



On the application of Simple Feedback Diagrams in nature and society: daisy topology and tipping points (part I)

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Abstract. Feedback loops are common in all kinds of complex systems and are a way to understand these systems' large-scale behaviour. Feedback diagrams are often used to illustrate such positive or negative loops qualitatively. The theory presented here allows these feedbacks to be quantified and their stability properties computed, in order to make the diagrams more widely applicable. In particular, the theory derived here can be used to describe the dynamics beyond the tipping point of a system and thereby to estimate fluctuations and extremes during the unfolding of the tipping process. To this end we derive simple rules for computing the Lyapunov exponents that determine stability from simple estimates of the processes that form a feedback. We exploit the simplification that follows from a loop structure, which is much more tractable than a situation in which every variable depends on every other variable. To make the diagrams useful across applications we (a) show that loops with multiple processes can be collapsed into two-process loops, (b) derive the relevant Lyapunov exponent from a loop's feedback strength and timescale, (c) derive rules for combining multiple loops into a system of loops sharing one central variable – a daisy topology and (d) derive rule to combine multiple daisy topologies. We show the existence of oscillatory instabilities in systems with at least 3 variables. The analysis allows the identification and quantification of thresholds, so-called tipping points, and the computation of their stability properties. It requires a minimum of information about the underlying system and is thereby applicable in physical, chemical and biological systems as well as in economic and societal ones. We illustrate the approach with examples from climate – the melt-elevation feedback of ice sheets and the water-vapour greenhouse feedback – and from society, using the feedback between wages and prices and a known business-cycle model. Finally, we provide a cookbook for a feedback and tipping-point analysis, with or without known governing equations.

1 Introduction: feedback loops, graphs, and dynamical stability

Feedbacks occur in all kinds of systems: the elevation-melt feedback that governs ice-sheet retreat – itself now represented directly inside integrated assessment models that couple the climate system to the economy (Ramme et al., 2025) – the water-vapour feedback that amplifies greenhouse warming, the overturning feedback in ocean circulation, and analogous amplifying or damping mechanisms in economic and social systems (Scheffer, 2009). In this paper we are interested in feedback loops of this kind. They arise in complex systems ranging from oceanic and atmospheric circulation, through chemical and biological processes, to economic and societal dynamics. The full dynamics of these systems is in general far more complicated than the



feedback loops discussed here; the merit of isolating a feedback loop lies in its self-amplifying or self-stabilizing effect, which can dominate the system's overall stability even if it does not capture all of its detail.

We do not aim here to capture all possible feedback-loop configurations comprehensively, since this would require a general topological discussion of interlinked network graphs. Instead we gradually construct more and more complicated feedback topologies. In this we aim to be as useful as possible to the eventual user; i.e. we aim to build a cookbook for approaching certain complex systems.

We first consider a single loop with an arbitrary number of processes. Then we show how such a loop can be collapsed into a loop with only two processes – our fundamental building block. Next we construct situations in which multiple two-process loops are interlinked at one central process, which allows a dynamical system dominated by one central variable to be analysed as a combination of independently characterised loops. This approach allows tipping points to be identified in such systems, and the relevant Lyapunov exponent – which determines the stability of the system as parameters change – to be computed easily. We illustrate the approach using Earth's global mean temperature as an example, and discuss the tipping of ice sheets through the melt-elevation feedback within this framework. Section 7 then gives a cookbook for a feedback and tipping-point analysis, illustrated with examples from climate and from society.

The present framework is conceptually related to several mathematical traditions that study how feedback-loop topology constrains the dynamics of complex systems. A particularly important precursor is the work of René Thomas on regulatory networks and multistationarity: Thomas conjectured that a positive feedback circuit is a necessary condition for multiple steady states, while a negative circuit is necessary for sustained oscillations (Thomas, 1981), conjectures that were later proven by Gouzé (1998) and Snoussi (1998) for the negative and by Soulé (2003) for the positive circuit. This establishes one of the first direct links between graph topology and qualitative dynamical behaviour; rather than requiring detailed equations, that theory derives dynamical constraints directly from the sign structure of an interaction graph – the same two qualitative outcomes, saddle-node tipping and Hopf 'wobbling', that re-appear below as the two routes by which a loop can lose stability.

Closely related is the theory of qualitative loop analysis developed by Levins and Puccia, in which a system is represented as a signed directed graph whose feedback structure determines its qualitative stability properties, translating Routh–Hurwitz-type criteria into combinations of graph loops and paths (Puccia and Levins, 1985; Justus, 2006). Loop analysis thereby infers stability and response patterns from causal structure alone, even when quantitative parameters are poorly known.

The present theory also connects to graph dynamical systems and network science, where differential equations are defined on a graph and the network topology constrains synchronization, stability, diffusion and cascade behaviour (Mesbahi and Egerstedt, 2010; Newman, 2010); the present framework can be understood as a reduction of such ideas to a small number of effective, interacting feedback modules.

A further connection is to the theory of qualitative stability, beginning with May's demonstration that the stability of a large, randomly coupled system is governed by simple combinations of its connectivity and the typical strength of its interactions (May, 1972), and to tipping-point theory: a tipping point is a bifurcation at which a perturbation ceases to decay and instead amplifies, often preceded by critical slowing down and classifiable into bifurcation-induced, noise-induced and rate-induced transitions (Scheffer et al., 2009; Ashwin et al., 2012). The present theory reformulates such instabilities directly in terms



of feedback-loop amplification: a tipping point occurs when the effective amplification around a loop exceeds the stabilizing damping set by the loop's own timescale. This is particularly relevant for interacting tipping elements in the Earth system, where climate tipping elements can form networks capable of triggering cascades through coupled feedbacks (Lenton et al., 2008; Wunderling et al., 2021); the daisy-topology formalism introduced here provides a reduced mathematical description of such interacting structures without requiring fully resolved Earth-system simulations.

The contributions of this paper are to put a selected set of widely applicable feedback diagrams on a solid mathematical basis, and to provide easy access to the feedback and tipping-point analysis of a dynamical system even without capturing it fully mathematically. The analysis itself is not particularly novel, nor is the idea of a feedback loop a new one; the point here is to allow a quantitative feedback analysis with minimal information about a system. All that is needed is (a) an estimate of the relations between the processes involved in the feedback, (b) an estimate of the partial derivative of one process with respect to another, i.e. how much one variable changes if another is changed, and (c) an estimate of the timescale associated with each process. With this one obtains (a) the tipping points of the system, (b) the stability of these tipping points, and (c) the timescale on which a tipping event would unfold.

Someone who merely wants to apply this method can skip the remaining sections of the present introduction and turn directly to Sect. 7, which provides a cookbook for a feedback and tipping-point analysis.

This is the first paper of a series. Here we establish the diagram formalism for loops combined at one shared central variable – the daisy topology – together with the resulting stability and model-building machinery, and we show that daisy topologies can themselves be nested or coupled (Sects. 5–6). A companion paper (part II) derives the characteristic equation of an arbitrary feedback diagram from its loops, shows that the rules of this paper are special cases of a general rule set, and applies it to loops that are coupled not only through a shared central variable but through a directly shared branch – the ‘rose’ topology. A daisy, in this sense, shares only a single point – the centre – between its loops. This is distinct from a topology in which two loops share an entire branch, i.e. a common process linking two variables, rather than only a single variable: we call this a ‘rose’ topology (Fig. 2), and it is the subject of the companion paper (part II) mentioned above.

2 General set-up of a feedback analysis

The analyses we consider here start with the construction of feedback diagrams, such as the one in Fig. 1. These graphs can form simple or complex loop structures which can be self-amplifying or damping. The purpose of this paper is to provide some means to analyse the stability of a dynamics that arises from the interaction between variables that are related through these graphs. We will try to build a collection of fundamental structures and rules for how to combine their stability properties into the stability property of the entire system.

To this end, we consider a set of N variables, A_i with $i \in \{1, \dots, N\}$, which depend on each other. Although we will make specific use of different kinds of loop structures in the interlinkage of the variables, we assume for a moment that they can all depend on each other.

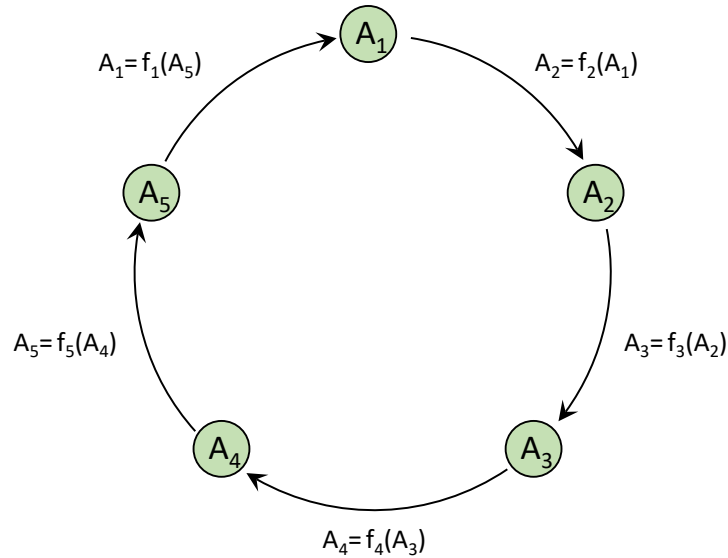


Figure 1. A single loop with 5 variables illustrates the fundamental building block of feedback diagrams. The feedback strength is the product of the partial derivatives (dependency strengths) along the loop, i.e. $S \equiv \prod_{i=1}^N \partial f_i / \partial A_{i-1}$. It is by definition dimensionless, i.e. has no unit. The Lyapunov exponents have, in principle, to be computed through the associated Jacobi matrix, but the Lyapunov exponent that determines stability can simply be computed as $(S - 1)/\tau$, as shown in the text. τ is the sum of the timescales associated with each of the relations between the variables, i.e. $\tau \equiv \sum_{i=1}^N \tau_i$.

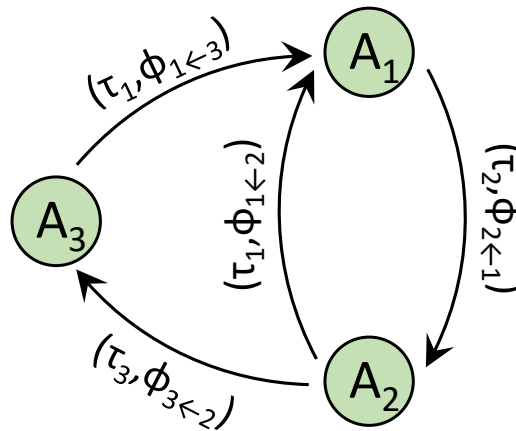


Figure 2. A rose topology, shown here for two loops sharing one branch: the usual two-variable loop between A_1 and A_2 is combined with a second, three-variable loop $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_1$ that shares the branch $A_1 \rightarrow A_2$ with the first. Unlike the daisy topology of this paper, where loops meet only at a single shared centre, loops here are linked through an entire shared process; this rose topology is the subject of the companion part II paper.



2.1 The time evolution

Regarding the dependencies, we consider two qualitatively different kinds of relations, plus a third case in which no equations are known at all. For simplicity we annotate three ‘knowledge levels’ to the respective equations and variables. A variable with an explicit prognostic equation has knowledge level 1 (KL1): the case in which the time evolution of a variable A_i is explicitly known,

$$\frac{dA_i}{dt} = F_i(A_1, \dots, A_N, t); \quad \forall i \in \{1, \dots, N\}. \quad (1)$$

We denote the functions that describe the time evolutions with F_i . They have the unit of the respective variable, A_i , over time.

The present formalism assumes intrinsically restoring local dynamics, i.e. $\partial F_i / \partial A_i < 0$, so that every variable possesses a positive local relaxation timescale,

$$\tau_i \equiv - \left(\frac{\partial F_i}{\partial A_i} \right)^{-1} > 0. \quad (2)$$

Systems with a vanishing or positive self-dependence require a generalized treatment, because their stability is then no longer determined solely by the feedback-loop structure but also by the intrinsic amplification or conservative dynamics of the individual variable; we return to a special, important case of this – a centre variable without its own restoring dynamics – in Sect. 4.4.

The case in which only the relations between the steady states of the variables, A_i^* , are known is denoted a diagnostic equation and has knowledge level 2 (KL2),

$$A_i^* = f_i(A_1^*, \dots, A_N^*); \quad \forall i \in \{1, \dots, N\}. \quad (3)$$

We denote the functions that describe these steady-state relations by f_i . They have the same unit as the respective variable, i.e. not divided by time, unlike F_i . In this case the stability of the system can also be determined, but doing so requires additional information: an estimate of the timescale $\tau_i > 0$ at which the variable A_i typically changes. We can then describe the time evolution near the steady state A_i^* by a linearized approximation of the (otherwise unknown) actual time evolution,

$$\frac{dA_i}{dt} = \frac{f_i(A_1, \dots, A_N) - A_i}{\tau_i}; \quad \forall i \in \{1, \dots, N\}. \quad (4)$$

For later convenience we define $r_i \equiv \tau_i^{-1}$, the reciprocal of the timescale. The steady state of Eq. (4), i.e. the time derivative set to zero, recovers Eq. (3). As we show later by example, not every prognostic equation F_i yields a steady-state function f_i valid for all values of $(A_1, \dots, A_N)$, but in the vicinity of the steady state a function f_i can generally be found such that the approximate time evolution of Eq. (4) is valid.

One restriction of the formalism needs to be emphasized, since it is the reason Eq. (2) above has to hold for every variable individually, before any loop is considered: we do not allow a variable to amplify itself; any amplification has to arise through the interaction *between* variables, i.e. it has to sit in the links of the diagram rather than inside a single node. This can generally



be arranged by introducing further variables where needed. Consider, for example, the reproduction of rabbits: we cannot simply write the time evolution of the number of rabbits N_r as

$$\frac{dN_r}{dt} = \beta_r \cdot N_r - \gamma_r \cdot N_r$$

with a reproduction rate β_r and a death rate γ_r , if $\beta_r > \gamma_r$, since this variable then amplifies itself and Eq. (2) would give a negative timescale. Instead we have to make the system one node richer, e.g. by separating out the bunnies that have not yet matured into reproducing rabbits,

$$\begin{aligned} \frac{dN_r}{dt} &= \mu_r \cdot N_b - \gamma_r \cdot N_r \\ \frac{dN_b}{dt} &= \beta_r \cdot N_r - \gamma_b \cdot N_b. \end{aligned}$$

Now neither variable depends on itself, and any net amplification of the system has to be supplied by the loop $N_r \rightarrow N_b \rightarrow N_r$ rather than by either variable alone – which is exactly the kind of amplification the present theory is built to quantify.

The main motivation for this paper, however, lies in treating a third case, in which no equations for the relations between the different variables are available at all. In these cases we again need to estimate the timescale associated with the change of the variable A_i , $\tau_i > 0$. In addition we need to estimate the change in A_i when another variable, A_j , changes. In a loop structure, one variable will generally depend only on one or two others. We denote this knowledge level 3 (KL3). The respective changes are mathematically represented by the partial derivatives

$$\phi_{i \leftarrow j} \equiv \frac{\partial f_i}{\partial A_j}, \quad (5)$$

but since we do not have the respective equations, these estimates can be made simply by estimating how much A_i changes if A_j changes, everything else being held fixed. We denote these entities, $\phi_{i \leftarrow j}$, as the dependency strength of A_j on A_i . In economics such entities are sometimes called the ‘marginal effects with respect to something’; for example, the marginal cost of climate-change mitigation is the macroeconomic cost of reducing CO₂ emissions by a given unit, e.g. one tonne of CO₂ per year. These specific estimates of $\phi_{i \leftarrow j}$ may arise from data, theory, or a general understanding of the system at play. We remain deliberately vague here to emphasize that the specific way in which an estimate of $\phi_{i \leftarrow j}$ is obtained does not matter for the following analysis – only its value does.

