Forward Observation Operator Library

The radar reflectivity factor represents a weighted integration of particle scattering properties over the PSD. In this study, a normalized gamma distribution is employed to characterize the hydrometeor PSD, utilizing the equivalent liquid mass content (M_c) and the mass-weighted mean diameter (D_m) as the state variables:

$$N(D) = M_c \frac{6}{\pi \rho_w D_m^4} \frac{(4+\mu)^{\mu+4}}{\Gamma(\mu+4)} \left(\frac{D}{D_m}\right)^\mu \exp\left(- (4+\mu) \frac{D}{D_m}\right), \quad (1)$$



To optimize forward modeling efficiency, the T-matrix method (Du et al., 2021) is utilized to precompute LUTs of backscattering and extinction cross-sections for various hydrometeors across different frequency bands. With D_m as the independent variable, polynomial fitting is applied to the equivalent reflectivity factor (Z_e) and specific attenuation coefficient (k) per unit mass content. This process derives the following radar forward operators (where $x = \log_{10} M_c$ and $y = D_m$):

$$Z = 10x + \sum_{i=0}^n c_i y^i, \quad (2)$$

$$10 \log_{10}(k) = 10x + \sum_{i=0}^n d_i y^i, \quad (3)$$

Consequently, forward observation operators for Z_e and k at the W and Ka bands are established for various hydrometeor types and phases. This comprehensive formulation adapts to diverse cloud and precipitation scenarios under varying a priori constraints. Fig. 2 illustrates the simulated Z and k forward operators at the W and Ka bands for hydrometeors across three phases. Specifically, the panels present the operators for: (a)-(d) liquid particles with varying shape parameters (μ); (e)-(h) mixed-phase particles with different water volume fractions (F_v); and (i)-(l) solid particles with distinct densities. Within each four-panel group, the subplots sequentially display Z_e at the W- and Ka- bands, followed by k at the W- and Ka-bands.

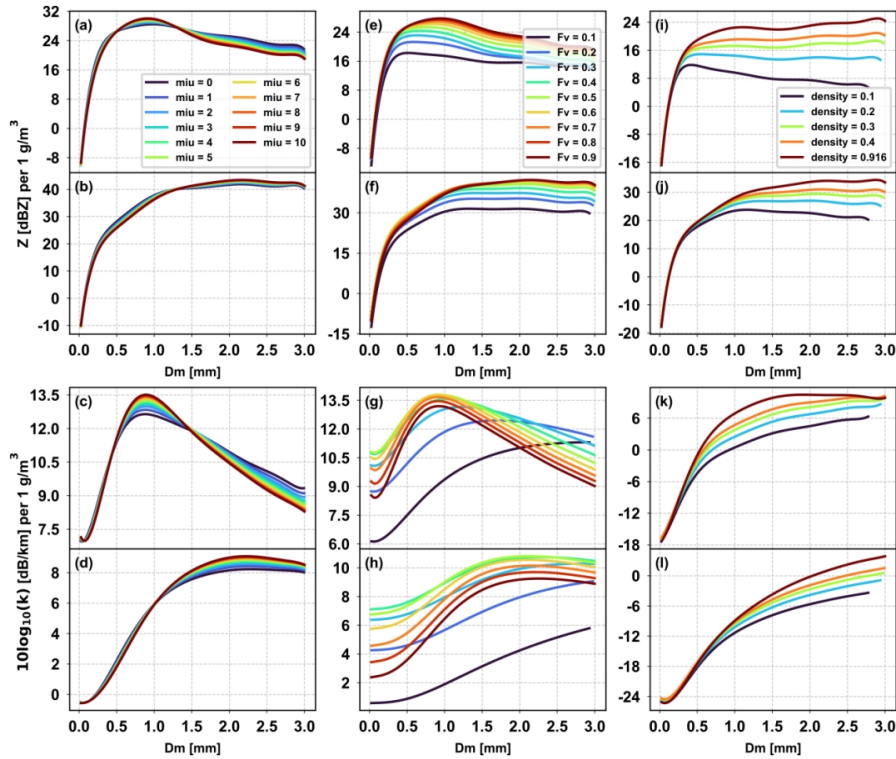


Figure 2: W- and Ka-band forward observation operators for three hydrometeor phases. Columns represent (a-d) liquid particles with varying shape parameters (μ), (e-h) mixed-phase particles with different liquid water volume fractions (F_v), and (i-l) solid particles with varying densities (ρ). Corresponding rows display the mass-normalized radar reflectivity factor (Z) at W- and Ka-bands (rows 1 and 2, respectively) alongside the mass-normalized attenuation coefficient (k) at W- and Ka-bands (rows 3 and 4, respectively).



2.2.2 Cost Function Construction

To establish a statistically optimal mapping between dual-frequency radar observations and the PSD, this study defines a state vector comprising D_m and M_c . Accordingly, a nonlinear cost function, $\mathbf{J}(\mathbf{x})$, is formulated in a vector-matrix representation as follows:

$$\mathbf{J}(\mathbf{x}) = [\mathbf{y} - H(\mathbf{x})]^T \mathbf{R}^{-1} [\mathbf{y} - H(\mathbf{x})] + [\mathbf{x} - \mathbf{x}_b]^T \mathbf{B}^{-1} [\mathbf{x} - \mathbf{x}_b] + [\mathbf{x} - \mathbf{x}_b]^T \mathbf{R}_{TT}^{-1} [\mathbf{x} - \mathbf{x}_b], \quad (4)$$

Here, $\mathbf{J}(\mathbf{x})$ represents the total cost function. The state vector, $\mathbf{x}$, comprises the profiles of M_c and D_m across n range bins along the radar beam. $\mathbf{x}_b$ denotes the a priori background state. The observation vector, $\mathbf{y}$, incorporates both the radar profile measurements and the PIA; in practice, reflectivities are converted into the dual-frequency ratio ($DFR = Z_m^{Ka} - Z_m^W$) to provide an auxiliary constraint for D_m . $H(\mathbf{x})$ is the forward operator mapping the state variables to the observation space. $\mathbf{B}$ is the background error covariance matrix characterizing prior uncertainties and spatial coherence (Rodgers, 2000), while $\mathbf{R}$ is a diagonal observation error covariance matrix accounting for instrumental noise and parameterization errors (Mahale et al., 2019). Additionally, $\mathbf{R}_{TT}$ denotes the Twomey-Tikhonov regularization matrix, introduced to suppress high-frequency vertical oscillations.

Notably, within this framework, $\mathbf{x}_b$ and $\mathbf{B}$ are dynamically extracted from the forward-simulation dataset. Using the attenuation-corrected Ka-band observations as a reference constraint, $\mathbf{x}_b$ is derived as the local neighborhood sample mean via Gaussian weighting with a 1-dBZ standard deviation. Concurrently, the weighted variance of the samples relative to this mean is computed to quantify the ill-posed uncertainty, which establishes the background error variances (σ_b^2). These variances populate the diagonal elements of $\mathbf{B}$, while its spatial correlation is modeled using a Gaussian function, yielding the general matrix element $\mathbf{B}_{ij} = \sigma_b^2 \exp \left[-\frac{1}{2} \left(\frac{r_{ij}}{r_L} \right)^2 \right]$. Here, the spatial decorrelation length (r_L) is determined by the actual altitude of the radar range gate (Waterman, 1971). In the practical algorithm, the retrieval framework synchronously simulates the expected observations using D_m and the attenuation operator across all range bins, thereby eliminating the need for prior static attenuation correction. The forward-modeled observation at each range bin is formulated as:

$$Z_m(r) = Z_c(r) - 2 \int_0^r k(s) ds, \quad (5)$$

$$DFR_{Ka/W}(r) = Z_c^{Ka}(r) - Z_c^W(r) + 2 \left[\int_0^r k^{Ka}(s) ds - \int_0^r k^W(s) ds \right], \quad (6)$$

Here, ds denotes the radar range resolution. The measured dual-frequency reflectivities (Z_m) and the dual-frequency ratio (DFR) intrinsically incorporate hydrometeor-induced attenuation. Fundamentally, the retrieval algorithm determines the particle size (D_m) by leveraging the reflectivity differences originating from non-Rayleigh scattering, while constraining M_c through the pronounced differential attenuation at higher frequencies.



2.2.3 Iterative formula and convergence criteria

Given the highly nonlinear nature of hydrometeor electromagnetic scattering, the conventional Gauss-Newton (GN) method is highly susceptible to an "operator overfitting" trap within minimal residual regimes, leading to non-physical state distortions (Rodgers, 2000). To mitigate this issue, this study reformulates an adaptive L-M optimization scheme regulated by a damping factor, λ (Huang, 2000; Kalnay, 2002). By incorporating a weighting matrix, $\mathbf{W}_k$, the state update equation at the k -th iteration is expressed as:

$$\Delta \mathbf{x}_k = (\mathbf{H}_k^T \mathbf{W}_k \mathbf{H}_k + \lambda \mathbf{I})^{-1} \mathbf{H}_k^T \mathbf{W}_k \Delta \mathbf{y}_k, \quad (7)$$

Here, $\mathbf{I}$ denotes the identity matrix, $\Delta \mathbf{y}_k$ is the observation residual vector, and $\mathbf{H}_k$ represents the Jacobian matrix comprising the local partial derivatives of the forward operator. During the differentiation of reflectivity, constrained by the propagation mechanisms of electromagnetic waves, the attenuation terms for the forward and backward profile integrations strictly exhibit lower and upper triangular matrix structures, respectively. If the cost function increases during an iteration—indicating a highly nonlinear loss surface or an ill-conditioned matrix—the algorithm exponentially amplifies the damping factor λ . This adaptively shifts the iteration step toward the steepest descent direction, enabling the algorithm to navigate through attenuation-induced perturbation regions. Conversely, upon a successful decrease in the cost function, λ is reduced to restore the quadratic convergence rate.