To summarize:

KL1: Knowledge Level 1: the functional form of the time evolution of a variable’s dependence on the other variables is known (Eq. (1)).

KL2: Knowledge Level 2: the functional form of the steady-state dependence of a variable on the other variables is known (Eq. (3)); a linearized time evolution near the steady state is constructed using a timescale estimate (Eq. (4)).

KL3: Knowledge Level 3: only the qualitative dependencies between variables are known. The dependency strength, $\phi_{i \leftarrow j}$, and the timescales of change, τ_i , are estimated from data or a general understanding of the system.



2.2 Setting up the Jacobian matrix for the stability analysis

Let us first consider a situation in which the system is completely KL2, i.e. we know the relations between the steady states of the variables (Eq. (3)) for all variables, have estimated the timescales τ_i , and assume a linearized time dependence around the steady state following Eq. (4). In this case the stability of the system is determined by the Jacobi matrix

$$155 \quad \mathbb{J}_{i,j} = r_i \cdot \begin{cases} \frac{\partial f_i}{\partial A_j} = \phi_{i \leftarrow j} & \forall (i,j) \mid j \neq i \\ -1 & \forall (i,j) \mid j = i, \end{cases} \quad (6)$$

where we again defined the rates $r_i \equiv \tau_i^{-1}$. Note that the Jacobian in Eq. (6) depends only on the timescales and the dependency strengths $\phi_{i \leftarrow j}$. For variables that have KL3, we can thus construct the Jacobian without the explicit equations, while keeping in mind that a mathematical structure as given by Eqs. (3) and (4) is an underlying assumption.

For equations of KL1, the Jacobian is given by

$$160 \quad \mathbb{J}_{i,j} = \left. \frac{\partial F_i}{\partial A_j} \right|_{A_k^*}, \quad \forall (i,j,k). \quad (7)$$

Generally, we can compute from here both the dependency strength $\phi_{i \leftarrow j}$ and the timescales τ_i . The dependency strength between two variables is then defined, using Eq. (2), as

$$\phi_{i \leftarrow j} \equiv \tau_i \cdot \left(\frac{\partial F_i}{\partial A_j} \right) = - \left(\frac{\partial F_i}{\partial A_j} \right) \cdot \left(\frac{\partial F_i}{\partial A_i} \right)^{-1}. \quad (8)$$

165 With these definitions the KL1 variables can be treated as KL2 variables. In KL1 cases the analysis can be carried out through the full Jacobian and the simplified feedback diagrams are not strictly necessary; the purpose of this paper is mainly KL2 and KL3 cases. We thus restrict ourselves to cases in which the Jacobian can be represented in the form

$$\mathbb{J}_{i,j} = \begin{cases} r_i \cdot \phi_{i \leftarrow j}, & j \neq i \\ -r_i, & j = i. \end{cases} \quad (9)$$

2.3 Loop structures

170 In order to make feedback loops useful for as many scientific disciplines as possible, we aim to reduce the necessary information about a system to a minimum. We will go through a number of geometries and provide rules to compute the relevant Lyapunov exponents, or at least their sign, in order to determine the stability of the system. Feedback systems can be very complicated and need not fall into the structural forms we investigate here; the structures to which we restrict the analysis nonetheless cover a number of systems and are, we hope, useful.

175 It is sometimes helpful to denote a system by a triplet (l, v, f) , where the first entry gives the number of loops, the second the number of variables in these loops, and the third the number of variables that are forced. For example, $(l1, v3, f0)$ is a system with one loop of three variables, none of which is forced, as illustrated by the case $N = 3$ of Fig. 1. Note that the number of



processes between the variables could in principle differ from the number of variables itself, but will generally coincide within one loop; it will differ once loops are combined.

We can combine triplets to specify systems with loops of different numbers of variables, e.g. $(l1, v2, f1)-(l4, v2, f0)$, which describes a system of five loops, one of which has two variables with a forcing on one of them, and the other four of which have two variables with no forcing. Since we will show below that a loop's processes, if they have no external forcing and are not directly linked to another loop, can generally be eliminated from the loop using two simple rules for its feedback strength and timescale, loops with two variables will be central to the analysis, and loops with many processes become correspondingly less important once this reduction has been applied. This nomenclature is neither unique nor comprehensive; it is merely a convenient shorthand.

3 One loop with N variables (pilea topology)

We first consider one-loop systems with an arbitrary number of processes. We emphasize again that we only consider systems here in which each variable possesses a restoring timescale, $\partial F_A / \partial A < 0$; any amplification should be separated into further variables so that it is captured by the loop structure rather than by a single variable's own dynamics. We discuss a special case with a centre-manifold dynamics, i.e. with a direction A_i for which $\partial F_i / \partial A_i = 0$, in Sect. 4.4 below; a single feedback loop with N variables constructed entirely from such centre-manifold dynamics has only zero Lyapunov exponents and is not of particular interest here. Let us thus proceed with the axiomatic set-up below.

3.1 Axiomatic set-up with diagnostic equations

Consider a system that is determined by N variables A_i^* with $i \in \{1, \dots, N\}$. Following Fig. 1, let us consider a loop of dependencies that can be expressed in terms of functions f_i that determine the state of A_i^* through the state of A_{i-1}^* , with a cyclic closing of the loop, i.e.

$$\begin{aligned} A_1^* &= f_1(A_N^*) \\ A_i^* &= f_i(A_{i-1}^*); \quad \forall i \in \{2, \dots, N\}. \end{aligned}$$

In order to simplify the notation we define $A_0^* \equiv A_N^*$, which transforms the above into

$$A_i^* = f_i(A_{i-1}^*); \quad \forall i \in \{1, \dots, N\}. \quad (10)$$

The steady-state solutions of this system are given by the solutions of the set of Eqs. (10). We approximate the time evolution near the steady state by

$$\frac{dA_i}{dt} = \frac{f_i(A_{i-1}) - A_i}{\tau_i} \equiv r_i \cdot (f_i(A_{i-1}) - A_i); \quad \forall i \in \{1, \dots, N\}, \quad (11)$$

again closing the loop by $\tau_0 \equiv \tau_N$ and $r_0 \equiv r_N$. Close to the steady-state solutions this is a reasonable approximation of any temporal dynamics that might be derived from other considerations for the evolution of A_j . We argue that if only the steady-state dependencies of an actual system are known, and the timescale of its dynamics, τ_j , can be approximated, this set of



equations is a reasonable time evolution for many systems. It has a number of helpful properties, e.g. a very robust numerical stability and an increasingly good approximation skill near the steady state.

3.2 Lyapunov exponents and stability

210 In order to compute the Lyapunov exponents of this system we compute $\det(\mathbb{J} - \lambda \cdot \mathbb{1})$ with the Jacobi matrix $\mathbb{J}$. For convenience we drop the second index in the dependency strength and define

$$\phi_i \equiv \phi_{i \leftarrow i-1} = f'_i(A_{i-1}^*); \quad \forall i \in \{1, \dots, N\}. \quad (12)$$

The determinant is readily expanded along its first row and column (Appendix A gives the short recursive argument in full) to give

$$215 \quad |\mathbb{J} - \lambda \cdot \mathbb{1}| = (-1)^N \left(\prod_{j=1}^N (\lambda + r_j) - \prod_{j=1}^N (r_j \cdot \phi_j) \right). \quad (13)$$

Thus the solutions of $|\mathbb{J} - \lambda \cdot \mathbb{1}| = 0$ are equivalent to the solutions of

$$\prod_{j=1}^N (\lambda + r_j) = \prod_{j=1}^N (r_j \cdot \phi_j), \quad (14)$$

or, since $r_j > 0$ for all j ,

$$S = \prod_{j=1}^N \left(1 + \frac{\lambda}{r_j} \right) = \prod_{j=1}^N (1 + \lambda \cdot \tau_j), \quad (15)$$

220 where the strength of the feedback loop,

$$S \equiv \prod_{j=1}^N \phi_{j \leftarrow j-1}, \quad (16)$$

has been introduced. The strength of the feedback loop is dimensionless, since it is the product of derivatives in a closed loop. Note that the specific choice of equilibrium solution A_j^* , in the case of multiple steady states, enters S , and only S , through the values of ϕ_j . The timescales are separated from the strength of the feedback loop. The Lyapunov exponents by definition have

225 the unit one over time. We know that Eq. (15) has N solutions, λ_i , $i \in \{1, 2, \dots, N\}$.

For $|S| < 1$, we can show that the real part of every solution λ has to be smaller than zero, and thus that the entire system has to be stable. This follows easily by contradiction: assume $\Re(\lambda) \geq 0$; then $|1 + \tau_j \lambda| \geq 1$ for all j , hence $|S| = \left| \prod_{j=1}^N (1 + \tau_j \lambda) \right| \geq 1$, which contradicts the assumption; thus all eigenvalues have a negative real part for $-1 < S < 1$.

The most important situation occurs at $S = 1$. In order to discuss the transition of S from below to above 1, define the
 230 complex function

$$p(\lambda; S) = \prod_{j=1}^N (1 + \tau_j \lambda) - S, \quad \tau_j > 0, S \in \mathbb{R}, \quad (17)$$



and note that for $S = 0$ all roots, i.e. λ that solve $p(\lambda; S = 0) = 0$, are real and negative, because they are given by the inverse of the timescales, $\lambda_i = -r_i = -1/\tau_i$. Hence, all roots lie on the negative real axis. By continuity of roots, the number of roots with $\Re(\lambda) > 0$ can change only when a root crosses the imaginary axis (Hurwitz, 1895). Writing such crossings as $\lambda = i\omega$,

235 $\omega \in \mathbb{R}$, and rewriting the polynomial in polar form,

$$p(\lambda; S) = \prod_{j=1}^N (1 + i\tau_j \omega) - S = \left(\prod_{j=1}^N \sqrt{1 + \tau_j^2 \omega^2} \right) \cdot e^{i \cdot \sum_{j=1}^N \tan^{-1}(\tau_j \omega)} - S, \quad (18)$$

we see that, since S is real, this expression can vanish only if $|S| = \prod_{j=1}^N \sqrt{1 + \tau_j^2 \omega^2} \geq 1$ and $\sum_{j=1}^N \tan^{-1}(\tau_j \omega) = m\pi$ with $m \in \mathbb{Z}$.

At $S = 1$, $\lambda = 0$ is a simple root, since $p(\lambda = 0, S = 1) = 0$. Furthermore, since $\partial p / \partial S = -1$,

$$240 \quad \frac{d\lambda}{dS} = -\frac{\partial_S p}{\partial_\lambda p} = \frac{1}{\partial_\lambda p}, \quad (19)$$

and thus

$$\left. \frac{d\lambda}{dS} \right|_{\lambda=0, S=1} = \left. \frac{1}{\partial_\lambda p} \right|_{\lambda=0, S=1} = \frac{1}{\sum_{j=1}^N \tau_j} > 0. \quad (20)$$

So, following the Routh–Hurwitz theorem, one real root crosses into the unstable regime, i.e. into $\Re(\lambda) > 0$, for $S > 1$ (Routh, 1877; Hurwitz, 1895). This transition is commonly referred to as a saddle-node bifurcation, or sometimes a tipping instability

245 (Andronov and Leontovich, 1937; Sotomayor, 1973).

In summary, for $|S| < 1$ the system is stable, and it becomes unstable at $S = 1$, acquiring one unstable, i.e. positive, real eigenvalue λ . These stability properties are completely independent of the values of the timescales τ_i , whether these have been computed from Eq. (1) or merely estimated. In words: S tells us *whether*, and at what threshold, a disturbance that travels once around the loop returns larger than it started (self-amplifying, $S > 1$) or smaller (self-damping, $S < 1$); the timescales tell us

250 only *how fast* this happens, not whether it happens at all.

With this knowledge we can now estimate the Lyapunov exponent λ^* that is relevant for the transition at $S = 1$. To this end we revisit Eq. (15) and approximate the left-hand side for λ^* close to zero. For $|\lambda^*| \ll r_j \forall j$,

$$S = \prod_{j=1}^N \left(1 + \frac{\lambda^*}{r_j} \right) \approx 1 + \lambda^* \cdot \sum_{j=1}^N \frac{1}{r_j} = 1 + \lambda^* \cdot \sum_{j=1}^N \tau_j. \quad (21)$$

Thus the Lyapunov exponent relevant for the system's stability, λ^* , can be estimated as

$$255 \quad \lambda^* = \frac{S - 1}{\sum_{j=1}^N \tau_j} = \frac{S - 1}{\tau}, \quad \tau \equiv \sum_{j=1}^N \tau_j. \quad (22)$$

This means that the stability itself is determined by S alone, not by the timescales, and that the relevant timescale of the loop is simply the sum of the timescales of the different processes that form the loop. The commonly considered tipping-point instabilities are crossings into the $S > 1$ regime, which are independent of τ_i . There are, however, other instabilities that do depend on the timescales, to which we turn next.



260 3.3 Oscillations, eigenfrequencies and ‘wobble’ instabilities

The discussion above focused on the dominant Lyapunov exponent near the saddle-node instability at $S = 1$, where the relevant eigenvalue approaches $\lambda = 0$. A feedback loop may, however, also support oscillatory modes associated with complex eigenvalues. The real part determines whether an eigenmode is stable (negative real part) or unstable (positive real part); the imaginary part is the (angular) frequency of the oscillation. Since non-zero imaginary parts always emerge in complex-conjugate pairs,

265 $\pm i\omega$, we only need to consider positive frequencies.

For a single loop with N variables, the eigenvalues satisfy Eq. (15). Explicitly distinguishing the real and imaginary parts allows us to compute the oscillation frequency of an eigenmode, $\mu = |\Im(\lambda)|/(2\pi)$. Such frequencies depend in general not only on the timescales within the loop but also on the feedback strength S , and the general calculation can become quite involved. It simplifies considerably if we ask only for the specific feedback strength at which a given mode becomes marginal, i.e. at which

270 the corresponding eigenvalues cross the imaginary axis, $\lambda = i\omega$. Expressing Eq. (15) in polar form, as in Eq. (18), gives

$$S = \left(\prod_{j=1}^N \sqrt{1 + \omega^2 \tau_j^2} \right) \cdot \exp \left(i \sum_{j=1}^N \tan^{-1}(\omega \tau_j) \right). \quad (23)$$

Since S has no imaginary part, solutions ω exist only where $\sum_{j=1}^N \tan^{-1}(\omega \tau_j)$ is an integer multiple of π ; whether this corresponds to an instability depends on whether the multiple is odd or even.

Crossings of roots into the unstable regime, $\Re(\lambda) > 0$, occur only at discrete values $\omega = \omega_k^-$ satisfying an *odd* multiple,

$$275 \sum_{j=1}^N \tan^{-1}(\tau_j \omega_k^-) = (2k+1)\pi, \quad k = 0, 1, 2, \dots, \quad (24)$$

which correspond to the discrete, *negative* feedback-strength values

$$S_k^- = - \prod_{j=1}^N \sqrt{1 + \tau_j^2 (\omega_k^-)^2}, \quad (25)$$

at which a conjugate pair $\lambda = \pm i\omega_k^-$ sits exactly on the imaginary axis. One can show that each such crossing point increases the number of roots in $\Re(\lambda) > 0$ by two as S decreases through S_k^- . The proper name for this kind of transition is a Hopf bifurcation (Hopf, 1942); we refer to it informally as a ‘wobble instability’. At $S = S_0^-$ – the least negative, and hence first-reached, of the thresholds in Eq. (25) – the system becomes oscillatory unstable through negative reinforcement: the feedback strength is so strongly negative that the system overcorrects any perturbation and thereby oscillates away from the steady state, rather than amplifying monotonically as at the ordinary tipping point.

The even multiples of Eq. (18) correspond to a second, distinct family of crossings. Defining ω_k^+ by

$$285 \sum_{j=1}^N \tan^{-1}(\tau_j \omega_k^+) = 2k\pi, \quad (26)$$

the corresponding feedback strengths are

$$S_k^+ = \prod_{j=1}^N \sqrt{1 + \tau_j^2 (\omega_k^+)^2}, \quad (27)$$



and each such crossing point increases the number of roots in $\Re(\lambda) > 0$ by two as S increases through S_k^+ . Since $S_k^+ > 1$, these crossings occur beyond the ordinary saddle-node instability at $S = 1$ and are therefore typically of less practical relevance than the negative-feedback Hopf bifurcations above, although they can still manifest as oscillatory modes if the system is perturbed from the outside.

To estimate the eigenfrequencies ω_m for both families ($m = 2k + 1$ or $m = 2k$), we return to their defining equation, $\sum_{j=1}^N \tan^{-1}(\omega_m \tau_j) = m\pi$, and define the arithmetic and harmonic means of the timescales,

$$\langle \tau \rangle_{\text{ari}} = \frac{1}{N} \sum_{j=1}^N \tau_j, \quad \langle \tau \rangle_{\text{harm}} = \left(\frac{1}{N} \sum_{j=1}^N \frac{1}{\tau_j} \right)^{-1} = \left(\frac{1}{N} \sum_{j=1}^N r_j \right)^{-1}. \quad (28)$$

The corresponding period $T_m \equiv 2\pi/\omega_m$ satisfies the bounds

$$\frac{2\pi}{\tan\left(\frac{m\pi}{N}\right)} \cdot \langle \tau \rangle_{\text{harm}} \leq T_m \leq \frac{2\pi}{\tan\left(\frac{m\pi}{N}\right)} \cdot \langle \tau \rangle_{\text{ari}}, \quad (29)$$

with equality throughout when all τ_j coincide. The harmonic mean is dominated by the smaller τ_j , the arithmetic mean by the larger ones.

Proof. Let $F(\omega) = \frac{1}{N} \sum_{j=1}^N \tan^{-1}(\omega \tau_j)$, so the eigenfrequency condition is $F(\omega_m) = m\pi/N$. Since $\tan^{-1}(\omega \tau)$ is concave in $\tau > 0$ for fixed $\omega > 0$, Jensen's inequality gives $F(\omega) \leq \tan^{-1}(\omega \langle \tau \rangle_{\text{ari}})$; evaluating at $\omega = \omega_m$ and inverting the (monotonic) $\tan^{-1}$ gives the lower bound on ω_m , and hence the upper bound on T_m . For the other direction, write $\tan^{-1}(\omega \tau_j) = \pi/2 - \tan^{-1}(1/(\omega \tau_j))$ and apply Jensen's inequality to the concave function $\tan^{-1}(u/\omega)$ with $u_j = 1/\tau_j$; this gives $F(\omega) \geq \tan^{-1}(\omega \langle \tau \rangle_{\text{harm}})$, and evaluating at $\omega = \omega_m$ gives the upper bound on ω_m and hence the lower bound on T_m . Combining the two bounds completes the proof. $\square$

Because only positive ω are considered and all $\tau_j > 0$, the sum $\sum_j \tan^{-1}(\omega \tau_j)$ is confined to $(0, N\pi/2)$, so only finitely many oscillatory modes exist, with

$$m = 1, \dots, \left\lfloor \frac{N-1}{2} \right\rfloor. \quad (30)$$

Thus a loop with $N = 2$ has no oscillatory mode at all, loops with $N = 3$ or $N = 4$ have one, loops with $N = 5$ or $N = 6$ have two, and so on. Once a mode frequency ω_m is known, the corresponding value S_m (from Eq. (25) or Eq. (27), depending on the parity of m) is the feedback strength at which that mode sits exactly on the imaginary axis; moving S away from S_m leaves the same mode in place but gives it a non-zero growth or decay rate, with the sign of S_m determining whether that side of the threshold is stable or unstable.

This separates the role of timescales from that of feedback strength: the timescales determine the characteristic oscillatory modes (their frequencies) of the loop, while the feedback strength determines whether a given mode is damped or amplified. A folk rule sometimes quotes the wobble threshold as simply $S = -1$; Eq. (25) shows this is in general not exact, and Sect. 3.4 gives $S_0^- = -8$, not -1 , for the smallest loop that can wobble at all ($N = 3$, equal timescales) – the value -1 is approached only in the limit of many comparably fast processes.



3.4 Analytic eigenfrequency for a three-variable loop

For a loop with $N = 3$ variables there is exactly one oscillatory mode, since Eq. (30) gives $m = 1$. Its frequency is determined
 320 by

$$\arctan(\omega\tau_1) + \arctan(\omega\tau_2) + \arctan(\omega\tau_3) = \pi. \quad (31)$$

This equation has an explicit algebraic solution.

3.4.1 Proposition.

For a three-variable loop, the oscillatory eigenfrequency is

$$325 \quad \omega^2 = \frac{\tau_1 + \tau_2 + \tau_3}{\tau_1\tau_2\tau_3}. \quad (32)$$

3.4.2 Proof.

Writing $x_i \equiv \omega\tau_i$, the frequency condition becomes $\arctan x_1 + \arctan x_2 + \arctan x_3 = \pi$. Taking the tangent of both sides and using the tangent addition theorem,

$$\tan(a + b + c) = \frac{\tan a + \tan b + \tan c - \tan a \tan b \tan c}{1 - \tan a \tan b - \tan a \tan c - \tan b \tan c}, \quad (33)$$

330 the left-hand side vanishes (since $\tan \pi = 0$), giving $x_1 + x_2 + x_3 - x_1x_2x_3 = 0$. Substituting $x_i = \omega\tau_i$ and dividing by $\omega \neq 0$ yields Eq. (32). $\square$

Equation (32) shows that the oscillation frequency is dominated by the *fast* processes in the loop. If one process is much slower than the other two, $\tau_3 \gg \tau_1, \tau_2$, then $\omega^2 \approx 1/(\tau_1\tau_2)$: the slow process drops out and the frequency is the geometric mean of the two fast rates, $\omega \approx 1/\sqrt{\tau_1\tau_2}$. Conversely, if one process is much faster than the other two, $\tau_3 \ll \tau_1, \tau_2$, then
 335 $\omega^2 \approx (\tau_1 + \tau_2)/(\tau_1\tau_2\tau_3) = \tau_3^{-1}(\tau_1^{-1} + \tau_2^{-1})$: the slow processes do not drop out, but the frequency is still dominated by the fast one.

For the special case of equal timescales, $\tau_1 = \tau_2 = \tau_3 \equiv \tau$, Eq. (32) gives $\omega = \sqrt{3}/\tau$ and a period $P = 2\pi/\omega = (2\pi/\sqrt{3})\tau \approx 3.6\tau$: the oscillation timescale is of the same order as an individual process timescale, larger by a factor of three to four. We can now also evaluate the corresponding threshold feedback strength from Eq. (25) with $k = 0$: since $\sqrt{1 + \tau^2\omega^2} = \sqrt{1 + 3} = 2$
 340 for each of the three (equal) factors,

$$S_0^- = - \prod_{j=1}^3 \sqrt{1 + \tau^2\omega^2} = -2^3 = -8, \quad (34)$$

which is the value already anticipated in Sect. 3.3: for the smallest loop that can wobble at all, the threshold is at $S = -8$, nowhere near the folk value of -1 .

For unequal timescales the threshold is even more negative. With $x_i \equiv \omega\tau_i$ and $x_1 + x_2 + x_3 = x_1x_2x_3$ from the proof above,
 345 $\prod_j (1 + ix_j) = 1 - (x_1x_2 + x_1x_3 + x_2x_3)$ is real, and Eq. (32) gives $x_1x_2 + x_1x_3 + x_2x_3 = (\sum_j \tau_j)(\sum_j \tau_j^{-1})$. Equation (25)



with $k = 0$ then becomes

$$S_0^- = 1 - \left(\sum_{j=1}^3 \tau_j \right) \left(\sum_{j=1}^3 \frac{1}{\tau_j} \right) \leq -8, \quad (35)$$

where the inequality between arithmetic and harmonic means gives $(\sum_j \tau_j)(\sum_j \tau_j^{-1}) \geq 9$, with equality only for equal timescales. Equal timescales thus give the least negative wobble threshold of any three-variable loop; unequal timescales make wobbling
 350 harder. For the hog cycle of Sect. 7.6, $\tau = (1, 0.5, 0.5)$ yr, Eq. (35) gives $S_0^- = 1 - 2 \cdot 5 = -9$.

3.5 General One-Loop Two-Variable System (1l2v)

We will now solve the two-process single loop as a building block for potentially more complex structures, and as an intuitive verification of the general calculation for all N . The equilibrium is defined by

$$A_1^* = f_1(A_0^*) \quad (36)$$

$$350 \quad A_0^* = f_0(A_1^*). \quad (37)$$

For later convenience, define the derivatives $\phi_{1 \leftarrow 0} = f_1'(A_0^*)$ and $\phi_{0 \leftarrow 1} = f_0'(A_1^*)$. The temporal evolution of the system near the equilibrium can be linearized as

$$\frac{dA_1}{dt} = \frac{f_1(A_0) - A_1}{\tau_1} \equiv r_1 \cdot (f_1(A_0) - A_1) \quad (38)$$

$$\frac{dA_0}{dt} = \frac{f_0(A_1) - A_0}{\tau_0} \equiv r_0 \cdot (f_0(A_1) - A_0), \quad (39)$$

360 where we defined the reciprocal timescales r_0 and r_1 . Without specifically computing the equilibrium solutions, we can evaluate the stability via the Jacobian

$$\mathbb{J} = \begin{pmatrix} -r_0 & r_0 \phi_{0 \leftarrow 1} \\ r_1 \phi_{1 \leftarrow 0} & -r_1 \end{pmatrix}. \quad (40)$$

The characteristic polynomial follows from the determinant,

$$\begin{aligned} |\mathbb{J} - \lambda \cdot \mathbb{1}| &= (\lambda + r_0)(\lambda + r_1) - r_0 r_1 S \\ 365 \quad &= \lambda^2 + (r_0 + r_1)\lambda - r_0 r_1 \cdot (S - 1) = 0, \end{aligned} \quad (41)$$

where we have defined the feedback strength $S = \phi_{1 \leftarrow 0} \cdot \phi_{0 \leftarrow 1}$. The two Lyapunov exponents of the system are thus

$$\lambda_{\pm} = -\frac{r_0 + r_1}{2} \pm \sqrt{\left(\frac{r_0 + r_1}{2}\right)^2 + r_0 r_1 \cdot (S - 1)} \quad (42)$$

$$= -\frac{r_0 + r_1}{2} \pm \sqrt{r_0 r_1 (S - S_W)}, \quad (43)$$

with the ‘wobble’ threshold defined as

$$370 \quad S_W \equiv -\frac{(r_0 - r_1)^2}{4r_0 r_1} \leq 0, \quad (44)$$



which indicates the feedback strength at which the eigenvalues become complex. For $S < S_W \leq 0$ both Lyapunov exponents are complex conjugates with the same negative real part, $\Re(\lambda_{\pm}) = -(r_0 + r_1)/2$. Note that this oscillatory solution in a two-process loop does *not* make the system unstable: it is oscillatory, but stable, i.e. a deviation from the steady state declines exponentially while the system oscillates around it with decreasing amplitude – exactly the behaviour anticipated for $N = 2$ in Sect. 3.3, since the phase condition of Eq. (24) has no solution for only two processes. For more processes these Hopf bifurcations can become genuine instabilities, as discussed above. For $S > S_W$ both Lyapunov exponents are real and $\lambda_- < 0$; a stability change occurs only for $S > 1$, where $\lambda_+ > 0$ and the system becomes unstable.

In order to derive a general rule for the relevant Lyapunov exponent near the threshold, consider S close to 1 and define $\delta S \equiv S - 1$. Expanding λ_+ in a Taylor series around $S = 1$,

$$\begin{aligned} \lambda_+ &= -\frac{r_0 + r_1}{2} + \sqrt{\left(\frac{r_0 + r_1}{2}\right)^2 + r_0 r_1 \cdot \delta S} \\ &\approx \frac{r_0 r_1}{r_0 + r_1} \cdot \delta S = \frac{\delta S}{\tau_0 + \tau_1}, \end{aligned} \quad (45)$$

which yields the approximation

$$\lambda_+ \approx \frac{S - 1}{\tau_0 + \tau_1}. \quad (46)$$

This is naturally the same result obtained from the path we took for general N above, since $S = (1 + \lambda\tau_0)(1 + \lambda\tau_1) \approx 1 + (\tau_0 + \tau_1)\lambda$ near $S = 1$.

3.6 Contraction of a multiple-process loop into a two-process loop

Generally, single feedback loops consist of more than two variables. For example, the extent of Arctic sea ice, A_I , is influenced by the temperature in the Arctic, T_A . The reflectivity, i.e. albedo α , of the Arctic surface is influenced by the sea-ice extent. The Arctic temperature is in turn again influenced by the Arctic albedo. This three-process loop (or three-variable loop) can be collapsed into a two-process loop by simply stating that the Arctic sea-ice extent influences the Arctic temperature, and vice versa (Fig. 3).

Conceptually, if we eliminate one variable and thereby one intermediate process, the partial derivatives of the formerly two processes multiply to give the strength of the final, contracted process. This follows from the chain rule of differentiation. From $A_i^* = f_i(A_{i-1}^*)$ and $A_{i-1}^* = f_{i-1}(A_{i-2}^*)$ we define the new composite process, $\tilde{f}$, via $A_i^* = f_i(f_{i-1}(A_{i-2}^*)) \equiv \tilde{f}_i(A_{i-2}^*)$, so that

$$\frac{\partial \tilde{f}_i}{\partial A_{i-2}^*} = \frac{\partial f_i}{\partial A_{i-1}^*} \cdot \frac{\partial f_{i-1}}{\partial A_{i-2}^*}. \quad (47)$$

The contraction of one variable, and thereby one process, is thus mathematically and conceptually consistent with the definition of the loop's feedback strength,

$$S \equiv \prod_{i=1}^N \left(\frac{\partial f_i}{\partial A_{i-1}^*} \right). \quad (48)$$



400 The timescale has to be additive, because one process occurs after the other,

$$\tau \equiv \sum_{i=1}^N \tau_i. \quad (49)$$

Thus, with respect to the overall stability regarding the transition across the $S = 1$ boundary, each single feedback loop with

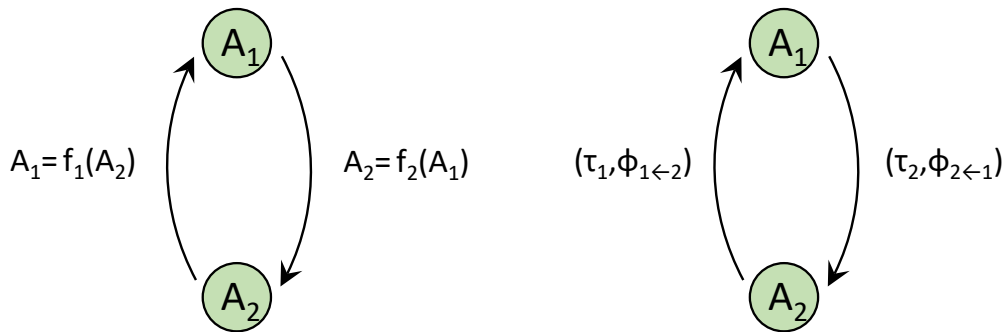


Figure 3. A single loop with 2 variables (daisy petal) for the fundamental stability analysis: single loops with multiple processes, as illustrated in Fig. 1, can be collapsed into loops with two variables. The full information about the relations between the different variables (left-hand labels, f_i) can be reduced to the corresponding timescale τ and the partial derivative of the functions $\phi_{i \leftarrow j} \equiv \partial f_i / \partial A_j$, denoted the dependency strength (right-hand labels). The feedback strength is then computed as the product of the dependency strengths along the loop, i.e. $S \equiv \prod_{i=1}^N \phi_{i \leftarrow i-1} = \phi_{2 \leftarrow 1} \cdot \phi_{1 \leftarrow 2}$.

multiple processes and variables can be contracted into a two-process (and thereby two-variable) loop. This can be used to dramatically simplify complex feedback loops. Note that, while the main instability, i.e. the saddle-node or tipping instability
 405 at $S = 1$, is preserved during the contraction process, one loses the ‘wobble instabilities’, i.e. the Hopf bifurcations, since a two-process feedback loop does not allow for these instabilities: both Lyapunov exponents are given by Eq. (42), and the Hopf-bifurcation criteria of Eqs. (24) and (25) are never fulfilled for $N = 2$. We conclude that any single feedback loop can be contracted into a two-variable (and thereby two-process) loop with strength S and timescale τ given by

$$S \equiv \prod_{i=1}^N \phi_{i \leftarrow i-1} = \prod_{i=1}^N \frac{\partial f_i}{\partial A_{i-1}^*} \quad (50)$$

$$410 \quad \tau \equiv \sum_{i=1}^N \tau_i. \quad (51)$$

4 N-loops system with a common centre (daisy topology)

Here we derive the solution for N loops with a common centre, which looks like a daisy. For illustrative purposes we first compute a two-loop system. Since we concluded from the previous section that single loops with multiple variables can be collapsed into a single loop with two variables when it comes to the quantification of the tipping instability (one eigenvalue



415 becoming positive), we restrict ourselves to the combination of two-variable loops. In words, before the algebra: a daisy is what happens when several independent feedback petals all push on, and are pushed by, the very same shared quantity – Sect. 7.4 develops this for Earth’s global mean temperature, with the ice-albedo, ice-sheet-elevation and water-vapour feedbacks as three such petals.