Establishing robust convergence criteria is essential for balancing computational efficiency and retrieval accuracy (Marquardt, 1963; Boukabara et al., 2011). Consequently, this framework implements a dual-convergence criterion based on the "observation residual" and the "cost increment":

$$\begin{cases} r_{\max}^{(k)} < \gamma_{\text{abs}} \\ \left| r_{\max}^{(k)} - r_{\max}^{(k-1)} \right| < \Delta \gamma \\ |J(\mathbf{x}_k) - J(\mathbf{x}_{k-1})| < \Delta J \end{cases}, \quad (8)$$

Let $\mathbf{r}_k = \mathbf{y} - H(\mathbf{x})$ denote the observation residual vector at the k -th iteration, where the maximum absolute residual of its first $3n$ components (i.e., reflectivity and dual-frequency ratio observations) is defined as $r_{\max}^{(k)} = \|\mathbf{r}_k[0 : 3n]\|$. Convergence is achieved when $k \geq 5$ and the joint tolerance criteria are satisfied—specifically, the absolute residual threshold $\gamma_{\text{abs}} = 0.1$ dB, the residual stationarity threshold $\Delta \gamma = 10^{-4}$ dB, and the cost function stationarity threshold $\Delta J = 1.0$ —subject to a maximum limit of $k_{\max} = 50$ iterations. This mechanism effectively circumvents pseudo-convergence trapped at flat saddle points and prevents infinite loops induced by strong nonlinearity. Consequently, it ensures both a high convergence success rate and robust computational efficiency when retrieving complex hydrometeor profiles.

2.2.4 Regularization constraint

In variational retrievals of cloud microphysics, the profound coupling between strong attenuation and nonlinear forward operators frequently triggers cascading error amplification and rapidly inflates the condition number of the Hessian matrix



(Rodgers, 2000). Traditional static climatological background covariance matrices ($\mathbf{B}$) fail to provide sufficient dynamic damping against localized abrupt changes, rendering them ineffective at suppressing high-frequency, non-physical vertical oscillations induced by cumulative integral effects. To mitigate such numerical instability, we innovatively introduce a regularization constraint applied directly to the state increments: $\mathbf{R}_{\text{TT}} = \begin{bmatrix} \tau \mathbf{T} & \mathbf{O} \\ \mathbf{O} & \tau \mathbf{T} \end{bmatrix}$, inspired by the Twomey-Tikhonov method (Delanoë and Hogan, 2008; Mroz et al., 2023). Here, τ serves as the regularization parameter balancing the observational data fit against increment smoothness, utilizing a discrete second-order difference operator matrix: $\mathbf{T} = \mathbf{L}_2^T \mathbf{L}_2$ (where $\mathbf{L}_2$ represents the discrete second-order difference operator acting on spatial node i : $(\mathbf{L}_2 \mathbf{x})_i = x_{i-1} - 2x_i + x_{i+1}$). Given that the extracted a priori background ($\mathbf{x}_b$) already possesses a physically reasonable structural evolution, the smoothing constraint is strategically shifted from the state variables to their increments, dictating that the iterative corrections must remain a slowly varying continuum. Operating independently on M_c and D_m , this second-order difference smoothing matrix ($\mathbf{R}_{\text{TT}}$) not only effectively filters out high-frequency oscillations associated with the matrix inversion but also preserves the physically sound a priori microphysical profiles, essentially embedding a robust physical constraint governed by fluid continuity into the system.

Furthermore, the vertical evolution of multi-phase hydrometeors frequently introduces numerical discontinuities in the forward operator at phase boundaries, a limitation referred to as operator fragmentation. To address this, we develop a physically consistent regularization framework based on the cross-phase continuity principles widely applied in satellite microphysical retrievals (Xu, 2020; Cazenave et al., 2024). The proposed algorithm applies adaptive vertical windows across hydrometeor phase boundaries to enforce smoothly weighted linear transitions. During the optimal estimation (OE) iterations, observation errors within these transitional zones are dynamically inflated to down-weight highly uncertain measurements, allowing the regularization smoothing term to dominate the cost function. As a result, the retrieved microphysical profiles achieve robust agreement with radar observations while strictly preserving the thermodynamic continuity inherent to gradual phase transitions. This methodology substantially improves the stability and physical realism of OE retrievals in complex mixed-phase cloud and precipitation regimes.

3 Sensitivity analysis of algorithms based on simulation

Although the adaptive regularized OE method mitigates the spurious inter-parameter correlations and error propagation typical of traditional algorithms (Munchak et al., 2022), incorporating dynamic constraints into a simultaneous retrieval framework inevitably amplifies the nonlinear complexity of the inversion. To address this, a quantitative sensitivity analysis based on WRF simulations is conducted. By utilizing WRF-derived synthetic profiles as a truth reference to isolate the compounding errors present in real-world observations, this framework independently quantifies key error sources, including radar observation errors, PIA, a priori background fields, and hydrometeor type variations. This systematic evaluation tests the structural stability and error tolerance of the improved algorithm under multi-source perturbations. Furthermore, it

characterizes model uncertainties and parameter sensitivities, facilitating the assessment of the algorithm's physical consistency, convergence stability, and retrieval reliability across complex cloud and precipitation regimes.

3.1 Simulation Scenario Introduction

Figure 3 illustrates a high-resolution three-dimensional (3D) numerical simulation of an East Pacific cyclone, conducted using the WRF model (v4.1.2) coupled with the Morrison two-moment microphysics scheme (Hogan, 2006). The simulation domain is centered at 15.0°N, 115.0°W, comprising 500×500 horizontal grid points at a 3-km resolution and 61 vertical levels, with a 15-s integration time step. The macrophysical structure and vertical phase stratification of the cyclone are characterized by the total water path (TWP) distribution (Fig. 3a) and the hydrometeor profiles (Fig. 3b). Collocated with an EarthCARE Cloud Profiling Radar (CPR) overpass, the forward-simulated W- and Ka-band reflectivity cross sections (Figs. 3d, e) show good spatial agreement with the CPR observations (Fig. 3c). Both the measured and simulated profiles capture the elevated eyewall cloud tops reaching the tropopause (14–16 km), alongside the low-reflectivity cyclone eye associated with downdrafts. Consistent with these dynamics, the simulated cross sections of D_m and hydrometeor M_c (Figs. 3f, g) exhibit a sharp contrast: deep, high-concentration hydrometeor columns maintained by strong updrafts in the eyewall, compared to depleted water content in the eye due to adiabatic subsidence warming. This synthetic scene serves as the basis for the subsequent sensitivity analysis, where the simulated D_m and M_c are used as truth to quantitatively assess the retrieval uncertainties and sensitivity limits of the model.

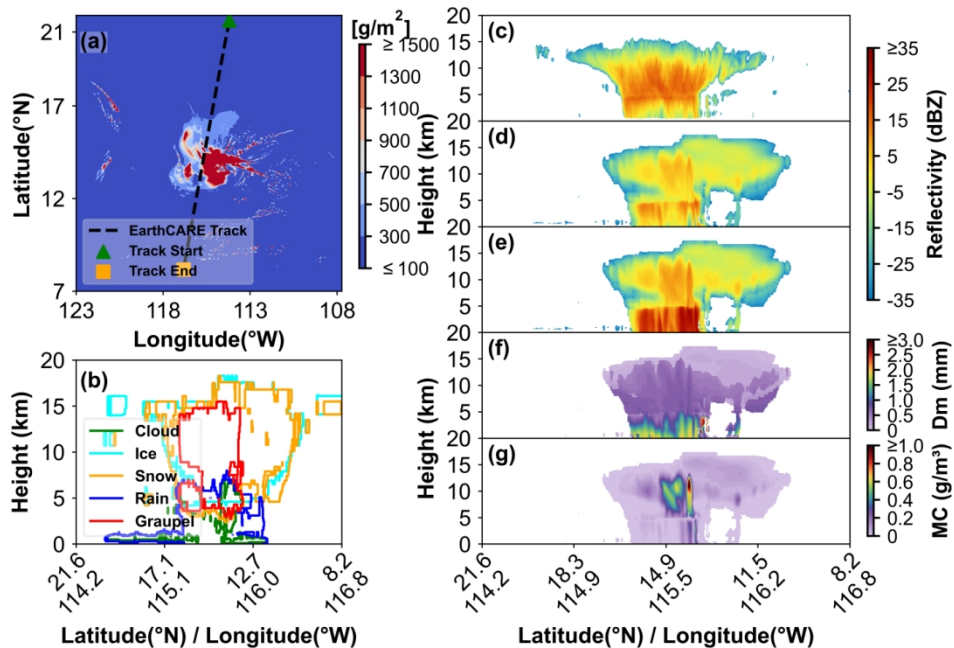


Figure 3: (a) WRF-simulated total water path (TWP) overlaid with the EarthCARE ground track. Vertical cross sections along this track illustrate (b) simulated hydrometeor types and (c) EarthCARE-observed W-band radar reflectivity. Forward-modeled along-track profiles depict (d) attenuated W-band and (e) Ka-band reflectivities, alongside (f) D_m and (g) M_c .

3.2 Algorithm Sensitivity Analysis

To establish a baseline for subsequent sensitivity analyses, the retrieval algorithm is initially evaluated within an idealized framework. To simplify hydrometeor mixtures, a single dominant hydrometeor type is assigned to each grid cell based on the WRF outputs and simulated reflectivities, as shown in the vertical cross section (Fig. 4). Observation uncertainties are set to 1 dB for both the reflectivity factor and the PIA. Attenuation-corrected Ka-band observations provide the a priori constraints. The background state vector is derived from the local neighborhood mean of the forward-modeled dataset using a Gaussian weighting function (1 dBZ standard deviation), while the background error covariance matrix is constructed from the corresponding weighted variance. To evaluate the algorithm across distinct microphysical regimes, two representative vertical profiles shown in Fig. 4 are selected. Profile 820 (blue dashed line) represents a mixed-phase precipitating column featuring complex phase transitions (cloud ice-snow-graupel-rain). This regime induces substantial W-band attenuation, thereby testing algorithmic stability under large PIA. In contrast, Profile 1070 (red dashed line) characterizes a non-precipitating cloud dominated by ice-phase particles. Featuring alternating layers of cloud ice and snow, this profile serves as an ideal case to evaluate fine-scale ice microphysical retrievals under observational perturbations.