4.1 Two-loop system with one common central variable

420 Consider a system, as depicted in Fig. 4 for the general N -petal case, in which two loops with two variables each are connected at one central variable C ,

$$\begin{aligned}\frac{dC}{dt} &= \frac{f_C(A_0, A_1) - C}{\tau_C} \equiv r_C \cdot (f_C(A_0, A_1) - C) \\ \frac{dA_1}{dt} &= \frac{f_1(A_0) - A_1}{\tau_1} \equiv r_1 \cdot (f_1(A_0) - A_1) \\ \frac{dA_0}{dt} &= \frac{f_0(A_1) - A_0}{\tau_0} \equiv r_0 \cdot (f_0(A_1) - A_0).\end{aligned}\quad (52)$$

425 This set-up yields the Jacobian

$$\mathbb{J} = \begin{pmatrix} -r_C & r_C \phi_{C \leftarrow 0} & r_C \phi_{C \leftarrow 1} \\ r_0 \phi_{0 \leftarrow C} & -r_0 & 0 \\ r_1 \phi_{1 \leftarrow C} & 0 & -r_1 \end{pmatrix}, \quad (53)$$

where we defined the rates r_i as above, as the inverse of the timescales τ_i , and the dependency strengths similarly to above,

$$\begin{aligned}\phi_{C \leftarrow 0} &\equiv \frac{\partial f_C}{\partial A_0}, & \phi_{C \leftarrow 1} &\equiv \frac{\partial f_C}{\partial A_1}, \\ \phi_{0 \leftarrow C} &\equiv \frac{\partial f_0}{\partial C}, & \phi_{1 \leftarrow C} &\equiv \frac{\partial f_1}{\partial C}.\end{aligned}$$

430 We can now define the feedback strengths of the two loops (petals) as

$$\begin{aligned}S_0 &\equiv \phi_{C \leftarrow 0} \cdot \phi_{0 \leftarrow C} = \frac{\partial f_C}{\partial A_0} \cdot \frac{\partial f_0}{\partial C}, \\ S_1 &\equiv \phi_{C \leftarrow 1} \cdot \phi_{1 \leftarrow C} = \frac{\partial f_C}{\partial A_1} \cdot \frac{\partial f_1}{\partial C},\end{aligned}$$

and compute the characteristic polynomial from $|\mathbb{J} - \lambda \cdot \mathbb{1}| = 0$ to

$$0 = -(r_C + \lambda)(r_0 + \lambda)(r_1 + \lambda) + r_C r_0 S_0 (r_1 + \lambda) + r_C r_1 S_1 (r_0 + \lambda). \quad (54)$$

435 In order to use the Routh–Hurwitz theorem (Routh, 1877; Hurwitz, 1895), we rewrite this as

$$\lambda^3 + (r_C + r_0 + r_1) \cdot \lambda^2 + (r_C r_0 (1 - S_0) + r_C r_1 (1 - S_1) + r_0 r_1) \cdot \lambda + r_C r_0 r_1 (1 - S_0 - S_1) = 0. \quad (55)$$



Since $r_C + r_0 + r_1 > 0$, the Routh–Hurwitz conditions (equivalent to the system’s stability, Gantmacher, 1959; Ogata, 2010) reduce to

$$S_0 + S_1 < 1 \quad (\text{RH-1})$$

$$440 \quad r_0 S_0 + r_1 S_1 < r_0 + r_1 + \frac{r_0 r_1}{r_C} \quad (\text{RH-2})$$

$$\frac{r_0}{r_C + r_1} S_0 + \frac{r_1}{r_C + r_0} S_1 < \frac{r_0 + r_1}{r_C}. \quad (\text{RH-3})$$

The first condition (RH-1) is the most relevant one: it describes the saddle-node bifurcation that is generally associated with the tipping point. The last condition (RH-3) describes the boundary at which the system undergoes a Hopf bifurcation, i.e. the emergence of two eigenvalues $\lambda = \pm i\omega_H$ with

$$445 \quad \omega_H^2 = r_C r_0 (1 - S_0) + r_C r_1 (1 - S_1) + r_0 r_1, \quad (56)$$

provided the right-hand side is positive, which is equivalent to condition (RH-2) being fulfilled.

To illustrate the limited practical relevance of the last two conditions, consider the common case in which both petals share a similar timescale, $\tau_0 = \tau_1$. The three conditions then collapse to

$$S_0 + S_1 < 1 \quad (\text{RH-1})$$

$$450 \quad S_0 + S_1 < 2 + \frac{r_0}{r_C} \quad (\text{RH-2*})$$

$$S_0 + S_1 < 2 + 2 \frac{r_0}{r_C}, \quad (\text{RH-3*})$$

of which (RH-2*) and (RH-3*) are automatically satisfied whenever (RH-1) is. The conditions (RH-2) and (RH-3) only become relevant if the two feedback petals are strongly asymmetric in time – i.e. if one petal’s timescale τ_0 or τ_1 differs from the other, and from τ_C , by a large factor.

455 To derive a simpler rule for the Lyapunov exponent near the saddle-node bifurcation, we reconsider the characteristic polynomial, Eq. (54), and multiply it by the product of the timescales $\tau_C \tau_0 \tau_1$,

$$0 = (1 + \tau_C \lambda) (1 + \tau_0 \lambda) (1 + \tau_1 \lambda) - S_0 (1 + \tau_1 \lambda) - S_1 (1 + \tau_0 \lambda). \quad (57)$$

Near the saddle-node bifurcation, i.e. near $\lambda \approx 0$, we approximate

$$1 + (\tau_C + \tau_0 + \tau_1) \lambda \approx S_0 (1 + \tau_1 \lambda) + S_1 (1 + \tau_0 \lambda), \quad (58)$$

460 which yields

$$\lambda \approx \frac{S_0 + S_1 - 1}{\tau_C + \tau_0 (1 - S_1) + \tau_1 (1 - S_0)} = \frac{S_0 + S_1 - 1}{\tau_C + \tau_0 + \tau_1 - (S_0 \tau_1 + S_1 \tau_0)}. \quad (59)$$

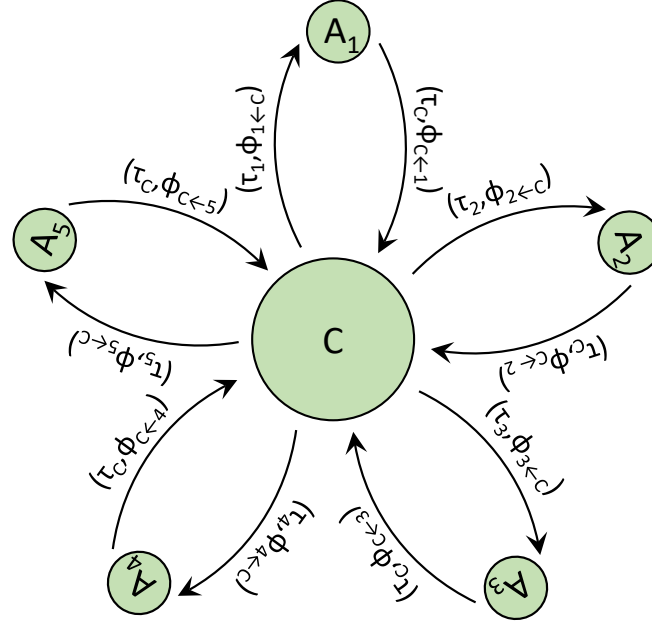


Figure 4. N loops with a common centre, illustrated here with $N = 5$: they form a daisy topology.

4.2 N-loop system with one common central variable

Consider a system in which N feedback loops are connected through one common central variable C . Each loop consists of
 465 the central variable C and one loop variable A_i , $i = 0, \dots, N - 1$. We write

$$\frac{dC}{dt} = \frac{f_C(A_0, \dots, A_{N-1}) - C}{\tau_C} \equiv r_C (f_C(A_0, \dots, A_{N-1}) - C) \quad (60)$$

$$\frac{dA_i}{dt} = \frac{f_i(C) - A_i}{\tau_i} \equiv r_i (f_i(C) - A_i), \quad i = 0, \dots, N - 1, \quad (61)$$

with $r_C \equiv \tau_C^{-1}$ and $r_i \equiv \tau_i^{-1}$ as before. The Jacobian is

$$\mathbb{J} = \begin{pmatrix} -r_C & r_C \phi_{C \leftarrow 0} & r_C \phi_{C \leftarrow 1} & \cdots & r_C \phi_{C \leftarrow N-1} \\ r_0 \phi_{0 \leftarrow C} & -r_0 & 0 & \cdots & 0 \\ r_1 \phi_{1 \leftarrow C} & 0 & -r_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{N-1} \phi_{N-1 \leftarrow C} & 0 & 0 & \cdots & -r_{N-1} \end{pmatrix}, \quad (62)$$

470 where $\phi_{C \leftarrow i} \equiv \partial f_C / \partial A_i$ and $\phi_{i \leftarrow C} \equiv \partial f_i / \partial C$, as before. We define the feedback strength of petal i as

$$S_i \equiv \phi_{C \leftarrow i} \phi_{i \leftarrow C} = \frac{\partial f_C}{\partial A_i} \frac{\partial f_i}{\partial C}. \quad (63)$$



The characteristic equation can be written compactly as

$$(1 + \tau_C \lambda) \prod_{j=0}^{N-1} (1 + \tau_j \lambda) - \sum_{i=0}^{N-1} S_i \prod_{\substack{j=0 \\ j \neq i}}^{N-1} (1 + \tau_j \lambda) = 0. \quad (64)$$

Near the saddle-node bifurcation, i.e. for $\lambda \approx 0$, we keep only terms linear in λ ,

$$1 + \left(\tau_C + \sum_{j=0}^{N-1} \tau_j \right) \lambda \approx \sum_{i=0}^{N-1} S_i \left(1 + \sum_{\substack{j=0 \\ j \neq i}}^{N-1} \tau_j \lambda \right), \quad (65)$$

so that

$$\lambda \approx \frac{\sum_{i=0}^{N-1} S_i - 1}{\tau_C + \sum_{j=0}^{N-1} \tau_j - \sum_{i=0}^{N-1} S_i \sum_{\substack{j=0 \\ j \neq i}}^{N-1} \tau_j}. \quad (66)$$

This can also be written as

$$\lambda \approx \frac{S - 1}{\tau} \quad (67)$$

with the *total feedback strength of the daisy* again simply given by the sum of the individual petal strengths,

$$S \equiv \sum_{i=0}^{N-1} S_i, \quad (68)$$

and a somewhat more involved effective timescale,

$$\tau \equiv \tau_C + \sum_{j=0}^{N-1} \tau_j (1 - S + S_j). \quad (69)$$

This additive rule for S – read off Fig. 4 simply by summing the strength of every petal – is the central practical result of this paper: the stability (at least with respect to the commonly relevant saddle-node instability) of a system dominated by one central variable is given by the feedback strength S alone, exactly as for the single loop of Sect. 3, except that S is now a *sum* over petals rather than a *product* over processes within one loop. The effective timescale, Eq. (69), is not needed to locate the tipping point, only to estimate the critical slowing down near it, or the recovery time after a sub-critical perturbation. If all petals are positive, the daisy cannot become oscillatory unstable, and $S < 1$ is then both necessary and sufficient for stability, independently of the timescales (part II, Appendix A2).

4.3 Oscillations, eigenfrequencies and wobble instabilities in the two-loop daisy topology

As in the single-loop case, oscillatory modes arise when a complex-conjugate pair of eigenvalues crosses the imaginary axis. For the two-loop daisy topology of Sect. 4.1, setting $\lambda = i\omega$ in Eq. (57) and separating real and imaginary parts gives, for non-trivial oscillations ($\omega \neq 0$),

$$\tau_C + \tau_0 + \tau_1 - \omega^2 \tau_C \tau_0 \tau_1 = S_0 \tau_1 + S_1 \tau_0, \quad (70)$$



so that the oscillation frequency at the stability boundary is

$$\omega_H^2 = \frac{\tau_C + \tau_0 + \tau_1 - (S_0\tau_1 + S_1\tau_0)}{\tau_C\tau_0\tau_1} = r_C r_0(1 - S_0) + r_C r_1(1 - S_1) + r_0 r_1, \quad (71)$$

identical to the Hopf frequency already obtained from the Routh–Hurwitz analysis of Sect. 4.1, as it must be. Crossing the corresponding boundary (RH-3) moves a conjugate eigenvalue pair through the imaginary axis, and the system undergoes a
500 Hopf bifurcation. Unlike the saddle-node instability at $S_0 + S_1 = 1$, which depends only on the total feedback strength, this wobble instability depends explicitly on the relative timescales of the two petals and the centre.

In the symmetric case $\tau_0 = \tau_1$ the Hopf condition reduces to $S_0 + S_1 = 2(1 + \tau_C/\tau_0) \geq 2$, far beyond the saddle-node threshold: for petals of comparable timescale the system always tips before it can wobble. Hopf bifurcations become relevant only for sufficiently asymmetric timescales. If, for example, the central variable responds much faster than the two petals,
505 $\tau_C \ll \tau_0, \tau_1$ (so $r_C \gg r_0, r_1$), condition (RH-3) reduces to

$$r_0 S_0 + r_1 S_1 = r_0 + r_1 \iff S_0 = 1 + \frac{r_1}{r_0} (1 - S_1), \quad (72)$$

which can bind at a combined strength $S_0 + S_1$ well below the saddle-node value of 1 whenever the faster petal is destabilizing, $S_1 > 1$, the slower petal is strongly stabilizing, $S_0 \ll 0$, and r_1/r_0 is large – i.e. a daisy with a fast centre and strongly asymmetric petals can wobble *before* it tips (cf. Appendix D1).