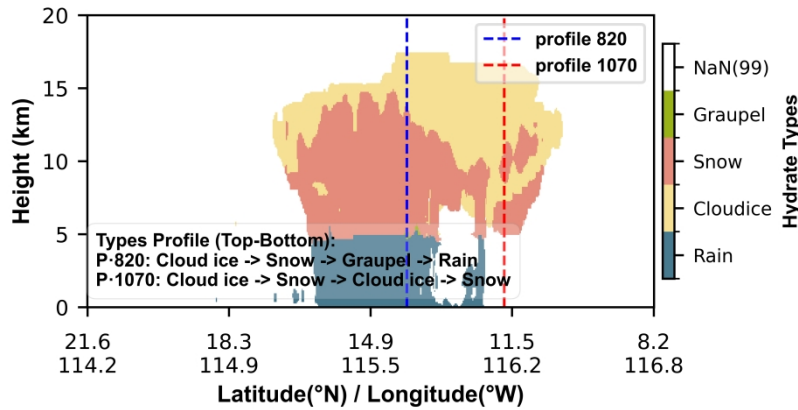


Figure 4: Vertical cross section of hydrometeors in the simulated scene, with dashed lines indicating representative test profiles (820 and 1070).

3.2.1 Sensitivity of background field constraints

In regularized retrieval frameworks, setting the background error standard deviation of D_m to 1 mm adequately captures its natural millimeter-scale fluctuations in precipitation (Williams et al., 2014). Conversely, given the wide dynamic range of M_e across varying precipitation regimes, an empirical variance of 0.5 (standard deviation of 0.707) provides an optimal balance (Mahale et al., 2019; Du et al., 2021). To evaluate the algorithmic sensitivity to the a priori constraints, Fig. 5 shows the retrieved microphysical profiles as the background error standard deviation (σ) is relaxed from 0.01 to 10.0. The left (Fig. 5a-d, Profile 820) and right (Fig. 5e-h, Profile 1070) panels display the retrievals within strongly attenuating and complex ice-phase environments, respectively, comparing single-frequency (top row) and dual-frequency (bottom row) performance.

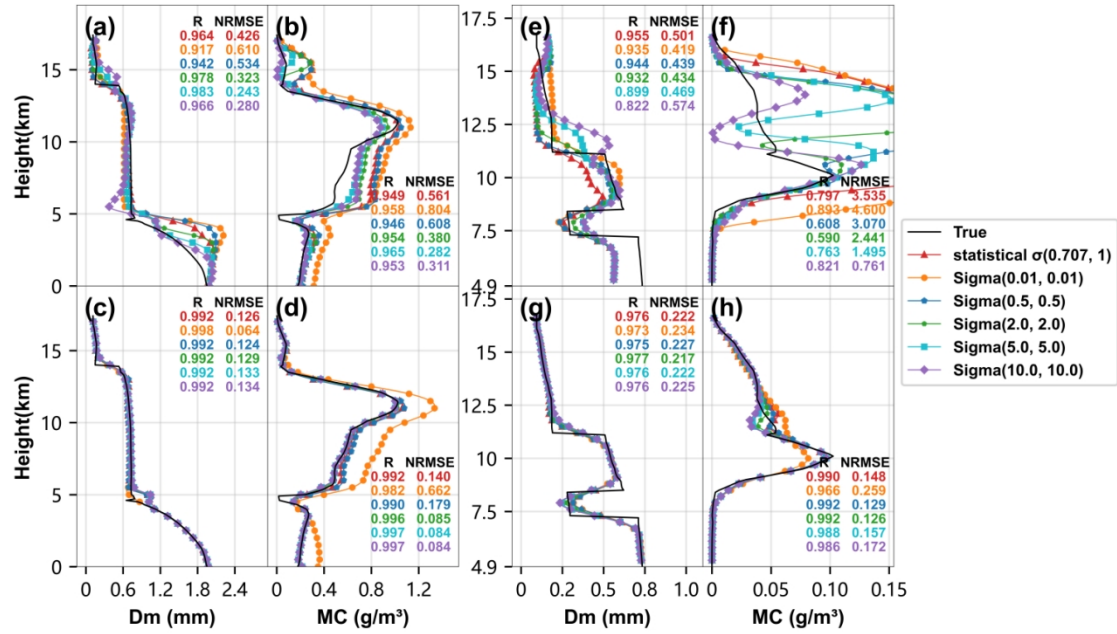


Figure 5: Sensitivity of single- and dual-frequency retrievals to background constraints. Retrieved vertical profiles of D_m and MC are shown for Profile 820 (a-d) and Profile 1070 (e-h). Upper panels (a, b, e, f) and lower panels (c, d, g, h) present single- and dual-frequency results, respectively. The synthetic truth (solid black lines) is evaluated against retrievals applying different background error standard deviations (σ , colored lines). Profile-level statistical metrics, including the correlation coefficient (R) and normalized root-mean-square error (NRMSE), are annotated in each panel to quantify the retrieval performance.

Limited by fewer observational degrees of freedom, single-frequency systems cannot independently resolve the PSD and number concentration. This renders the cost function ill-conditioned and highly dependent on a priori constraints. Consequently, when background error standard deviations (σ) vary, single-frequency retrievals deviate substantially from the synthetic truth. Such high sensitivity is quantitatively evident in the pure-ice regime (Fig. 5f), where a misspecified σ degrades the MC correlation (R) to approximately 0.6 and significantly amplifies the normalized RMSE ($NRMSE(x) = RMSE(x)/SD(x)$, where $SD(x)$ denotes the standard deviation of x). In contrast, the dual-frequency approach mitigates this ambiguity by incorporating the DFR . It leverages non-Rayleigh scattering signatures to constrain D_m and high-frequency differential attenuation to bound MC . As a result, dual-frequency retrievals demonstrate high stability across a wide range of σ values, consistently maintaining $R > 0.96$ and suppressing NRMSEs largely below 0.3 (Fig. 5d, h). Furthermore, theoretical analyses based on the forward operator library indicate that the W-/Ka-band configuration optimally resolves small-to-medium particles ($0.5 \leq D_m \leq 2.0$ mm), explaining the robust performance shown in Fig. 5. For $D_m > 2.0$ mm, however, DFR saturation caused by Mie scattering inevitably increases retrieval uncertainties. Ultimately, the dual-frequency algorithm minimizes the impact of a priori errors, accurately resolving fine-scale microphysics across both strongly attenuating precipitating columns and weakly attenuating pure-ice clouds.

3.2.2 Sensitivity of observational errors

OE algorithms exhibit nonlinear responses to observation noise. As highlighted by L'Ecuyer and Stephens (2002) and Mroz et al. (2021), minor errors in regions with a low signal-to-noise ratio (SNR) or strong attenuation can propagate vertically, causing the cost function to converge at local minima. To quantitatively assess the sensitivity of the improved algorithm to such perturbations, independent realizations of Gaussian white noise (mean of 0, standard deviation σ ranging from 0.5 to 10.0 dB) are randomly added to the synthetic W- and Ka-band reflectivity profiles. Fig. 6 shows the retrieval sensitivity to observation errors for a heavily attenuated regime (Profile 820) and a weakly attenuated, high-level ice-phase regime (Profile 1070). Simulated observation errors (Fig. 6a, b) are applied to assess the retrieved D_m and M_c profiles using the forward (Fig. 6c, e, h, j) and backward (Fig. 6d, f, i, k) algorithms, which are then compared against the true reference profiles (solid black lines).

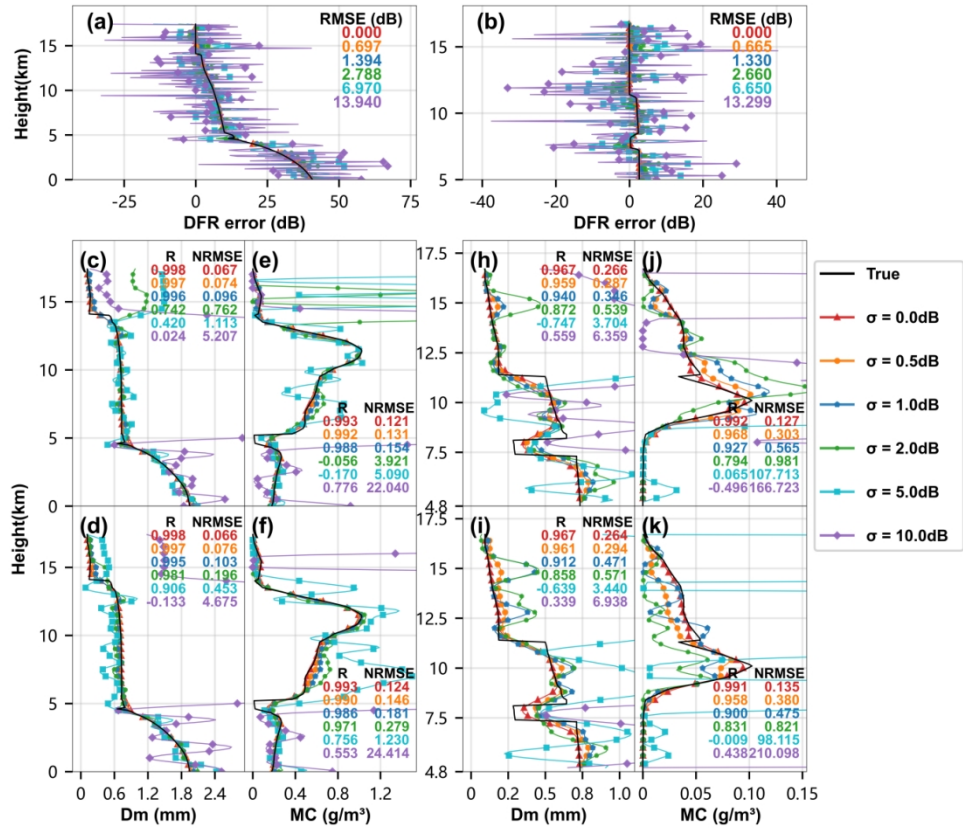


Figure 6: Sensitivity analysis of Profiles 820 and 1070 to observation errors. Panels (a) and (b) present the applied dual-band observation and DFR errors for Profiles 820 and 1070, respectively. For Profile 820, the D_m and M_c retrieved via the forward algorithm are shown in (c) and (e), alongside corresponding backward algorithm results in (d) and (f). Analogously, forward and backward retrievals for Profile 1070 are depicted in (h, j) and (i, k), respectively. Solid black lines denote the synthetic truth, while colored lines indicate retrievals evaluated under varying observation error standard deviations (σ) from 0.0 to 10.0 dB. To quantify performance, profile-level statistical metrics are annotated in each panel: root-mean-square errors (RMSE) for the simulated observation errors in (a, b), and R alongside NRMSE for the retrieval outcomes.



Under low-noise conditions ($\sigma \leq 2.0$ dB), the internal covariance constraints act as a low-pass filter, yielding stable
300 microphysical retrievals that closely approximate the truth (consistently maintaining $R > 0.9$ and $\text{NRMSE} < 0.8$). However, at
noise levels of $\sigma \geq 5.0$ dB, the true physical *DFR* gradients are obscured, inducing non-physical high-frequency oscillations
that significantly degrade accuracy. The forward and backward algorithms exhibit distinct error propagation characteristics
under these perturbations. In strongly attenuating scenarios (Profile 820), forward retrievals lack a posterior boundary
constraint and are highly susceptible to downward error amplification; for instance, at $\sigma = 5.0$ dB, the forward M_c retrieval
305 completely diverges ($R = -0.170$, $\text{NRMSE} = 5.090$, Fig. 6e). Conversely, the backward retrieval leverages the PIA as a robust
integral constraint to limit error propagation, dramatically enhancing noise tolerance (restoring R to 0.756 and reducing
 NRMSE to 1.230, Fig. 6f). Nevertheless, in weakly attenuating pure-ice regions (Profile 1070) characterized by low SNR,
uniform Gaussian noise dominates the signal. Here, the PIA constraint becomes negligible, nullifying the posterior correction
advantage of the backward approach and rendering it as vulnerable as the forward method—both exhibit catastrophic M_c
310 NRMSEs approximating 100 at $\sigma = 5.0$ dB (Fig. 6j, k). Ultimately, while the backward PIA effectively suppresses error
accumulation in strongly attenuating profiles, mitigating observational noise in weak-signal regimes remains a major
algorithmic challenge.

3.2.3 Sensitivity of PIA errors

As demonstrated in the previous section, an accurate PIA enables the backward retrieval to outperform the forward approach
315 by effectively suppressing error accumulation. However, this advantage is highly contingent on PIA reliability. Liao and
Meneghini (2019) noted that high-frequency backward integrations are acutely sensitive to PIA uncertainties, where a mere 2
dB error can trigger nonlinear error amplification, severe hydrometeor underestimation, or numerical instability. In this context,
we evaluate the sensitivity and retrieval stability of the standalone backward algorithm compared to the regularized bi-
directional approach, introducing prescribed systematic biases (1–5 dB at Ka-band and 2–10 dB at W-band) into the total PIA
320 boundary constraints. Fig. 7 illustrates the retrieval performance under these biases for a strongly attenuated precipitating
column (Profile 820) and a weakly attenuated, high-level anvil (Profile 1070). The top row (Fig. 7a, b, e, f) displays the
standard single-pass backward retrievals for D_m and M_c , while the bottom row (Fig. 7c, d, g, h) shows the proposed dual-pass
framework, both compared against the true profiles (solid black lines).

The standalone backward algorithm enforces the total-column PIA as a hard constraint, rendering retrievals highly susceptible
325 to nonlinear error amplification. In strongly attenuating precipitating regimes, negative PIA biases induce an attenuation deficit
during upward integration, severely underestimating mid- to upper-level D_m and M_c (e.g., degrading M_c correlation R to 0.056
and inflating NRMSE to 2.444, Fig. 7b). Conversely, in weakly attenuating pure-ice clouds, positive PIA biases force the
algorithm to allocate unrealistic hydrometeor mass to satisfy the constraint, triggering numerical divergence evident as sharp
upper-level deviations with catastrophic NRMSEs exceeding 110 (Fig. 7f). To overcome this, the bi-directional framework
330 reformulates the PIA as an a priori soft constraint within a "backward-then-forward" strategy, dynamically refining a priori
information to mitigate cumulative propagation errors. Under identical perturbations, the bi-directional profiles (Fig. 7c, d, g,



h) successfully suppress these sharp structural anomalies. However, while physical profile continuity improves, the overall statistical metrics (R and NRMSE) exhibit relatively modest enhancements. This discrepancy arises because profile-level statistics evaluate cumulative, layer-wide variances; although the bi-directional soft constraint effectively eliminates extreme localized numerical instability, uncompensated systematic PIA biases still impose a distributed baseline shift across the vertical column, dominating the squared-error penalty. Residual sensitivities remain mostly confined to the lowest precipitating layers, suggesting that future retrievals require additional auxiliary constraints.

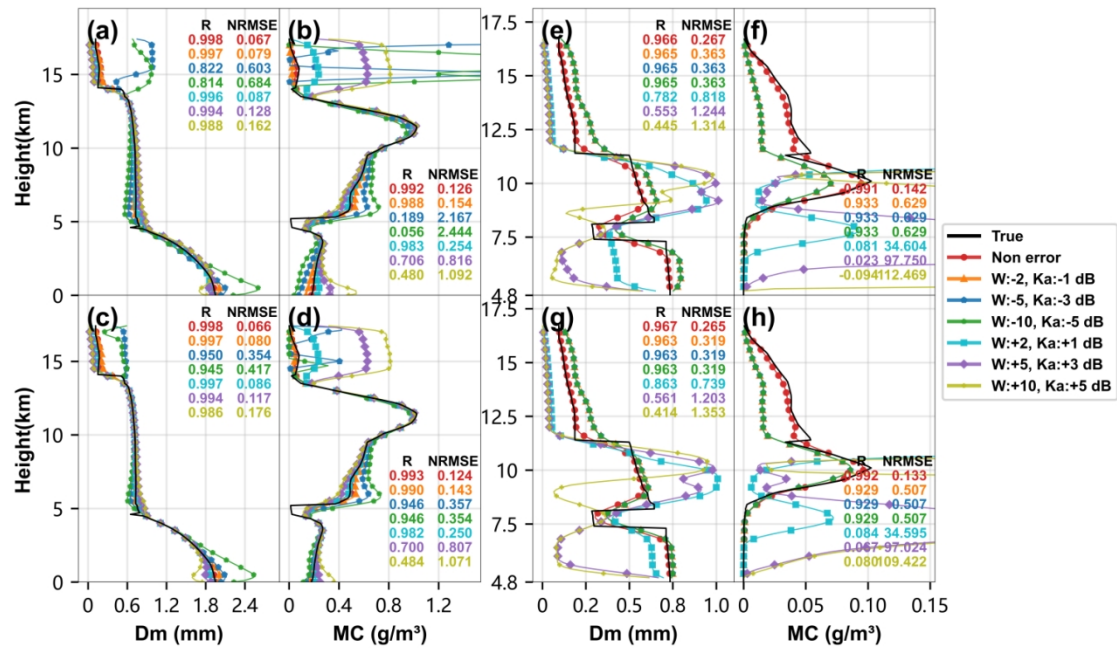


Figure 7: Sensitivity of dual-frequency retrievals to PIA errors for Profiles 820 (a-d) and 1070 (e-h). For Profile 820, the D_m and M_c retrieved via the standalone backward algorithm are shown in (a, b), alongside corresponding bi-directional algorithm results in (c, d). Analogously, the standalone backward and bi-directional retrievals for Profile 1070 are depicted in (e, f) and (g, h), respectively. Solid black lines denote the synthetic truth, while colored lines indicate retrievals evaluated under various W- and Ka-band PIA error combinations. To quantify algorithmic stability, profile-level statistical metrics, including R and NRMSE, are annotated within each panel.

3.2.4 Sensitivity of initial value errors

The sensitivity to a priori initial state errors is a key indicator of retrieval stability. Although nonlinear OE conventionally relies on a priori constraints to stabilize the inversion (Austin and Stephens, 2001), incorporating physical constraints—such as multi-frequency synergy or PIA—can effectively decouple the final solution from background uncertainties (Leinonen et al., 2016). To evaluate this sensitivity within the proposed dual-frequency dual-pass framework, Gaussian multiplicative perturbations (ranging from 0 % to 80 %) are applied to the a priori background fields. Fig. 8 compares the resulting retrieved vertical profiles of D_m and M_c against the true values (solid black lines) for Profile 820 (Fig. 8a, b) and Profile 1070 (Fig. 8c, d). The algorithm exhibits substantial stability against these perturbations. Quantitatively, even under extreme 80 % initial



errors, the retrievals remain accurate: for Profile 820, R remains above 0.99 with NRMSE below 0.13 (Fig. 8a, b), while Profile 1070 similarly sustains $R > 0.94$ and $\text{NRMSE} < 0.35$ (Fig. 8c, d). Slight discrepancies are mostly confined to the complex mixed-phase layers (5–8 km), where the highly nonlinear mapping between radar observables and microphysical properties inherently elevates retrieval uncertainty. This algorithmic stability is attributed to the framework's adaptive regularization: within the cost function, dual-frequency signatures rigorously constrain the nonlinear optimization, while the bi-directional iterative scheme dynamically synchronizes state variable estimation with attenuation correction. By reducing the inversion ambiguity typical of traditional retrievals, this mechanism closely ties the optimal solution to the measurements, ensuring that the final retrievals are largely independent of the initial background uncertainties.