510 4.4 Centre manifold: daisy topology without self-restoring centre dynamics

We now consider a special but important case in which the central variable of the daisy topology does *not* exhibit self-restoring dynamics – precisely the exception to Eq. (2) flagged in Sect. 2.1. Instead of Eq. (60) we consider

$$\frac{dC}{dt} = F_C(A_0, \dots, A_{N-1}), \quad (73)$$

so that $\partial F_C / \partial C = 0$, while the petals remain as before,

$$515 \quad \frac{dA_i}{dt} = \frac{f_i(C) - A_i}{\tau_i}. \quad (74)$$

Such a situation can occur, for example, when the variables A_i represent fluxes into C , such that $dC/dt = \sum_{i=0}^{N-1} A_i$. In many cases this can be converted back into the standard format used throughout this paper, $\partial F_C / \partial C < 0$, simply by recognizing that at least one of the fluxes is itself stabilizing and absorbing it into the time dependence of C ; we use exactly this device in the ice-sheet example of Sect. 7.3. Where this is not obvious, however, it is convenient to treat the centre-manifold case directly.

520 To preserve the dimensional structure of the feedback strengths, we introduce an arbitrary normalization timescale τ_C and define

$$\phi_{C \leftarrow i} \equiv \tau_C \frac{\partial F_C}{\partial A_i}. \quad (75)$$



The petal sensitivities $\phi_{i \leftarrow C}$ remain unchanged, so the loop strengths are again $S_i = \phi_{C \leftarrow i} \phi_{i \leftarrow C}$. Linearizing about equilibrium gives the Jacobian

$$525 \quad \mathbb{J} = \begin{pmatrix} 0 & \tau_C^{-1} \phi_{C \leftarrow 0} & \cdots & \tau_C^{-1} \phi_{C \leftarrow N-1} \\ \tau_0^{-1} \phi_{0 \leftarrow C} & -\tau_0^{-1} & & 0 \\ \vdots & & \ddots & \\ \tau_{N-1}^{-1} \phi_{N-1 \leftarrow C} & 0 & & -\tau_{N-1}^{-1} \end{pmatrix}, \quad (76)$$

with a zero in the upper-left corner in place of $-r_C$. The characteristic equation becomes

$$\tau_C \lambda \cdot \prod_{j=0}^{N-1} (1 + \tau_j \lambda) - \sum_{i=0}^{N-1} S_i \cdot \prod_{j \neq i} (1 + \tau_j \lambda) = 0, \quad (77)$$

which differs from the ordinary daisy's characteristic equation, Eq. (64), only by the replacement $1 + \tau_C \lambda \rightarrow \tau_C \lambda$, reflecting the absence of an intrinsic restoring term for the centre. Expanding near $\lambda \approx 0$ and keeping only linear terms,

$$530 \quad \tau_C \lambda \approx \sum_i S_i + \lambda \sum_i S_i \sum_{j \neq i} \tau_j, \quad (78)$$

so that

$$\lambda \approx \frac{\sum_i S_i}{\tau_C - \sum_i S_i \sum_{j \neq i} \tau_j}. \quad (79)$$

This appears to depend on the arbitrarily chosen normalization τ_C , but it does not: every S_i is, by Eq. (75), linear in τ_C , so defining

$$535 \quad s_i \equiv \frac{S_i}{\tau_C} = \frac{\partial F_C}{\partial A_i} \frac{\partial f_i}{\partial C} \quad (80)$$

removes the dependence on τ_C (at the price that s_i now carries units of inverse time rather than being dimensionless). With this,

$$\lambda \approx \frac{\sum_i s_i}{1 - \sum_i s_i \sum_{j \neq i} \tau_j}. \quad (81)$$

The stability conditions are now

$$540 \quad \sum_i s_i < 0 \quad \text{and} \quad \sum_i s_i \sum_{j \neq i} \tau_j < 1, \quad (82)$$

in place of $\sum_i S_i < 1$ for the standard daisy – reflecting the absence of a self-stabilizing centre. The second condition is automatically satisfied whenever the petal timescales are not too disparate: for uniformly equal $\tau_j = \tau_{\text{const}}$ it reads $\sum_i s_i < ((N-1)\tau_{\text{const}})^{-1}$, which is always implied by the first condition.



4.5 Gain factors: connection to equilibrium response to finite forcing

545 In order to reconcile our notation with traditional equilibrium feedback theory, consider an external perturbation applied directly to the central variable, ΔC_0 . The linearized system response of Eq. (60) is then

$$\Delta C = \Delta C_0 + \sum_{i=1}^N \left. \frac{\partial f_C}{\partial A_i} \right|_0 \Delta A_i. \quad (83)$$

The perturbation in each petal variable A_i must itself be evaluated using the final (equilibrium) perturbation of C , in order to capture its full equilibrium response,

550 $\Delta A_i = \left. \frac{df_i}{dC} \right|_0 \Delta C,$ (84)

which, substituted back, gives

$$\Delta C = \frac{\Delta C_0}{1 - \sum_{i=1}^N g_i} = \frac{\Delta C_0}{1 - \sum_{i=1}^N S_i}, \quad (85)$$

where the commonly used *gain factors*,

$$g_i \equiv \left(\frac{\partial f_C}{\partial A_i} \right) \left(\frac{df_i}{dC} \right) = S_i, \quad (86)$$

555 are simply equal to the respective feedback strengths of the individual petals. This is the same amplification familiar from classical climate-feedback analysis, where the equilibrium response to a forcing is amplified by a factor $1/(1 - \sum_i g_i)$; we return to this connection quantitatively for the water-vapour feedback in Sect. 7.4.

5 Combining two daisy topologies by replacing one petal by another daisy centre

560 We now consider a hierarchical extension of the daisy topology, in which one petal of a daisy is itself replaced by the centre of a second daisy. This structure represents a ‘daisy-of-daisies’ topology and provides a simple example of nested feedback amplification.

Consider a first daisy with central variable C_1 and N_1 petals. One of its petals is replaced by the centre C_2 of a second daisy with N_2 petals; the resulting structure is illustrated schematically in Fig. 5. The topology is no longer a simple daisy, because one branch attached to C_1 now itself contains an internal feedback structure.



565 5.1 KL2 formulation

Following the KL2 formalism introduced above, the dynamics are approximated near equilibrium by

$$\frac{dC_1}{dt} = \frac{f_{C_1}(C_2, A_1^{(1)}, \dots, A_{N_1-1}^{(1)}) - C_1}{\tau_{C_1}}, \quad (87)$$

$$\frac{dA_i^{(1)}}{dt} = \frac{f_i^{(1)}(C_1) - A_i^{(1)}}{\tau_i^{(1)}}, \quad i = 1, \dots, N_1 - 1, \quad (88)$$

$$\frac{dC_2}{dt} = \frac{f_{C_2}(C_1, A_1^{(2)}, \dots, A_{N_2}^{(2)}) - C_2}{\tau_{C_2}}, \quad (89)$$

$$570 \quad \frac{dA_j^{(2)}}{dt} = \frac{f_j^{(2)}(C_2) - A_j^{(2)}}{\tau_j^{(2)}}, \quad j = 1, \dots, N_2. \quad (90)$$

We define the feedback strengths of the ordinary petals attached to C_1 as $S_i^{(1)} = \phi_{C_1 \leftarrow A_i^{(1)}} \phi_{A_i^{(1)} \leftarrow C_1}$, $i = 1, \dots, N_1 - 1$, with total contribution $S^{(1)} = \sum_{i=1}^{N_1-1} S_i^{(1)}$. Similarly, the petals of the second daisy define $S_j^{(2)} = \phi_{C_2 \leftarrow A_j^{(2)}} \phi_{A_j^{(2)} \leftarrow C_2}$, $j = 1, \dots, N_2$, with total $S^{(2)} = \sum_{j=1}^{N_2} S_j^{(2)}$. Finally, the direct coupling between the two centres defines the inter-centre loop strength

$$S^{(1,2)} = \phi_{C_1 \leftarrow C_2} \phi_{C_2 \leftarrow C_1}. \quad (91)$$

575 5.2 Reduced characteristic equation

Near the saddle-node bifurcation we eliminate the leaf variables exactly as in the contraction procedure of Sect. 3.6. The resulting reduced linearized system for the two central variables is

$$\left[(1 - S^{(1)}) + \lambda \tau^{(1)} \right] \delta C_1 = \phi_{C_1 \leftarrow C_2} \delta C_2, \quad (92)$$

$$\left[(1 - S^{(2)}) + \lambda \tau^{(2)} \right] \delta C_2 = \phi_{C_2 \leftarrow C_1} \delta C_1, \quad (93)$$

580 where, to leading order in the slow eigenvalue λ , the petal contractions generate the effective coefficients

$$\tau^{(1)} = \tau_{C_1} + \sum_{i=1}^{N_1-1} S_i^{(1)} \tau_i^{(1)}, \quad \tau^{(2)} = \tau_{C_2} + \sum_{j=1}^{N_2} S_j^{(2)} \tau_j^{(2)}. \quad (94)$$

The characteristic equation follows from the determinant condition

$$\left[(1 - S^{(1)}) + \lambda \tau^{(1)} \right] \left[(1 - S^{(2)}) + \lambda \tau^{(2)} \right] - S^{(1,2)} = 0. \quad (95)$$

5.3 Effective feedback strength and instability condition

585 Expanding near the instability threshold, and neglecting terms of order $\mathcal{O}(\lambda^2)$, yields

$$(1 - S^{(1)})(1 - S^{(2)}) + \lambda \left[\tau^{(1)}(1 - S^{(2)}) + \tau^{(2)}(1 - S^{(1)}) \right] - S^{(1,2)} = 0, \quad (96)$$



so that the dominant Lyapunov exponent is approximately

$$\lambda \approx \frac{S^{(1,2)} - (1 - S^{(1)})(1 - S^{(2)})}{\tau^{(1)}(1 - S^{(2)}) + \tau^{(2)}(1 - S^{(1)})} = \frac{S^{(1)} + S^{(2)} + (S^{(1,2)} - S^{(1)}S^{(2)}) - 1}{\tau^{(1)}(1 - S^{(2)}) + \tau^{(2)}(1 - S^{(1)})}. \quad (97)$$

This suggests defining a total effective feedback strength,

$$S_{\text{eff}} = S^{(1)} + S^{(2)} + (S^{(1,2)} - S^{(1)}S^{(2)}), \quad (98)$$

such that the saddle-node instability again occurs at $S_{\text{eff}} = 1$. The threshold itself is exact: evaluating the characteristic equation at $\lambda = 0$ involves no approximation (part II, Sect. 3.2), and only the effective timescale, Eq. (99), is a leading-order result.

This format lets us interpret the combination of the two daisies directly: to obtain the effective feedback strength, add the strengths of the two daisies' own petals, $S^{(1)} + S^{(2)}$, and then add the direct centre-to-centre coupling $S^{(1,2)}$ reduced by the product $S^{(1)}S^{(2)}$ of the two daisies' own strengths. This correction term arises because each daisy's own petals modify how a perturbation propagates through to the other centre, so the combined system behaves like a renormalized two-centre loop, each centre already internally amplified by its own petals before interacting with the other. The effective timescale of the combined system follows an analogous rule: each daisy's own effective timescale is reduced by the *other* daisy's feedback strength and the two are summed,

$$\tau_{\text{eff}} = \tau^{(1)}(1 - S^{(2)}) + \tau^{(2)}(1 - S^{(1)}). \quad (99)$$

This timescale is evaluated at the threshold; away from it, the exact linear coefficient of the characteristic equation contains additional terms that vanish at $S_{\text{eff}} = 1$ (part II, Sect. 3.3).

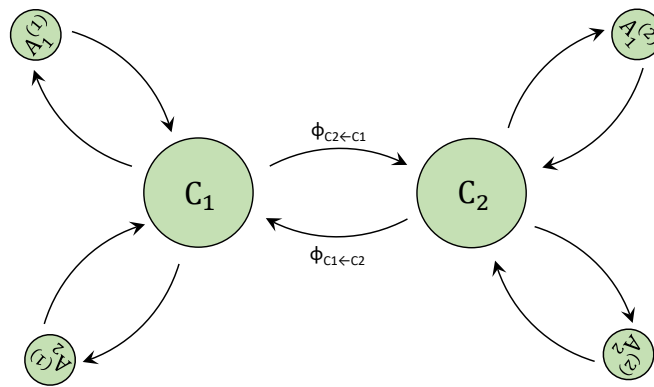


Figure 5. Two coupled daisy topologies (a ‘daisy of daisies’): the centre of one daisy, C_1 , is itself coupled to the centre of a second daisy, C_2 , while each centre retains its own ordinary petals $A_i^{(1)}$, $A_j^{(2)}$. Section 6’s bouquet rule gives the combined feedback strength of such a structure.



6 Modular rules for feedback strength and effective timescale

The preceding sections suggest a modular rule system for computing the dominant Lyapunov exponent near the saddle-node
 605 instability. In every case considered, the relevant Lyapunov exponent can be written as

$$\lambda \approx \frac{S_{\text{eff}} - 1}{\tau_{\text{eff}}}, \quad (100)$$

where S_{eff} is an effective total feedback strength and τ_{eff} the corresponding effective timescale, with the instability threshold always at $S_{\text{eff}} = 1$. The feedback strength locates the stability boundary; the effective timescale sets the rate of recovery or divergence near it.

610 6.1 Pile rule

A pilea topology is one closed feedback loop with variables $A_1, \dots, A_N$ and dependency strengths $\phi_i \equiv \phi_{i \leftarrow i-1} = \partial f_i / \partial A_{i-1}$ (cyclic indexing). Processes arranged in series around the loop *multiply* their dependency strengths and *add* their timescales (Sects. 3.2 and 3.6),

$$S_{\text{pilea}} = \prod_{i=1}^N \phi_i, \quad \tau_{\text{pilea}} = \sum_{i=1}^N \tau_i, \quad \lambda_{\text{pilea}} \approx \frac{S_{\text{pilea}} - 1}{\tau_{\text{pilea}}}. \quad (101)$$

615 6.2 Daisy rule

A daisy topology consists of N independent petals attached to one common central variable C , each petal A_i forming a two-variable loop with C of strength $S_i = \phi_{C \leftarrow i} \phi_{i \leftarrow C}$. Independent petals around a common centre simply *add* their feedback strengths (Sect. 4.2),

$$S_{\text{daisy}} = \sum_{i=1}^N S_i, \quad \tau_{\text{daisy}} = \tau_C + \sum_{i=1}^N \tau_i (1 - S_{\text{daisy}} + S_i), \quad \lambda_{\text{daisy}} \approx \frac{S_{\text{daisy}} - 1}{\tau_{\text{daisy}}}. \quad (102)$$

620 6.3 Bouquet rule for coupled daisy topologies

Finally, consider a bouquet topology in which two daisies are coupled through a direct interaction between their centres (Fig. 5): the first daisy has centre C_1 and ordinary petals $A_i^{(1)}$, $i = 1, \dots, N_1 - 1$, of total strength $S^{(1)} = \sum_i S_i^{(1)}$; the second has centre C_2 and petals $A_j^{(2)}$, $j = 1, \dots, N_2$, of total strength $S^{(2)} = \sum_j S_j^{(2)}$; and the direct two-centre loop has strength $S^{(1,2)} = \phi_{C_1 \leftarrow C_2} \phi_{C_2 \leftarrow C_1}$ (Sect. 5). The effective total feedback strength of the combined, bouquet system is

$$625 \quad S_{\text{bouquet}} = S^{(1,2)} + S^{(1)} + S^{(2)} - S^{(1)} S^{(2)}, \quad (103)$$

equivalently written as the instability condition $S^{(1,2)} > (1 - S^{(1)})(1 - S^{(2)})$, i.e. the stability margins of the two daisies, $1 - S^{(1)}$ and $1 - S^{(2)}$, combine *multiplicatively* rather than additively. With the renormalized centre timescales $\tau^{(1)} = \tau_{C_1} + \sum_i S_i^{(1)} \tau_i^{(1)}$ and $\tau^{(2)} = \tau_{C_2} + \sum_j S_j^{(2)} \tau_j^{(2)}$, the effective timescale and Lyapunov exponent are

$$\tau_{\text{bouquet}} = \tau^{(1)}(1 - S^{(2)}) + \tau^{(2)}(1 - S^{(1)}), \quad \lambda_{\text{bouquet}} \approx \frac{S_{\text{bouquet}} - 1}{\tau_{\text{bouquet}}}. \quad (104)$$



630 6.4 Toward a modular rule system

The three rules above can be summarized as three ways of combining feedback structures:

Serial rule. Processes in series around a single loop multiply their dependency strengths and add their timescales: $S_{\text{series}} =$

$$\prod_i \phi_i, \tau_{\text{series}} = \sum_i \tau_i.$$

635 **Parallel-petal rule.** Independent petals attached to the same centre add their feedback strengths, $S_{\text{parallel}} = \sum_i S_i$, with a timescale contribution weighted by the complementary strength of the remaining petals, $\tau_{\text{parallel}} = \tau_C + \sum_i \tau_i (1 - S_{\text{parallel}} + S_i)$.