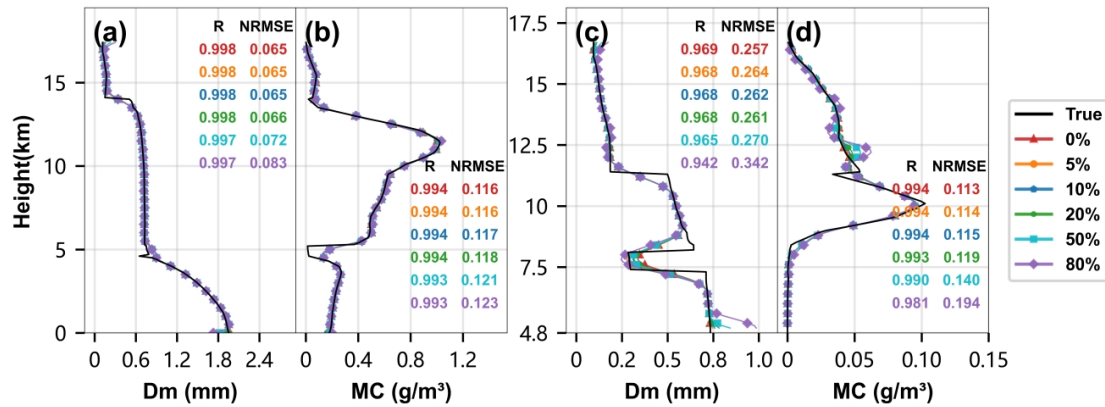


Figure 8: Sensitivity of the dual-frequency bi-directional retrieval algorithm to initial a priori errors. Retrieved D_m and M_c are depicted in (a) and (b) for Profile 820, alongside corresponding results in (c) and (d) for Profile 1070. Solid black lines denote the true reference profiles, while colored lines indicate retrievals evaluated under initial a priori error perturbations ranging from 0 % to 80 %. To quantify algorithmic stability, profile-level statistical metrics, including R and NRMSE, are annotated within each panel.

3.2.5 Sensitivity of hydrometeor type identification errors

In OE retrievals, accurate hydrometeor classification is essential to mitigate errors. Although the dominant hydrometeor type in each grid cell (Fig. 4a) is uniquely determined by its characteristic reflectivity, natural atmospheric volumes frequently contain complex hydrometeor mixtures. Previous studies have highlighted the high sensitivity of retrieval algorithms to phase assumptions, where erroneous parameterizations can introduce substantial biases ranging from -64% to $+122\%$ (Cooper et al., 2017). In this study, the baseline is configured using hydrometeor types derived from the WRF model (Fig. 4a), alongside a comparative three-phase distribution based on ambient temperature and hydrometeor phase identifications. To quantify the impact of forward operator misclassification, Fig. 9 presents a sensitivity analysis of the dual-frequency bi-directional algorithm. It compares the retrieved D_m and M_c profiles for Profile 820 (Fig. 9a, b) and Profile 1070 (Fig. 9c, d), using background colors and hatching to denote hydrometeor types and phase states. The evaluation first establishes an accurately aligned operator combination (red lines). This four-class operator suite is constructed by referencing the exact microphysical parameter settings from the simulation: rain with $\mu = 8$, snow with $F_v = 0.1$, graupel with $F_v = 0.5$, and cloud ice with a density



of 0.5 g cm^{-3} . Against this baseline, the sensitivity tests (other colored lines) apply three-class phase-dependent operators to evaluate varying assumptions. These variations include the liquid gamma shape parameter ($\mu \in [1, 10]$), the mixed-phase liquid water volume fraction ($F_v \in [0.1, 0.9]$), and the dry snow density ($\rho_{dry} \in [0.1, 0.916] \text{ g cm}^{-3}$). For instance, the profile labeled ' $\mu = 1, F_v = 0.1, \rho_{dry} = 0.1 \text{ g cm}^{-3}$ ' illustrates a retrieval assuming a broad liquid drop size distribution ($\mu = 1$), a 10 % liquid volume fraction for mixed-phase particles ($F_v = 0.1$), and a low snow density ($\rho_{dry} \in 0.1 \text{ g cm}^{-3}$).

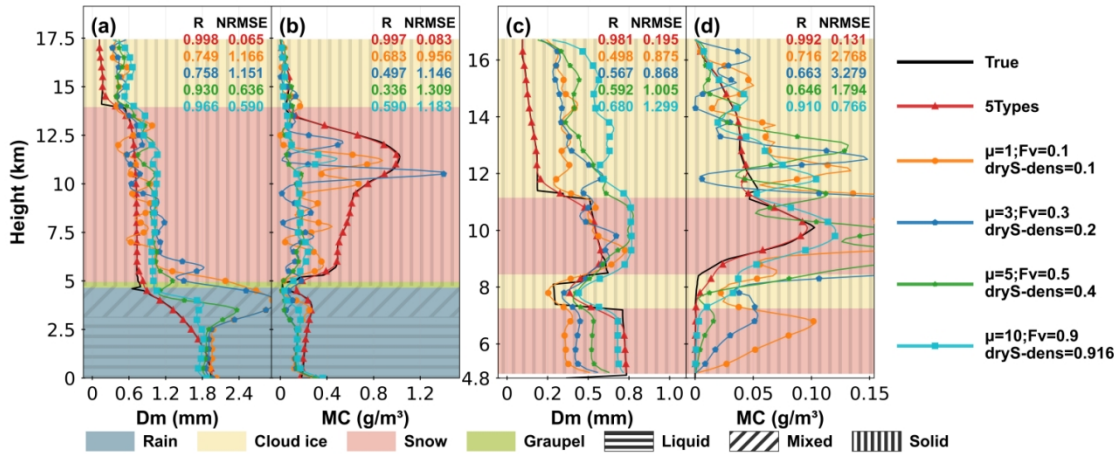
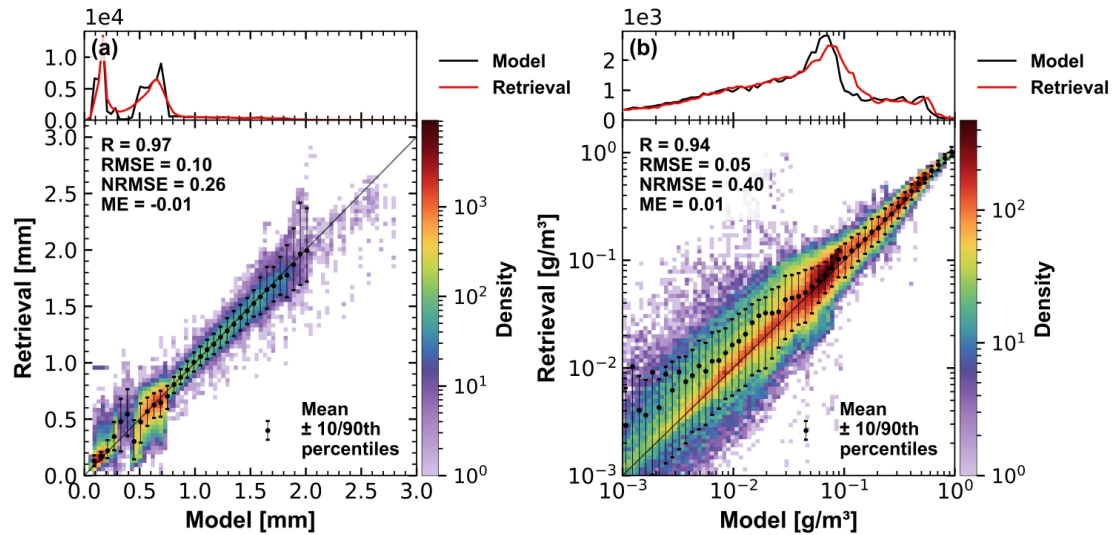


Figure 9: Sensitivity of the dual-frequency bi-directional retrieval algorithm to hydrometeor observation operator configurations. Retrieved D_m and M_C are depicted in (a) and (b) for Profile 820, alongside corresponding results in (c) and (d) for Profile 1070. Solid black lines denote the true reference profiles. Red lines indicate results utilizing operators based on a four-class hydrometeor classification consistent with the simulated species settings, while other colored lines represent retrievals using operators under different phase-specific microphysical assumptions based on a three-class phase identification. Profile-level statistical metrics, including R and NRMSE, are annotated in the upper right of each panel to quantify the retrieval performance. Background colors and hatching patterns illustrate the specific hydrometeor types and phase states, respectively.

As demonstrated, utilizing the four-class operators aligned with the simulated species robustly reproduces the true D_m and M_C profiles. Theoretically, applying three-class phase-dependent operators with hydrometeor matching errors distorts the inherent mass-to-reflectivity relationship, fundamentally degrading retrieval accuracy. For instance, underestimating bulk density forces the optimal estimation algorithm to mathematically compensate by unrealistically inflating the hydrometeor mass to fit the observed radar backscatter. This mechanism drives the massive M_C overestimations clearly visible in the ice- and mixed-phase regimes (8–16 km). This degradation is strongly supported by the profile-level statistics. The accurately matched baseline (red lines) consistently yields correlation coefficients (R) exceeding 0.98 and normalized root-mean-square errors (NRMSE) below 0.2 for both profiles. Conversely, mismatched assumptions trigger severe statistical deterioration; notably, in the pure-ice cloud (Profile 1070, Fig. 9d), incorrect density constraints cause the M_C NRMSE to surge past 3.2. These findings indicate that prior hydrometeor classification can significantly improve the stable profile retrievals. Therefore, incorporating operational hydrometeor classification products, such as those from EarthCARE and the upcoming GPM Dual-frequency Precipitation Radar (DPR) V8 will supply these critical a priori constraints.

3.3 Multi-source error sensitivity assessment

405 The systematic sensitivity analysis demonstrates that the improved dual-frequency OE algorithm is most sensitive to observation noise, followed by hydrometeor misclassification and extreme PIA biases at the base of deep convection. However, compared to conventional single-frequency or hard-boundary architectures, integrating a backward PIA soft constraint and a dual-pass iterative scheme significantly reduces the sensitivity to a priori background error covariance, initial state perturbations, and typical PIA biases. This effectively improves the retrieval stability and reduces the solution non-uniqueness
 410 typical of traditional methods. Nevertheless, due to the weak scattering signatures of ice particles and naturally low SNRs, microphysical retrievals in ice-phase, weak-precipitation regimes retain a residual sensitivity to large observational perturbations. This inherent physical limitation necessitates future optimizations, potentially by refining small-particle covariance matrices or incorporating auxiliary observations. Furthermore, this study highlights the critical impact of hydrometeor operator selection on OE retrievals. We show that precise four-class hydrometeor identification effectively
 415 decouples the nonlinear scattering ambiguities caused by variations in dielectric properties. This emphasizes the importance of accurate phase classification for ensuring cost function convergence and physical consistency.



420 **Figure 10: Performance evaluation and multi-source error sensitivity assessment of the retrieval algorithm using simulated fields. Panels (a) and (b) present density scatterplots comparing the retrieved D_m and M_c against the model reference, respectively, with corresponding marginal PDFs displayed in the top sub-panels. Black markers with error bars denote the binned statistical means and their 10th/90th percentiles.**

To evaluate the framework's retrieval stability in complex observational environments, a full-scene simulation coupled with multi-source errors is conducted. The experiment introduces realistic perturbations: independent Gaussian white noise (mean of 0, standard deviation $\sigma = 2$ dB) is superimposed onto each band of the dual-frequency reflectivity profiles; a 10 % to 20 % random fluctuation is applied to the total PIA; and a priori information is reconstructed solely from empirical physical
 425 relationships (e.g., $DFR-D_m$ and $Z-M_c$). Fig. 10 presents the two-dimensional (2D) scatter density distributions of the true



versus retrieved D_m (Fig. 10a) and M_c (Fig. 10b), accompanied by their marginal probability density functions (PDFs) in the top panels. The scatter analysis demonstrates the algorithm's error tolerance under multi-source perturbations. Despite the injected errors, the high-density cores for both parameters tightly converge along the 1:1 reference line, with the binned statistical means showing no notable systematic offsets. Quantitative evaluation further confirms this performance: D_m exhibits a strong linear correlation with the true values ($R = 0.97$), characterized by a low RMSE of 0.10 mm and a relative error NRMSE of 26 %. Similarly, M_c yields consistent retrievals across multiple logarithmic orders of magnitude, maintaining a high correlation ($R = 0.94$) with an absolute RMSE = $0.05 \text{ g} \cdot \text{m}^{-3}$ and a relative error of 40 %. Furthermore, the retrieved PDFs closely track the model distributions. These results indicate that combining fine-scale hydrometeor classification with the backward PIA iterative mechanism effectively mitigates the nonlinear structural distortions induced by low signal-to-noise ratios. This approach reduces multi-source errors and enforces the microphysical consistency of the retrievals, highlighting the numerical stability of the proposed framework for practical applications.

4 Retrieval and Analysis of Observed Cases

4.1 Retrieval results for Ground-based CloudCube radar

This study validates a W-/Ka-band dual-frequency microphysical retrieval algorithm using ground-based CloudCube radar observations from the DOE ARM Eastern Pacific Cloud Aerosol Precipitation Experiment (EPCAPE) in La Jolla, California (March 2023; data available at <https://adc.arm.gov/discovery>). Situated in a coastal environment driven by seasonal land-ocean temperature gradients, the site consistently generates low-altitude marine clouds containing characteristically small hydrometeors, providing an ideal physical baseline for subsequent cloud particle constraints. Figure 11 illustrates the retrieval performance for a representative marine stratocumulus event observed on 30 March 2023. Time-height cross sections of the observed equivalent radar reflectivity factor (Z_e) at W and Ka bands are shown in Figs. 11a, 11b, respectively. Based on these dual-frequency radar measurements, the vertical profiles of D_m and M_c were retrieved (Figs. 11c, d). For column-integrated evaluation, the temporal evolution of liquid water path (LWP) derived from both single- and dual-frequency retrievals is compared against independent microwave radiometer (MWR) measurements (Fig. 11e). Furthermore, to assess self-consistency, the retrieved D_m and M_c were fed into the radar forward model to simulate the W- and Ka-band reflectivities. The forward-simulated reflectivities are evaluated against the observations using density scatterplots and marginal PDFs for the W band (Fig. 11f) and Ka band (Fig. 11g), along with statistical error metrics. Based on temperature soundings and the local springtime climatology, this event is entirely dominated by warm-cloud liquid-phase processes, with the cloud top located below the 0°C isotherm. In the upper cloud layer (1.2–2.0 km), the hydrometeor population predominantly consists of supercooled cloud liquid droplets, as indicated by high M_c values within this active condensation zone. In contrast, larger D_m values occur primarily below the 0°C level and extend toward the cloud base, signifying the formation of drizzle drops. This vertical structure reflects the classical microphysical evolution of drizzling marine stratocumulus: descending droplets grow

efficiently via collision-coalescence, depleting cloud liquid water while shifting the drop size distribution toward low-level drizzle.

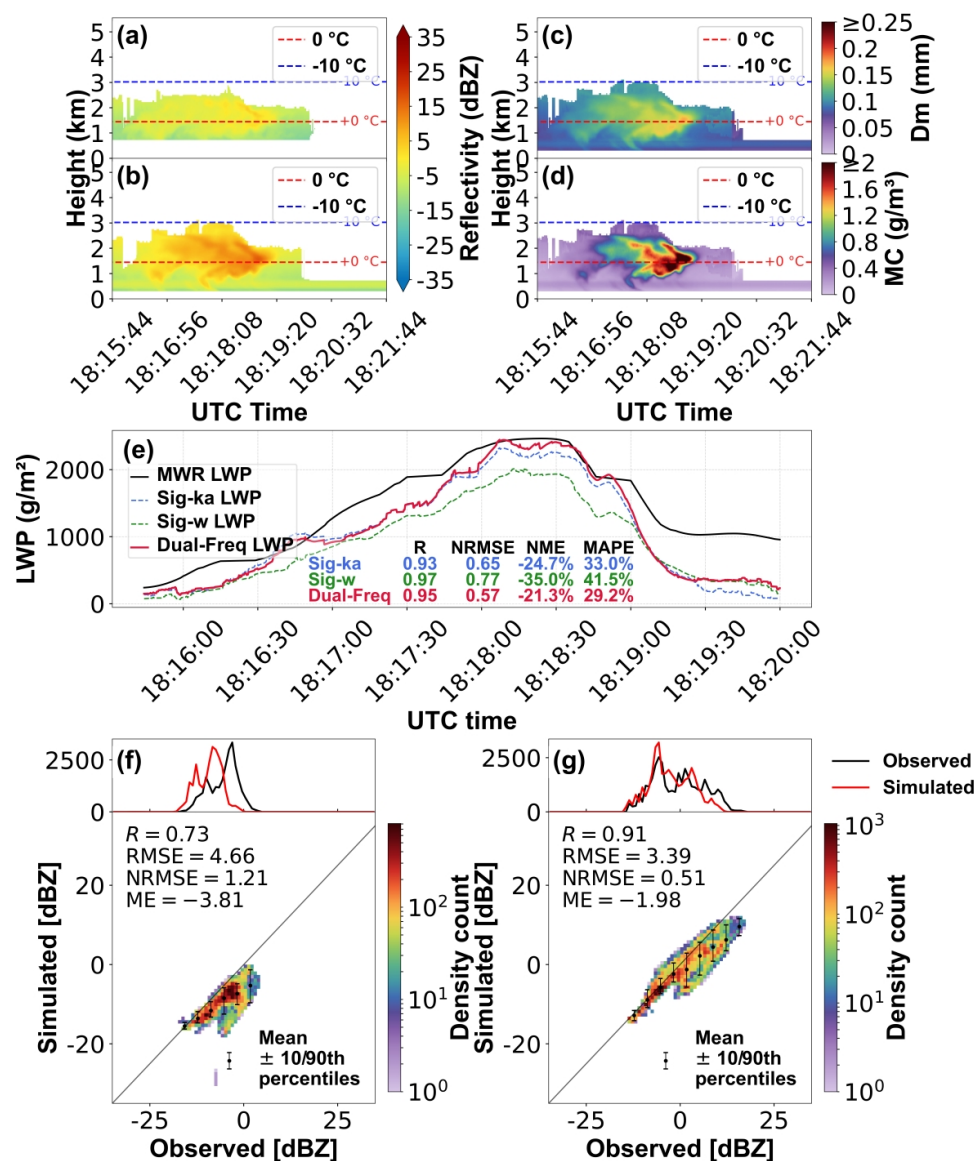


Figure 11: Observational validation of retrieval performance for a marine stratocumulus case using the ground-based CloudCube radar. Time-height cross sections display the observed reflectivities at W-band (a) and Ka-band (b), alongside the retrieved D_m (c) and M_c (d). Panel (e) evaluates the temporal evolution of LWP derived from single- and dual-frequency retrievals against independent MWR measurements, with statistical metrics annotated to quantify retrieval accuracy. Furthermore, density scatterplots in (f) and (g) evaluate the simulated W- and Ka-band reflectivities, respectively, against observations, incorporating marginal probability density distributions and statistical evaluations.