Bouquet rule. Two daisy modules coupled through a direct centre-to-centre feedback loop combine their stability margins, $1 - S$, *multiplicatively*, corrected by the direct coupling: $1 - S_{\text{bouquet}} = (1 - S^{(1)})(1 - S^{(2)}) - S^{(1,2)}$, equivalently $S_{\text{bouquet}} = S^{(1,2)} + S^{(1)} + S^{(2)} - S^{(1)}S^{(2)}$.

640 This suggests that complex feedback diagrams can be reduced recursively: first, serial chains are contracted into single effective branches (Sect. 3.6); second, parallel petals around a common centre are summed (Sect. 4.2); third, bouquet modules are combined through their remaining stability margins, $1 - S$, together with the direct coupling strength between the participating centres (Sect. 5). Part II of this series shows that these modular rules are special cases of a single loop expansion of the characteristic equation, which applies to any diagram of the form of Eq. (4), and extends them to loops that share a process directly, rather than only a common centre.

645 7 Cookbook with examples for stability analysis and model building

7.1 Feedback analysis

We now collect the preceding sections into an explicit, step-by-step recipe. Figure 6 summarizes the route from a feedback diagram to a stability and tipping-point statement; Sect. 7.5 below adds the further steps needed to turn the same diagram into an explicit temporal model.

- 650 1. **Construct the feedback diagram.** Identify the variables involved and draw an arrow $A_j \rightarrow A_i$ for every direct dependency $\partial f_i / \partial A_j \neq 0$ (Sect. 2). It is enough to know the qualitative structure of the diagram at this stage.
2. **Determine the knowledge level of each link**, KL1, KL2 or KL3 (Sect. 2.1): do we have the full time-dependent equation, only a steady-state relation, or only a qualitative judgement about the sign and rough size of the dependency?
- 655 3. **Collapse the diagram to its minimal form.** Any serial chain of processes not directly linked to another loop is contracted into a single two-variable petal using the product rule for S and the sum rule for τ (Sect. 3.6); any set of petals sharing one central variable is then combined additively (Sect. 4.2), and, if required, further combined using the nested or bouquet rules of Sects. 5–6.



4. **Assign dependency strengths and timescales** to what remains, either by differentiating known equations (KL1/KL2) or by direct estimation (KL3, Sect. 2.1).

660 5. **Steady-state analysis.** For KL1/KL2 systems, solve Eqs. (1) or (3) for the equilibrium A_i^* at which the dependency strengths are to be evaluated; for a purely KL3 diagram this step is skipped; one works directly with the estimated $\phi_{i \leftarrow j}$ and τ_i .

665 6. **Stability analysis.** Read the total feedback strength S and timescale τ off the collapsed diagram (a product around a loop, Eq. (50); a sum over petals, Eq. (68); or the corresponding nested/bouquet rule). The tipping point is at $S = 1$; the dominant Lyapunov exponent near it is $\lambda \approx (S - 1)/\tau$ (stable for $S < 1$, tipping for $S > 1$); and, for loops of three or more processes, check whether a wobble instability (Sect. 3.3) is reached first.

This already answers the two questions motivating this paper: where is the tipping point of the system (at $S = 1$), and how fast would the instability unfold once triggered (at rate $\lambda \approx (S - 1)/\tau$) – all from qualitative, easily obtainable information about the diagram, without ever needing the full governing equations. Sections 7.2–7.4 apply this recipe to three examples from the climate system; Sects. 7.5–7.6 extend it with the additional steps needed to build a full temporal model, illustrated with two examples from economics.

7.2 1st example of a daisy with three variables: the Stommel model of Atlantic overturning

As a first, simple illustration of a daisy with real governing equations, consider Stommel (1961)’s classic box model of the Atlantic Meridional Overturning Circulation (AMOC), in the simplified two-box form widely used for conceptual studies of its stability. Two boxes – polar and equatorial – exchange heat and salt; the relevant state is the pole-minus-equator contrast in temperature, $\Delta T \equiv T_{\text{pole}} - T_{\text{eq}}$, and in salinity, $\Delta S \equiv S_{\text{pole}} - S_{\text{eq}}$, both negative under present-day conditions, since the pole is both colder and fresher than the equator. The density contrast $\rho_{\text{pole}} - \rho_{\text{eq}} \propto (\beta \Delta S - \alpha \Delta T)$ drives an overturning flow of strength

$$\psi = k(\beta \Delta S - \alpha \Delta T), \quad (105)$$

680 with respective thermal and haline expansion coefficients $\alpha, \beta > 0$. We treat ψ as a third dynamical variable that relaxes towards this diagnostic target, with a timescale τ_ψ that corresponds to the internal information transport of the density gradient, while ΔT and ΔS are each driven by an external forcing and eroded by the overturning itself, which mixes the two boxes together,

$$\begin{aligned} \frac{d\psi}{dt} &= \frac{k(\beta \Delta S - \alpha \Delta T) - \psi}{\tau_\psi}, \\ \frac{d\Delta T}{dt} &= \frac{\Delta T_a - \Delta T}{\tau_{T0}} - \frac{\psi \Delta T}{V}, \\ 685 \quad \frac{d\Delta S}{dt} &= -F_0 - \frac{\psi \Delta S}{V} \end{aligned} \quad (106)$$

where ΔT_a is the atmospherically set equilibrium pole-minus-equator temperature contrast (negative, since the pole is radiatively colder than the equator), $F_0 > 0$ is a constant net freshwater forcing (precipitation and runoff freshening high latitudes,

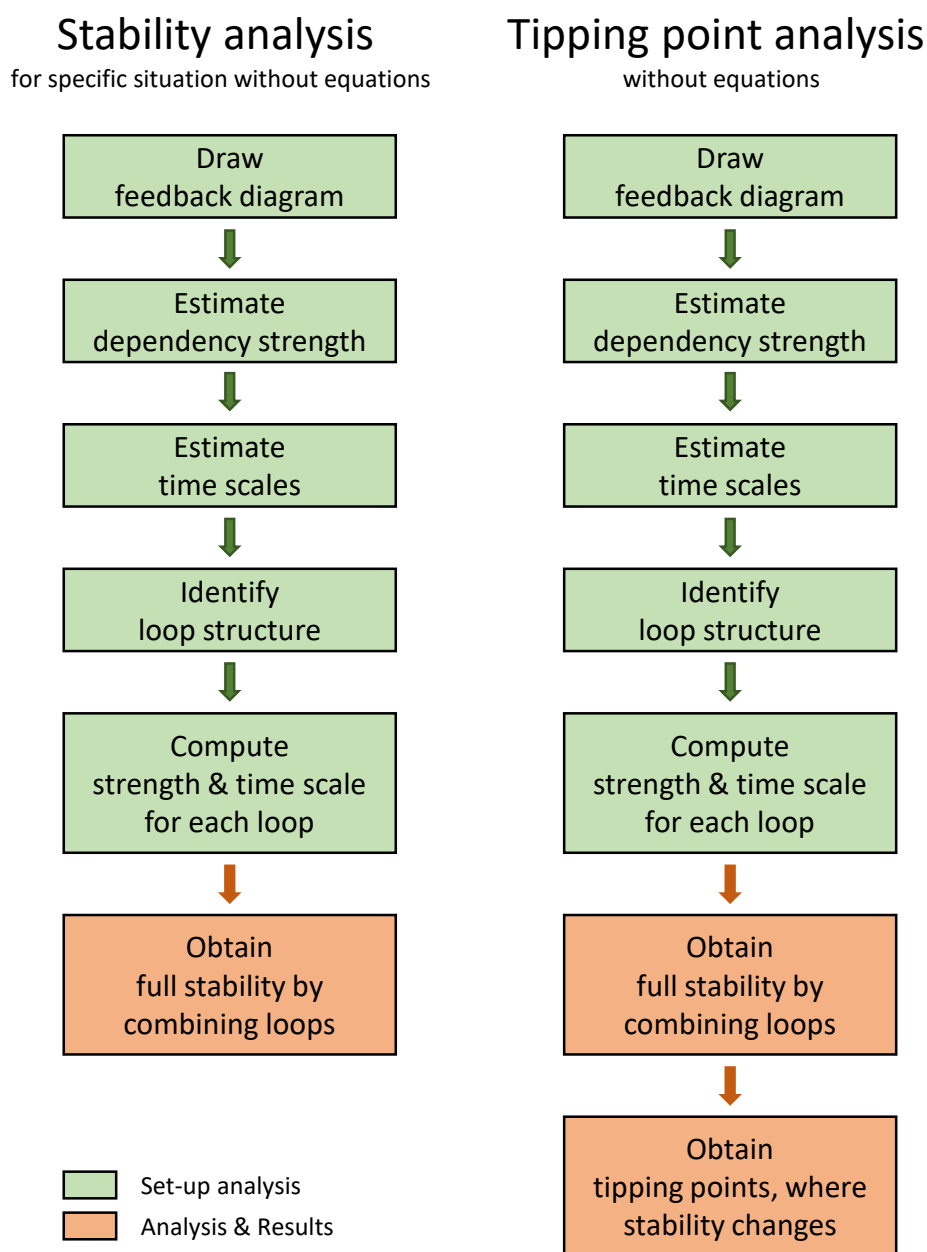


Figure 6. Cookbook for a stability and tipping-point analysis, with or without known governing equations (Sect. 7.1): the same set-up steps (green) feed both a stability analysis (left) and a tipping-point analysis (right), the latter adding one further step to locate where S crosses 1.



evaporation making low latitudes saltier, which drives ΔS further negative), and V is a box volume. Crucially, ΔS has *no* restoring term of its own: the overturning is the only process that removes a salinity anomaly, exactly the feature that makes this system able to tip.

This is a daisy (Sect. 4, Fig. 7) with the overturning strength ψ as the shared centre and ΔT , ΔS as its two petals. Following the general KL1-to-KL2 recipe of Sect. 2.2, the temperature petal has restoring timescale $\tau_T = V\tau_{T0}/(V + \psi\tau_{T0})$ and dependency strengths $\phi_{\psi \leftarrow T} = -k\alpha$, $\phi_{T \leftarrow \psi} = -\tau_T \Delta T/V$, giving a petal strength

$$S_T = \phi_{\psi \leftarrow T} \phi_{T \leftarrow \psi} = \frac{k\alpha\tau_T \Delta T}{V} < 0 : \quad (107)$$

stronger overturning drives the temperature contrast back toward its (negative) equilibrium value faster than it can drift away from it, so this petal is always *stabilizing* – the ordinary, familiar negative feedback of heat transport. The salinity petal, by contrast, has restoring timescale $\tau_S = V/\psi$ (supplied entirely by the flow itself) and dependency strengths $\phi_{\psi \leftarrow S} = k\beta$, $\phi_{S \leftarrow \psi} = -\Delta S/\psi$, giving

$$S_S = \phi_{\psi \leftarrow S} \phi_{S \leftarrow \psi} = -\frac{k\beta \Delta S}{\psi} > 0 : \quad (108)$$

a weaker flow transports less salt poleward, which freshens (and hence further lightens) the polar box – driving ΔS further negative – which weakens the flow still further – the self-reinforcing salt-advection feedback. The total feedback strength is $S = S_T + S_S$, and the AMOC tips once S crosses 1. Equation (108) shows why this is dangerous: at the steady state $\Delta S = -F_0 V/\psi$, so $S_S = k\beta F_0 V/\psi^2$ grows as the flow ψ weakens, for instance under an increasing high-latitude freshwater input F_0 from melting ice sheets – precisely the destabilizing, self-amplifying mechanism that places the AMOC among the Earth system’s recognized tipping elements (Lenton et al., 2008; Wunderling et al., 2021).

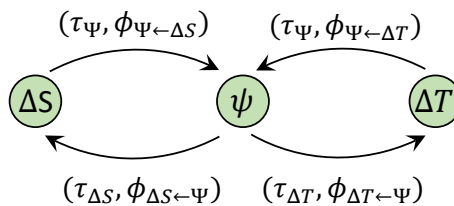


Figure 7. The Stommel ocean-circulation model as a daisy: the overturning strength ψ is the shared centre; the temperature contrast ΔT is a stabilizing petal ($S_T < 0$), and the salinity contrast ΔS is a destabilizing one ($S_S > 0$, the salt-advection feedback), with the AMOC tipping once $S_T + S_S = 1$.

It is instructive to check the daisy decomposition above against the classical closed-form analysis of the same two-box system (Stommel, 1961), which is normally solved not by linearizing around a single equilibrium but by eliminating ΔT and ΔS in favour of ψ and solving the resulting polynomial directly.

Setting the time derivatives in Eqs. (106) to zero gives

$$\Delta T^* = \frac{\Delta T_a V}{V + \psi^* \tau_{T0}}, \quad \Delta S^* = -\frac{F_0 V}{\psi^*}, \quad (109)$$



which, substituted into Eq. (105), yields a single polynomial equation for the equilibrium overturning strength ψ^* ,

$$\tau_{T0}(\psi^*)^3 + V(\psi^*)^2 + (k\beta F_0 V \tau_{T0} + k\alpha \Delta T_a V) \psi^* + k\beta F_0 V^2 = 0. \quad (110)$$

This is *cubic* in ψ^* , so it admits up to three real roots – the same multiplicity as the classical model’s bistable structure, with a strong (thermally dominated) circulation, an unstable saddle, and a reversed (haline-dominated) circulation (Stommel, 1961).

715 The nonlinearity responsible is not, as in the usual two-box formulation, the use of $|\psi|$ in the box-exchange terms; here it arises instead from ψ^* appearing in the denominator of ΔT^* and ΔS^* , since ψ is simultaneously the shared centre variable and the rate that erodes both petals. The daisy formalism therefore reproduces the classical multiplicity of equilibria, even though every individual link in Fig. 7 is itself a simple linear relaxation.