Without in situ data to directly validate the retrieved D_m and M_c , M_c is vertically integrated into LWP for comparison with independent MWR observations (Fig. 11e). Note that both single- and dual-frequency methods employ the identical adaptively



regularized, bi-directional retrieval framework. Both single-frequency retrievals systematically underestimate LWP, yielding
 470 negative normalized mean errors (NME = -24.7% for Ka-band and -35.0% for W-band) alongside elevated mean absolute
 percentage errors (MAPE = 33.0% and 41.5% , respectively). The dual-frequency approach effectively mitigates this
 underestimation, reducing the NME to -21.3% , NRMSE to 0.57, and MAPE to 29.2% , representing an approximately 30 %
 error reduction compared to single-frequency estimates, while maintaining a high linear correlation ($R = 0.95$). Furthermore,
 the dual-frequency LWP accurately tracks the temporal evolution of MWR data, successfully capturing the peak liquid water
 475 ($> 2000 \text{ g}\cdot\text{m}^{-2}$) at approximately 18:18 UTC. To verify physical closure, a forward operator is applied to the retrieved
 microphysical profiles to simulate equivalent radar reflectivities, presented as density scatterplots in Fig. 11f and g. Simulated
 Ka-band reflectivities (Fig. 11g) agree well with observations, clustering tightly along the 1:1 line
 ($R = 0.91$, RMSE = 3.39 dB, NRMSE = 0.51) with a slight negative mean error (ME = -1.98 dB). This confirms that the
 retrieved drop size spectra effectively constrain Rayleigh-dominated backscattering. In contrast, W-band simulations (Fig. 11f)
 480 exhibit larger scatter ($R = 0.73$, RMSE = 4.66 dB, NRMSE = 1.21) and a more pronounced underestimation (ME = -3.81 dB).
 This discrepancy primarily stems from cumulative two-way PIA in the dense liquid layer, compounded by non-Rayleigh (Mie)
 resonance effects of larger droplets. Ultimately, the superior performance of the dual-frequency algorithm arises from
 combining differential scattering and attenuation properties across both bands, which constrains the droplet size distribution
 more rigorously than single-frequency approaches, thereby ensuring robust physical closure.

485 4.2 Retrieval results for EarthCARE-GPM matched pseudo-dual-frequency spaceborne radar

4.2.1 Overview of the EarthCARE-GPM Colocation Scenario

In the absence of an operational spaceborne W- and Ka-band dual-frequency radar, we constructed a synthetic dual-frequency
 dataset using the EarthCARE-GPM spatiotemporal collocation algorithm (Aoki et al., 2026). The nadir-pointing EarthCARE
 CPR and the cross-track scanning GPM DPR possess distinct spatial geometries: CPR features a fine ~ 1.0 km horizontal
 490 footprint, whereas DPR has a broader ~ 5.0 km footprint. Figure 12 illustrates a collocation event on 23 March 2025, where
 the CPR track (orbit 04644D) intersects the DPR swath center (orbit 062822) over the target domain ($31.0\text{--}34.0^\circ\text{N}$, 55.9--
 56.7°E). These tracks are overlaid on the GPM Microwave Imager (GMI) 18.7 GHz brightness temperature (T_B) field.
 Combined with the radar morphology, this T_B signature indicates an extensive stratiform precipitation system. Within this
 cross section, the collocation identifies 326 valid matching profiles. The spatial displacement (Δd) between matched footprints
 495 averages 1.99 ± 0.76 km (maximum: 3.71 km). Because this maximum Δd remains smaller than the DPR footprint, fine-scale
 CPR measurements are entirely encapsulated within the corresponding DPR resolution cells, allowing the raw observational
 data to be preserved without artificial smoothing. Temporally, the overpasses are highly synchronized, with a pixel-level time
 difference (Δt) averaging only 0.39 ± 0.22 min (maximum: 0.79 min). Such stringent matching constraints ($\Delta d \leq 5$ km,
 $\Delta t \leq 5$ min) minimize uncertainties from hydrometeor advection and system evolution, establishing a physically consistent
 500 framework for direct multi-frequency reflectivity and microphysical comparisons.

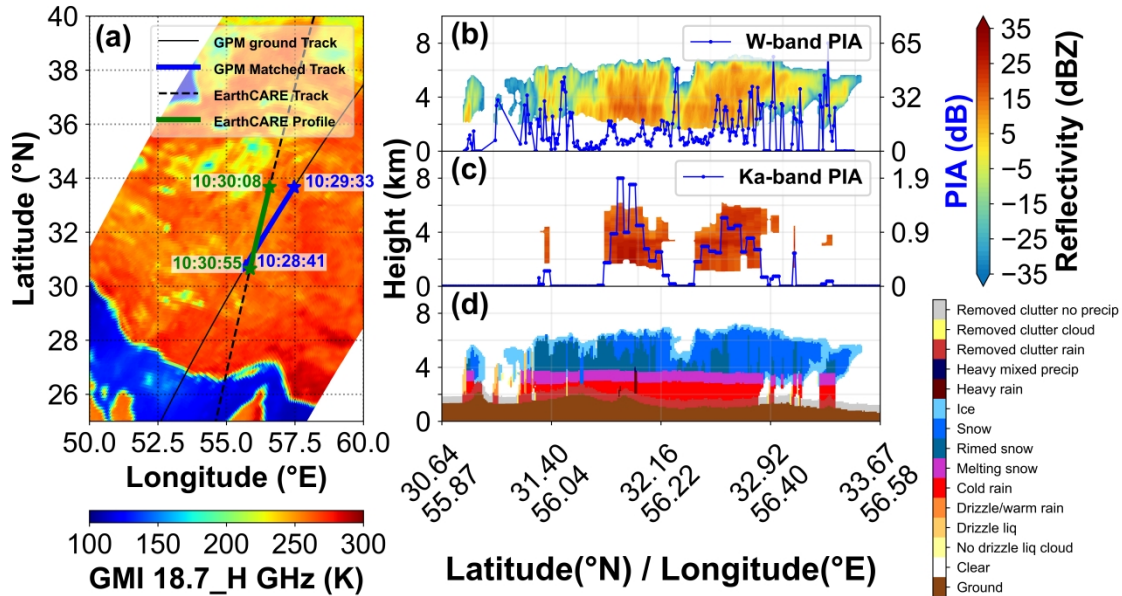


Figure 12: Collocated observations of a stratiform precipitation system by the GPM and EarthCARE satellites. Panel (a) presents the intersecting ground tracks overlaid on the GMI 18.7 GHz brightness temperature field. Matched vertical cross sections along these tracks illustrate the EarthCARE W-band (b) and GPM Ka-band (c) radar reflectivity factors, with solid blue lines indicating the PIA for each respective band. (d) Corresponding EarthCARE hydrometeor classification.

4.2.2 Retrieval results for pseudo-dual-frequency spaceborne radar

The proposed algorithm employs an adaptively regularized, bi-directional retrieval framework constrained by the 3D hydrometeor classification from the EarthCARE CPR 2A-CLD standard product (Fig. 12d). Within this classification scheme, a unified liquid forward operator is applied across all rain categories due to their negligible microphysical differences, whereas distinct scattering operators are prescribed for other hydrometeor types based on their specific microphysical properties. This unified architecture performs dual-frequency (W- and Ka-band) retrievals in spatially collocated regions, transitioning to single-frequency retrievals where observational footprints do not overlap. Figure 13 compares the along-track vertical profiles of microphysical parameters for the collocated stratiform precipitation event. The left column displays the spatial distribution of D_m derived from the official EarthCARE product (Fig. 13a), the matched GPM product (Fig. 13b), and our proposed retrieval (Fig. 13c). The right column shows the corresponding M_c fields from EarthCARE (Fig. 13e) and the proposed retrieval (Fig. 13f).

Official single-platform satellite products exhibit inherent limitations dictated by their respective operating frequencies (Battaglia et al., 2020). The W-band EarthCARE CPR provides exceptional sensitivity to small hydrometeors near cloud tops, yet suffers from severe attenuation and non-Rayleigh scattering in deep liquid layers, often underestimating D_m within lower-tropospheric heavy precipitation cores. Conversely, the GPM DPR reliably resolves moderate-to-heavy precipitation at mid-to-lower levels but lacks the sensitivity required to detect tenuous clouds and small ice particles (Toyoshima et al., 2015). Addressing these distinct detection mechanisms, the proposed retrieval effectively captures the complete vertical evolution of

hydrometeors. Near cloud tops and in weak precipitation regimes, the algorithm applies moderate spectral smoothing to mitigate nonlinear mapping effects, yielding slightly larger D_m values compared to the official EarthCARE product. In heavy precipitation cores, integrating mid-to-lower-level GPM observations allows the dual-frequency retrieval (Fig. 13c) to accurately reproduce the spatial structure and magnitude of the GPM product (Fig. 13b), effectively mitigating the D_m underestimation inherent to single-frequency W-band algorithms. Furthermore, the retrieved M_c (Fig. 13f) demonstrates strong spatial agreement with the EarthCARE product (Fig. 13e), confirming the algorithm's reliability in estimating macroscopic water budgets.

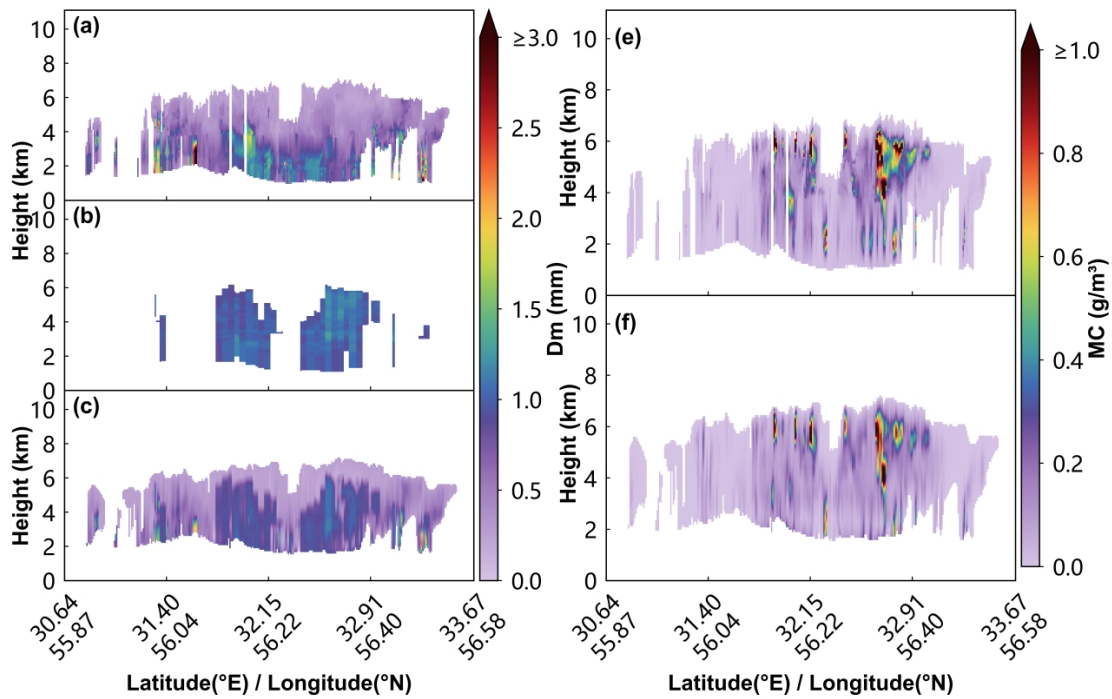


Figure 13: Comparison of matched vertical cross sections for the stratiform precipitation case. The panels compare spatial distributions of D_m (left column) and corresponding M_c profiles (right column) derived from the EarthCARE official product (a, e), the matched GPM product (b), and the proposed retrieval algorithm (c, f).