The daisy stability condition, $S = S_T + S_S = 1$ (Sect. 4), is by construction the statement that the Jacobian of the full
 720 three-variable system acquires a zero eigenvalue – exactly the condition used classically to locate the fold at which the strong and the saddle branches merge. The two must therefore agree, and this can be checked explicitly in the limit in which the temperature contrast relaxes fast compared to the overturning, $\tau_{T0} \rightarrow 0$ (the limit in which the classical box model is usually solved, with ΔT held at its externally imposed target ΔT_a). In this limit $\tau_T = V \tau_{T0} / (V + \psi^* \tau_{T0}) \rightarrow 0$, so the thermal petal strength vanishes, $S_T \rightarrow 0$ (Eq. (107)), and Eq. (110) collapses to the quadratic

$$725 \quad (\psi^*)^2 + k\alpha \Delta T_a \psi^* + k\beta F_0 V = 0, \quad (111)$$

which is, up to the sign convention for ΔT_a , the same quadratic classically solved for the two branches of the thermally dominated regime (Stommel, 1961). Its discriminant vanishes – the classical fold – at

$$F_{0,c} = \frac{k\alpha^2 (\Delta T_a)^2}{4\beta V}, \quad \psi_c^* = -\frac{k\alpha \Delta T_a}{2}, \quad (112)$$

and substituting ψ_c^* into the salinity petal strength, Eq. (108), gives $S_S = k\beta F_{0,c} V / (\psi_c^*)^2 = 1$ exactly – confirming, in this
 730 explicit limit, that the classical saddle-node point and the daisy’s $S = 1$ threshold are one and the same condition, reached here along the single petal that survives the fast- ΔT limit.

Two features of the classical picture lie outside the linear daisy framework as used here. First, Eqs. (106) use ψ rather than $|\psi|$ in the box-exchange terms; this is immaterial for the local stability analysis above, which only ever probes the sign-definite neighbourhood of today’s branch ($\psi^* > 0$), but it means the equations as written do not, by themselves, correctly describe
 735 a circulation reversal. Second, the linear Lyapunov exponent $\lambda \approx (S - 1)/\tau$ characterizes only the onset and initial growth rate of the instability at the strong-circulation branch; it does not determine the saddle (ψ^* between roots) or the reversed, haline-dominated branch that Eq. (110) also admits, nor the width of the hysteresis loop in F_0 between the point where the strong branch disappears and the point where the system could recover it. Those global features require solving Eq. (110) in full, exactly as in the classical treatment; the daisy decomposition’s contribution is to attribute the local loss of stability of
 740 the present-day branch to two separately interpretable, named mechanisms – the stabilizing thermal-transport petal and the destabilizing salt-advection petal – rather than to a single opaque discriminant.



7.3 2nd example of a daisy petal: melt-elevation feedback of ice sheets

7.3.1 Generic approach: feedback between ice height and surface mass balance

As a second, nonlinear example, we consider the melt-elevation feedback of continental ice sheets (Fig. 8a), using the simplified
 745 model of Levermann and Winkelmann (2016). The ice-sheet thickness h evolves according to the balance between ice discharge
 and surface mass balance,

$$\frac{dh}{dt} = -B_{\text{ice}} \left(\frac{h}{h_c} \right)^m + a \equiv F_h(h, a), \quad (113)$$

where B_{ice} is an ice-softness parameter, a is the surface mass balance (snowfall minus melting), and m describes the nonlinear
 dependence of discharge on ice thickness ($m = 8$ for a Vialov profile, Vialov, 1958). We choose the elevation scale h_c in
 750 Eq. (113) for later convenience as

$$h_c \equiv \frac{mB_{\text{ice}}}{\gamma\Gamma} \equiv mB_{\text{ice}}\tau_m, \quad (114)$$

which is simply a convenient re-parametrization of B_{ice} , with $\tau_m \equiv 1/(\gamma\Gamma)$ the timescale of melting due to surface-elevation
 changes. The surface mass balance relaxes towards a value set by the ice-sheet's own height through the atmospheric lapse
 rate,

$$\frac{da}{dt} = \frac{a_0 + \gamma\Gamma h - a}{\tau_a} \equiv \frac{f_a(h) - a}{\tau_a}, \quad (115)$$

where a_0 is the surface mass balance at $h = 0$, γ is the melt rate per degree of surface warming, and $\Gamma \approx 6 \times 10^{-3} \text{ K m}^{-1}$ is the
 lapse rate. The interaction graph (Fig. 8a) is a two-variable loop: a taller ice sheet has a colder (higher-elevation) surface and
 hence a more positive mass balance, which builds the ice sheet further, while a shorter ice sheet warms and loses mass faster –
 the classic melt-elevation feedback, which can run away once discharge is too weak to counteract it.

760 The intrinsic restoring timescale of h follows from the KL1 equation (113),

$$\tau_h = - \left(\frac{\partial F_h}{\partial h} \right)^{-1} = \left(\frac{h_c}{h} \right)^{m-1} \frac{h_c}{mB_{\text{ice}}} = \left(\frac{h_c}{h} \right)^{m-1} \tau_m, \quad (116)$$

and the dependency strengths are $\phi_{h \leftarrow a} = \tau_h$ and $\phi_{a \leftarrow h} = \gamma\Gamma = \tau_m^{-1}$, so that the (dimensionless) loop strength is

$$S = \phi_{h \leftarrow a} \phi_{a \leftarrow h} = \frac{\tau_h}{\tau_m} = \left(\frac{h_c}{h} \right)^{m-1}, \quad (117)$$

which explains our choice of h_c : the system is stable ($S < 1$) exactly when $h > h_c$, so h_c is the *critical, minimum* elevation
 765 below which the melt-elevation feedback runs away. A thick ice sheet has strong, nonlinear discharge ($\propto h^m$) that dominates
 the (constant-strength) melt-elevation amplification and restores any perturbation quickly; once the ice sheet has thinned below
 h_c , discharge is too weak to do so, and the feedback self-amplifies. The dominant Lyapunov exponent is, as always, $\lambda =$
 $(S - 1)/(\tau_h + \tau_a)$, i.e.

$$\lambda = \frac{(h_c/h)^{m-1} - 1}{\tau_m (h_c/h)^{m-1} + \tau_a}, \quad (118)$$



770 positive (unstable, self-amplifying collapse) for $h < h_c$ and negative (stable) for $h > h_c$. This is the same critical-elevation mechanism identified for the Greenland Ice Sheet by Levermann and Winkelmann (2016): once ice loss pushes the surface below the critical elevation, melting feeds on itself.

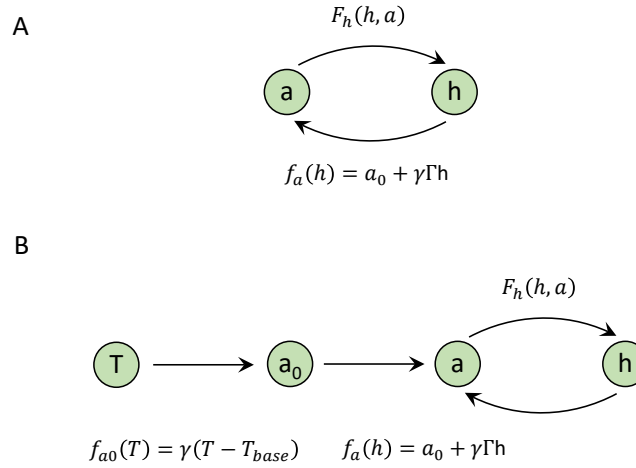


Figure 8. The melt-elevation feedback (panel A) as a two-variable loop between ice height h and surface mass balance a (Sect. 7.3) incorporating the discharge as a stabilizing effect within the function F_h and not as an independent variable as in Fig. 9. Global warming can easily be incorporated as a forcing via the temperature T , which sets a_0 (panel B).

7.3.2 Center-manifold approach: feedback between ice height, discharge and surface mass balance

The generic approach above folded discharge directly into $F_h(h, a)$, so that h has its own (discharge-supplied) restoring timescale τ_h . In reality, discharge often adjusts with some delay of its own – ice streams take time to speed up or slow down – so it can be useful to treat the discharge D as a third, separately relaxing variable, with D approaching its height-dependent target $f_D(h) = B_{ice}(h/h_c)^m$ on a timescale τ_D , rather than tracking it instantaneously. The height equation then becomes

$$\frac{dh}{dt} = -D + a \equiv F_h(D, a), \quad (119)$$

780 which has *no explicit dependence on h at all*: h is driven entirely by its two petals D and a , i.e. $\partial F_h / \partial h = 0$. This is exactly the centre-manifold case of Sect. 4.4 (Fig. 9), with

$$\frac{dD}{dt} = \frac{f_D(h) - D}{\tau_D}, \quad \frac{da}{dt} = \frac{f_a(h) - a}{\tau_a} \quad (120)$$

with f_a as in equation (115). Using the centre-manifold formula $s_i = (\partial F_h / \partial A_i)(\partial f_i / \partial h)$,

$$s_D = (-1) \cdot \frac{\partial f_D}{\partial h} = -\frac{1}{\tau_h}, \quad s_a = (+1) \cdot \gamma\Gamma = \frac{1}{\tau_m}, \quad (121)$$

with $\tau_h = (h_c/h)^{m-1}\tau_m$ exactly as before (Eq. (117) shows $\tau_h/\tau_m = S$). The stability condition of Eq. (82), $\sum_i s_i < 0$,
 785 becomes

$$s_D + s_a = \frac{1}{\tau_m} - \frac{1}{\tau_h} < 0 \iff \tau_h < \tau_m \iff h > h_c, \quad (122)$$

recovering exactly the same critical elevation h_c as the simpler two-variable treatment above – a useful consistency check,
 and a reminder that the two formulations (discharge folded into h 's own dynamics, or treated as its own relaxing petal) agree
 whenever discharge is fast compared to h itself ($\tau_D \ll \tau_h$), and differ only in how they resolve the transient on timescale
 790 τ_D . A reduced feedback representation of exactly this kind – fast enough to run inside the economic models used for policy
 analysis, yet still capturing the essential melt-elevation and discharge feedbacks – is precisely what is needed to couple ice-
 sheet behaviour to integrated assessment models of the climate–economy system, as has recently been implemented explicitly
 (Ramme et al., 2025).

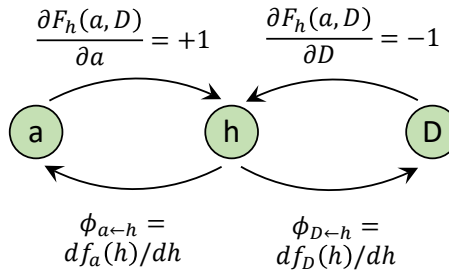


Figure 9. Ice height as a centre without self-restoring dynamics: discharge D and surface mass balance a are its two petals (Sect. 4.4); h itself appears in neither petal's partner equation directly, only through $f_D(h)$ and $f_a(h)$.

7.4 3rd example of a daisy petal: water-vapour greenhouse effect

795 As a third example, we formulate the greenhouse effect in feedback terms via the global energy balance

$$C_p \frac{dT}{dt} = R_{SW} - R_{LW}$$

where T is the global mean surface temperature, C_p the effective heat capacity per unit area, R_{SW} the absorbed shortwave radiation, and R_{LW} the outgoing longwave radiation.

7.4.1 Basic water-vapour feedback

800 First we set up a two-variable loop in which the global mean surface temperature interacts with the infrared optical depth of the atmosphere (Fig. 10), following the same recipe. The global energy balance is

$$\frac{dT}{dt} = \frac{R_{SW} - R_{LW}}{C_p} \equiv F_T(T, \chi), \quad (123)$$

where F_T is, as required by Eq. (1), the time derivative of T itself. $R_{\text{SW}} = (1 - \alpha)R_{\text{Sun}}$ with solar insolation $R_{\text{Sun}} = S_0/4 \approx 340 \text{ W m}^{-2}$ and planetary albedo α . The outgoing longwave radiation follows a grey Stefan–Boltzmann law,

$$805 \quad R_{\text{LW}} = \frac{\sigma T^4}{1 + \epsilon\chi}, \quad (124)$$

with χ the infrared optical depth and ϵ an opacity parameter; the factor $(1 + \epsilon\chi)^{-1}$ represents the suppression of outgoing radiation by a more opaque atmosphere. The optical depth itself relaxes towards a value set by the (temperature-dependent) water-vapour content,

$$\frac{d\chi}{dt} = \frac{f_\chi(T) - \chi}{\tau_\chi}, \quad f_\chi(T) = \chi_0 \exp\left[-\frac{L}{R_v} \left(\frac{1}{T} - \frac{1}{T_0}\right)\right], \quad (125)$$

810 using the Clausius–Clapeyron relation for saturation water vapour (latent heat L , gas constant R_v , reference values χ_0, T_0) and assuming constant relative humidity and $\chi \propto$ water-vapour content. A warmer surface holds more water vapour, which raises the optical depth and suppresses outgoing radiation, which warms the surface further – the water-vapour feedback.

The intrinsic (Planck) restoring timescale follows from Eq. (2) applied to Eq. (123),

$$\tau_T = -\left(\frac{\partial F_T}{\partial T}\right)^{-1} = \frac{C_p(1 + \epsilon\chi)}{4\sigma T^3} \equiv \frac{C_p}{\kappa_P}, \quad \kappa_P \equiv -\frac{\partial(R_{\text{SW}} - R_{\text{LW}})}{\partial T} = \frac{4\sigma T^3}{1 + \epsilon\chi}, \quad (126)$$

815 where κ_P is the Planck response of the radiation budget to surface warming (in $\text{W m}^{-2}\text{K}^{-1}$). The dependency strengths follow from Eq. (8), $\phi_{T \leftarrow \chi} = \tau_T \partial F_T / \partial \chi$ and $\phi_{\chi \leftarrow T} = \partial f_\chi / \partial T$, so that the loop strength is

$$S_\chi = \phi_{T \leftarrow \chi} \phi_{\chi \leftarrow T} = \frac{\tau_T}{C_p} \cdot \frac{\epsilon\sigma T^4}{(1 + \epsilon\chi)^2} \cdot \frac{\partial f_\chi}{\partial T} = \frac{\kappa_\chi}{\kappa_P}, \quad \kappa_\chi \equiv \frac{\partial(R_{\text{SW}} - R_{\text{LW}})}{\partial \chi} \frac{\partial f_\chi}{\partial T}, \quad (127)$$

where $\kappa_\chi = \epsilon\sigma T^4(1 + \epsilon\chi)^{-2} \partial f_\chi / \partial T$ for the grey-gas law of Eq. (124). The last form is worth stating generally, because it will carry every further petal below: for any petal variable A_i that acts on T only through the radiation budget, the petal strength is

$$820 \quad S_i = g_i = \frac{\kappa_i}{\kappa_P}, \quad \kappa_i \equiv \frac{\partial(R_{\text{SW}} - R_{\text{LW}})}{\partial A_i} \frac{\partial f_i}{\partial T}, \quad (128)$$

with κ_i the radiative feedback parameter of that petal (in $\text{W m}^{-2}\text{K}^{-1}$; the climate-feedback literature denotes these λ_i , a symbol reserved here for Lyapunov exponents). Equation (128) is exactly the classical decomposition of climate feedbacks into additive radiative feedback parameters (Bony et al., 2006), and the daisy sum rule of Eq. (68), $\sum_i S_i = 1$, is exactly the classical runaway condition $\sum_i \kappa_i = \kappa_P$, at which the net feedback parameter of the radiation budget vanishes. The dominant

825 Lyapunov exponent of the water-vapour loop is, once again, simply

$$\lambda \approx \frac{S_\chi - 1}{\tau_T + \tau_\chi}. \quad (129)$$

The full stability problem follows directly from the local loop strength and the two intrinsic timescales, without ever assembling or diagonalizing the 2×2 Jacobian explicitly – exactly the simplification the diagrammatic approach is designed to provide.

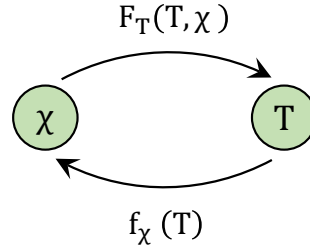


Figure 10. The water-vapour greenhouse feedback as a two-variable loop between global mean temperature T and infrared optical depth χ .