Building upon the aforementioned profile analysis, Figure 14 quantitatively evaluates the retrieved microphysical parameters against the official satellite products using density scatterplots. The retrieved M_c demonstrates robust consistency with EarthCARE (Fig. 14c), yielding a high correlation ($R = 0.90$) and a negligible mean error ($ME = -0.01 \text{ g} \cdot \text{m}^{-3}$). For the D_m , the algorithm exhibits a slight positive bias against EarthCARE (Fig. 14a, $ME = 0.04 \text{ mm}$) and a negative bias relative to GPM (Fig. 14b, $ME = -0.12 \text{ mm}$). The positive ME relative to EarthCARE occurs because the synergistic framework actively compensates for W-band attenuation in deeper layers, thereby mitigating the previously discussed underestimation inherent to single-frequency algorithms. Conversely, the negative bias against GPM arises from the inherent insensitivity of the DPR to small hydrometeors, which physically skews the GPM official product toward larger drop sizes. Incorporating highly sensitive



W-band observations effectively constrains the small-particle end of the size spectrum, naturally shifting the retrieved D_m downward. Furthermore, the moderate correlation coefficients for D_m ($R = 0.62$ and 0.53 , respectively) are physically expected. These moderate correlations are driven by inevitable spatiotemporal uncertainties during cross-platform collocation, alongside the intrinsic behavior of the bi-directional optimal estimation framework. Rather than forcing an overfit to either individual baseline, the proposed algorithm optimally weights the divergent W- and Ka-band scattering signatures to converge on a physically consistent intermediate state.

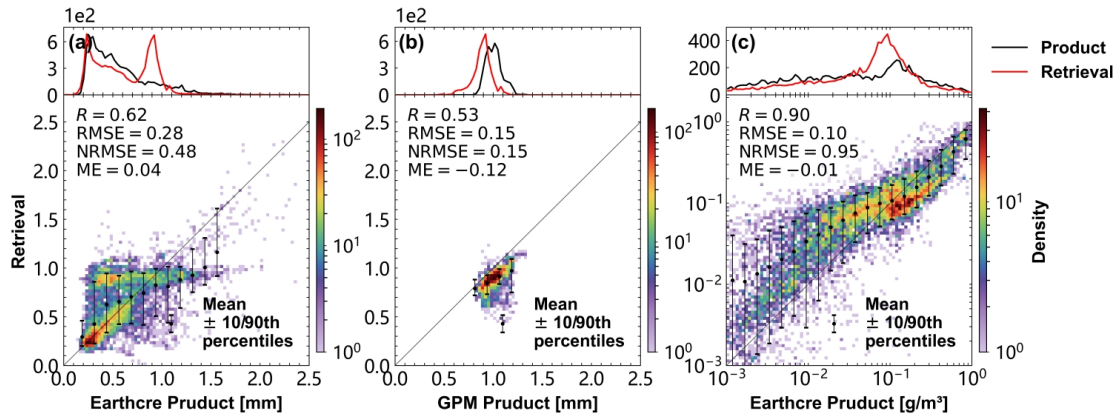


Figure 14: Density scatterplots evaluating retrieved microphysical parameters against official satellite products for the collocated stratiform precipitation scene. Panels compare retrieved (a) D_m versus EarthCARE, (b) D_m versus GPM, and (c) M_c versus EarthCARE. Top sub-panels display corresponding marginal probability density distributions. Statistical metrics and error bars indicating the mean and 10th/90th percentiles are annotated within each panel.

5 Conclusions

This study develops an adaptively regularized dual-frequency radar retrieval framework and presents a comprehensive sensitivity analysis. Conditioned on a priori hydrometeor classification, the algorithm employs a Levenberg-Marquardt optimization with dynamic damping. It incorporates a multiphase forward operator and applies Twomey-Tikhonov regularization to stabilize the optimization by constraining state increments, mitigating hydrometeor uncertainties, and preserving operator continuity. Finally, a bi-directional iterative strategy formulates PIA as a soft constraint, enabling simultaneous attenuation correction and state parameter optimization.

To characterize the impacts of multiple perturbation terms on the regularized optimal retrieval, sensitivity analyses were conducted using WRF-simulated marine cyclones to decouple key error sources. Results show that the algorithm effectively suppresses a priori and initial-guess perturbations while mitigating standard PIA biases. Incorporating the DFR reduces the NRMSE of M_c from > 3.0 (in single-frequency retrievals) to < 0.3 . The adaptive regularization tolerates up to 80 % initial state errors ($R > 0.94$, $NRMSE < 0.35$), while the soft-constrained PIA prevents numerical divergences common in hard-constrained backward integration. However, accuracy remains sensitive to hydrometeor misclassification and observation noise. The soft PIA constraint suppresses downward error accumulation in attenuating columns (reducing M_c NRMSE from



5.09 to 1.23 under 5.0 dB observation noise), weak pure-ice signals remain susceptible to observation noise. To assess the retrieval stability, full-scene simulations were conducted under combined multi-source uncertainties, specifically incorporating independent Gaussian white noise ($\sigma = 2$ dB) on reflectivity profiles, 10–20 % random fluctuations in total PIA, and a priori fields reconstructed solely from empirical relationships. Under these prescribed perturbations, the relative uncertainties for D_m and M_c remain within 26 % (RMSE = 0.10 mm) and 40 % (RMSE = 0.05 g·m⁻³).

Validations utilizing collocated W- and Ka-band measurements from ground-based (CloudCube) and spaceborne (EarthCARE, GPM) platforms demonstrate the framework's retrieval reliability and physical consistency. For the ground-based marine stratocumulus case, retrievals accurately capture collision—coalescence microphysical signatures. Dual-frequency LWP retrievals effectively mitigate the systematic underestimations inherent in single-frequency methods, yielding an NRMSE of 0.57 and a MAPE of 29.2 % against independent microwave radiometer data—an approximate 30 % error reduction. This performance improvement stems from the synergistic integration of differential scattering and attenuation properties across both bands. Unlike single-frequency retrievals, which suffer from cumulative two-way PIA and non-Rayleigh resonance in dense liquid layers, the dual-frequency approach rigorously constrains the droplet size distribution to ensure physical closure. For collocated spaceborne observations, the dual-frequency retrieval reproduces EarthCARE M_c profiles closely ($R = 0.90$) while showing a slight positive D_m bias against EarthCARE ($ME = 0.04$ mm) and a negative bias relative to GPM ($ME = -0.12$ mm). These opposite offsets stem from the complementary physical characteristics of the two single-frequency products: W-band observations are subject to path-integrated attenuation in dense cloud layers, whereas GPM DPR measurements weight larger drops more heavily due to lower small-particle sensitivity. By balancing W- and Ka-band scattering signatures, the dual-frequency retrieval accounts for deep-layer attenuation while constraining the small-particle spectrum, yielding a balanced intermediate state without overfitting to either baseline.

In conclusion, conditioned on an a priori hydrometeor classification, the proposed adaptively regularized bi-directional framework employs Twomey-Tikhonov regularization to mitigate non-physical vertical oscillations driven by observation noise during simultaneous microphysical retrievals. Furthermore, treating PIA as a soft constraint within the bi-directional iteration alleviates the numerical divergences typical of hard-constrained backward integrations, thereby facilitating stable attenuation correction and enhancing retrieval stability even under substantial initial-state uncertainties. Nevertheless, inherent physical limitations of the W-/Ka-band combination persist, with Rayleigh scattering of small particles yielding a near-zero DFR, and non-Rayleigh resonances in larger particles causing DFR saturation. These scattering characteristics, compounded by severe W-band attenuation in heavy precipitation and complex melting-layer transitions, further exacerbate retrieval ambiguities. Future research will explore triple-frequency (Ku/Ka/W) synergies to compensate for severe deep-layer attenuation, extending continuous profiling capabilities from non-precipitating clouds to intense rainfall. Moreover, integrating sub-millimeter radars (e.g., G-band), Doppler spectra, and in situ observations will help alleviate the Rayleigh limitations of current frequency bands and refine melting-layer parameterizations, ultimately optimizing retrieval architectures for next-generation spaceborne missions.



Data availability. The NCEP FNL reanalysis data for driving WRF model simulation are available at <https://rda.ucar.edu/datasets/ds083.2>. The CloudCube radar data are available at <https://adc.arm.gov/discovery>. The GPM DPR data are available at <https://gpm.nasa.gov/data/directory>. The EarthCARE data are available at <https://earth.esa.int/eogateway/missions/earthcare/data>.

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Author contributions. JR and LL carried out the sensitivity analysis, weather case simulation experiment and data analysis; ZY constructed the scattering property database; WY carried out the WRF model simulation and provided the input; J provided technique support and analysis methods. JR wrote the original article with feedback from all the co-authors.

610 *Competing interests.* The authors declare that they have no conflict of interest.

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