7.4.2 Additional feedbacks and the runaway greenhouse instability

830 The climate system remains stable as long as $S_\chi < 1$; at the critical threshold $S_\chi = 1$ the destabilizing greenhouse amplification exactly cancels the intrinsic (Planck) radiative restoring, and the recovery timescale $\tau = (\tau_T + \tau_\chi)/(1 - S_\chi)$ diverges – critical slowing down. Equation (127) shows that this threshold can be reached along two different routes: either the water-vapour feedback parameter κ_χ grows, or the Planck response κ_P in the denominator shrinks, i.e. the centre’s own restoring timescale $\tau_T = C_p/\kappa_P$ diverges. The second route is the Simpson–Nakajima runaway-greenhouse limit (Simpson, 1928; Nakajima et al., 1992): the outgoing radiation saturates at a finite value regardless of surface temperature, so that $\kappa_P \rightarrow 0$ and every petal strength $S_i = \kappa_i/\kappa_P$ diverges at once. The grey-gas model of Eq. (124) only hints at this, through the factor $(1 + \epsilon\chi)^{-1}$ in κ_P ; it cannot produce the saturation itself, because $f_\chi(T)$ in Eq. (125) is bounded and κ_P therefore never vanishes. Physically, the saturation occurs because the flux escaping to space is set not by the surface temperature itself but by the temperature at the level from which that radiation actually leaves the atmosphere; once the atmosphere is warm and moist enough, Clausius–

840 Clapeyron ties the water-vapour path above that level to the *local* temperature there, so the level’s own temperature – and hence the outgoing flux – becomes nearly independent of the (much warmer) surface below it (derived in detail in Appendix E). The relevant control parameter is therefore the absorbed *solar* radiation relative to this fixed ceiling, $R_{\text{LW}}^{\text{max}} \approx 280\text{--}310 \text{ W m}^{-2}$ (Nakajima et al., 1992), not the greenhouse-gas forcing: the limit is approached on planets receiving high stellar insolation (e.g. early Venus, or a planet orbiting closer to its star) or, for Earth, only once the Sun brightens by a few percent over the

845 coming ~ 1 Gyr. It cannot happen on Earth today because the present absorbed shortwave radiation, $R_{\text{SW}} \approx 240 \text{ W m}^{-2}$, sits well below $R_{\text{LW}}^{\text{max}}$; since anthropogenic CO_2 warms the surface by suppressing R_{LW} at a given R_{SW} rather than by increasing R_{SW} itself, even burning all of Earth’s fossil-fuel reserves is too small a perturbation to close that gap (Goldblatt et al., 2013).

Further petals attach to the same centre T in the same way (Fig. 11): each is a variable A_i relaxing towards a temperature-dependent target, exactly as χ does in Eq. (125),

$$850 \quad \frac{dA_i}{dt} = \frac{f_i(T) - A_i}{\tau_i}, \quad A_i \in \{\alpha, A_{\text{PF}}, A_{\text{ARF}}, A_{\text{BF}}\}, \quad (130)$$



and each acts back on T only through the radiation budget, which extends Eq. (123) to

$$\frac{dT}{dt} = \frac{R_{SW}(\alpha) - R_{LW}(T, \chi) + \sum_i \Delta R_i(A_i)}{C_p} \equiv F_T(T, \chi, \alpha, A_{PF}, A_{ARF}, A_{BF}), \quad (131)$$

with $\Delta R_i(A_i)$ the change in the net radiation budget caused by petal i . Every petal strength then follows from Eq. (128), $S_i = \kappa_i / \kappa_P$, and the system is a daisy in the sense of Sect. 4, tipping where $\sum_i S_i = 1$. Two kinds of petal appear.

855 The planetary albedo α is already present in $R_{SW} = (1 - \alpha)R_{Sun}$, so no additional term ΔR_α is needed. With the linearized target $f_\alpha(T) = \alpha^* - \gamma(T - T^*)$, $\gamma > 0$, around the present-day state (α^*, T^*) , Eq. (128) gives

$$\kappa_\alpha = \frac{\partial R_{SW}}{\partial \alpha} \frac{\partial f_\alpha}{\partial T} = R_{Sun} \gamma, \quad S_\alpha = \frac{R_{Sun} \gamma}{\kappa_P}. \quad (132)$$

The three biogeophysical carbon-cycle feedbacks of Steffen et al. (2018) that are naturally expressed as an area – the fraction of permafrost A_{PF} , Amazon rainforest A_{ARF} and boreal forest A_{BF} still intact, each normalized to 1 at the present-day state
860 – have targets $f_i(T) = 1 - k_i(T - T^*)$, with $k_i > 0$ the fractional area lost per degree of warming, and act on T through the carbon they release: area loss raises atmospheric CO_2 and thereby the radiative forcing, $\Delta R_i(A_i) = -\eta_i(A_i - 1)$, with $\eta_i > 0$ the forcing added per unit of area lost. Equation (128) gives

$$\kappa_i = \frac{\partial \Delta R_i}{\partial A_i} \frac{\partial f_i}{\partial T} = \eta_i k_i, \quad S_i = \frac{\eta_i k_i}{\kappa_P}, \quad i \in \{PF, ARF, BF\}. \quad (133)$$

None of the sensitivities γ , η_i , k_i is needed individually, because the literature reports the combined feedback parameters κ_i
865 directly – knowledge level KL2 of step 4 in Sect. 7.1: the diagnostic relation is known even where the time-dependent equation is not. One point of care is the common normalization: all petals must be referred to the same κ_P , otherwise the sum rule of Eq. (68) does not apply. For the fast petals this is immediate. The IPCC’s assessment of the radiative feedbacks gives $\kappa_P \approx 3.2 \text{ W m}^{-2} \text{ K}^{-1}$, a combined water-vapour–lapse-rate feedback $\kappa_\chi \approx 1.3 \text{ W m}^{-2} \text{ K}^{-1}$ and a surface-albedo feedback $\kappa_\alpha \approx 0.35 \text{ W m}^{-2} \text{ K}^{-1}$ (Forster et al., 2021), i.e. $S_\chi \approx 0.40$ and $S_\alpha \approx 0.11$. The water-vapour value deserves a remark. Taken on
870 its own, the water-vapour feedback is about $1.8 \text{ W m}^{-2} \text{ K}^{-1}$ and would give $S \approx 0.55$; but it is accompanied by a lapse-rate feedback of about -0.5 to $-0.8 \text{ W m}^{-2} \text{ K}^{-1}$ with which it is strongly anti-correlated across models, since both arise from the same moistening and warming of the upper troposphere, and the two are therefore conventionally assessed together (Bony et al., 2006; Forster et al., 2021). The grey-gas petal χ cannot separate them: a single bulk optical depth tied to the surface temperature describes the column’s entire longwave response to surface warming, lapse-rate change included, so the petal
875 has to carry the combined value, $S_\chi \approx 0.4$. For the carbon petals, Steffen et al. (2018) report instead the additional warming ΔT_i each feedback contributes by 2100 on top of an external warming of $\Delta T_{ext} \approx 2^\circ \text{C}$. That ΔT_{ext} already contains the fast feedbacks, so by Eq. (85) the ratio $\Delta T_i / \Delta T_{ext}$ measures κ_i against the *net* response $\kappa_{net} = \kappa_P - \kappa_\chi - \kappa_\alpha - \dots$, not against κ_P . For an equilibrium climate sensitivity of $\approx 3 \text{ K}$ per CO_2 doubling (forcing $\approx 3.7 \text{ W m}^{-2}$), $\kappa_{net} \approx 1.2 \text{ W m}^{-2} \text{ K}^{-1}$, and the daisy-consistent petal strengths are

$$880 \quad S_i = \frac{\kappa_i}{\kappa_P} \approx \frac{\Delta T_i}{\Delta T_{ext}} \cdot \frac{\kappa_{net}}{\kappa_P} \approx 0.4 \frac{\Delta T_i}{\Delta T_{ext}}, \quad i \in \{PF, ARF, BF\}. \quad (134)$$

The relaxation times τ_i , finally, are available independently as the transition timescales of the corresponding tipping elements (Lenton et al., 2008). Table 1 collects the result for all five petals. Two checks follow from Table 1. First, the fast petals alone



Table 1. The five petals of the T -daisy of Fig. 11: radiative feedback parameter κ_i (in $\text{W m}^{-2}\text{K}^{-1}$, order of magnitude) and petal strength $S_i = \kappa_i/\kappa_P$ with $\kappa_P \approx 3.2 \text{ W m}^{-2}\text{K}^{-1}$ (χ , α from Forster et al., 2021; carbon petals from Steffen et al., 2018 via Eq. (134), with their reported ranges), the petal’s own relaxation time τ_i (Lenton et al., 2008), and the resulting recovery time $\tau_i/(1 - S_i)$ of that petal alone.

Petal	κ_i	$S_i = \kappa_i/\kappa_P$	τ_i	$\tau_i/(1 - S_i)$
χ (water vapour incl. lapse rate)	1.3	0.40	weeks	weeks
α (ice-albedo)	0.35	0.11	10 yr	11 yr
A_{PF} (permafrost)	0.05	0.017 (0.008–0.03)	100 yr	100 yr
A_{ARF} (Amazon rain forest)	0.03	0.009 (0.006–0.02)	50 yr	50 yr
A_{BF} (boreal forest)	0.035	0.011 (0.004–0.02)	50 yr	50 yr

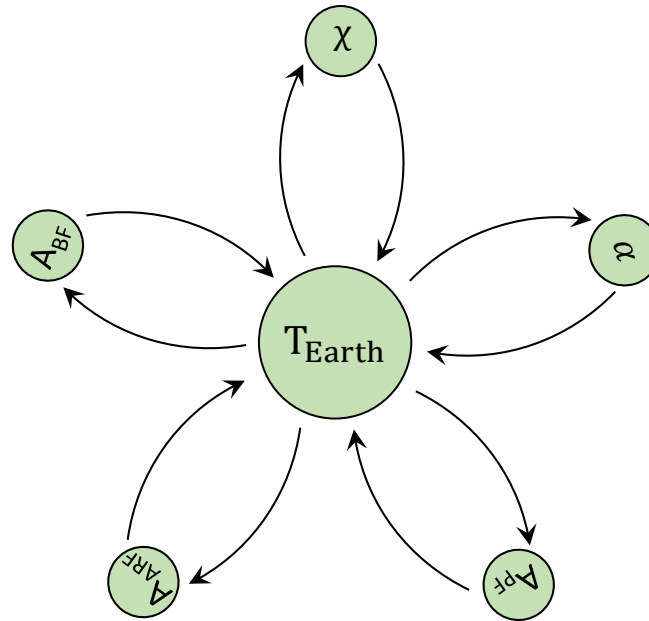


Figure 11. Earth’s global mean temperature as a five-petal daisy centre: the water-vapour optical depth χ (Eq. (127)), the planetary albedo α (Eq. (132)), and the permafrost (A_{PF}), Amazon-rainforest (A_{ARF}) and boreal-forest (A_{BF}) area petals (Eq. (133)). All five act on T only through the radiation budget, Eq. (131), so their strengths $S_i = \kappa_i/\kappa_P$ add (Table 1).



give $S_\chi + S_\alpha \approx 0.5$, i.e. an equilibrium gain $1/(1 - \sum_i S_i) \approx 2$ (Eq. (85)) over the Planck-only response of $3.7/3.2 \approx 1.2$ K per CO₂ doubling. The remaining fast feedback is that of clouds, $\kappa \approx 0.4 \text{ W m}^{-2} \text{ K}^{-1}$ or $S \approx 0.13$ (Forster et al., 2021), which
885 the grey-gas model has no variable for; adding it raises the sum to ≈ 0.64 and the gain to ≈ 2.8 , i.e. an equilibrium climate sensitivity of ≈ 3 K, as assessed. Second, the three carbon petals add only ≈ 0.04 to that sum: the full daisy, clouds included, sits at $\sum_i S_i \approx 0.7$, well below the tipping threshold of 1, as it must for the present climate to be stable. Each petal's own Lyapunov exponent, $\lambda_i = (S_i - 1)/(\tau_T + \tau_i)$, is dominated by its own relaxation time whenever $\tau_i \gg \tau_T$ – which holds for all but the water-vapour petal, since $\tau_T = C_p/\kappa_P \approx 3$ yr for a ~ 70 m ocean mixed layer – so that with today's $S_i \ll 1$ a perturbation of
890 any single slow petal decays essentially on τ_i itself. These numbers put the ‘Hothouse Earth’ hypothesis of Steffen et al. (2018) in perspective. Steffen et al. propose a planetary threshold near 2°C of warming beyond which the biogeophysical feedbacks of their Table 1, rather than human forcing, would carry the Earth System into a much warmer stable state. In the language of the daisy this would require $\sum_i S_i$ to approach 1. The daisy stands at ≈ 0.7 , and the three carbon petals that Steffen et al. invoke contribute ≈ 0.04 of that – almost an order of magnitude less than the remaining margin of ≈ 0.3 , and still only about a
895 third of it at the upper end of their own uncertainty ranges (≈ 0.1). Nor does a warming of two or several degrees change this: it moves the fast petals within their assessed ranges and the carbon petals by fractions of their small values. A self-propelled, feedback-driven departure of the global mean temperature – a runaway in the sense of Sect. 7.4 – is therefore excluded by the feedback strengths themselves, independently of, and for a different reason than, the radiative ceiling of Appendix E. What the daisy analysis leaves open is a different mechanism, and it is on this that the Hothouse argument in fact rests: individual petals
900 with a threshold of their own, $S_i = 1$ – an ice sheet thinning below h_c (Sect. 7.3), an overturning weakening until $S_S \rightarrow 1$ (Sect. 7.2) – can tip while $\sum_i S_i$ stays far below 1. Such a tip shifts T by a finite amount to a new equilibrium, hysteretic and slow but bounded; whether a sequence of such local tips adds up to a new, warmer basin of attraction is a question about the thresholds of the individual elements and their direct couplings, not about the feedback strength of the centre. Steffen et al. (2018) assert such a cascade but do not quantify it, and the feedbacks of their Table 1 cannot carry it. Two caveats bound the
905 present treatment accordingly. The daisy assumes that petals interact only through the shared centre T ; a forest dieback that changes the albedo directly, or permafrost thaw that affects the boreal forest, links two petals without passing through T – a rose topology, the subject of part II. And the linear petal strengths describe only the onset of an instability; whether a local tip in one petal propagates through T_{Earth} into the others is the question addressed by network analyses of interacting tipping elements (Wunderling et al., 2021), for which the daisy topology provides the corresponding reduced mathematical description.

910 7.5 1st example of model building: a wage–price feedback

The three examples above all started from known governing equations (KL1) or (KL2) and used the diagram only to shortcut the stability analysis. We now go one step further and show how the *same* diagram can be used to build an explicit, time-dependent model even when no governing equations exist at all (KL3) – the situation the present framework was mainly designed for (Sect. 1). The extra step is simple: once dependency strengths $\phi_{i \leftarrow j}$ and timescales τ_i have been estimated for every link (step



915 4 of Sect. 7.1), write the generic linear relaxation equation (4) for every variable, with f_i taken to be linear in its argument(s),

$$f_i(A_j) = A_i^* + \phi_{i \leftarrow j} (A_j - A_j^*), \quad (135)$$

and combine these generic relations for every link in the diagram into one coupled system of ordinary differential equations. This is, by construction, a fully specified KL1 model: it can be integrated forward in time, perturbed, or driven by an external forcing, exactly like the explicit climate models of Sects. 7.2–7.4 – even though no one ever wrote down a fundamental equation of motion for the system it describes. Figure 12 summarizes this extended recipe. As a societal example, and precisely the KL3

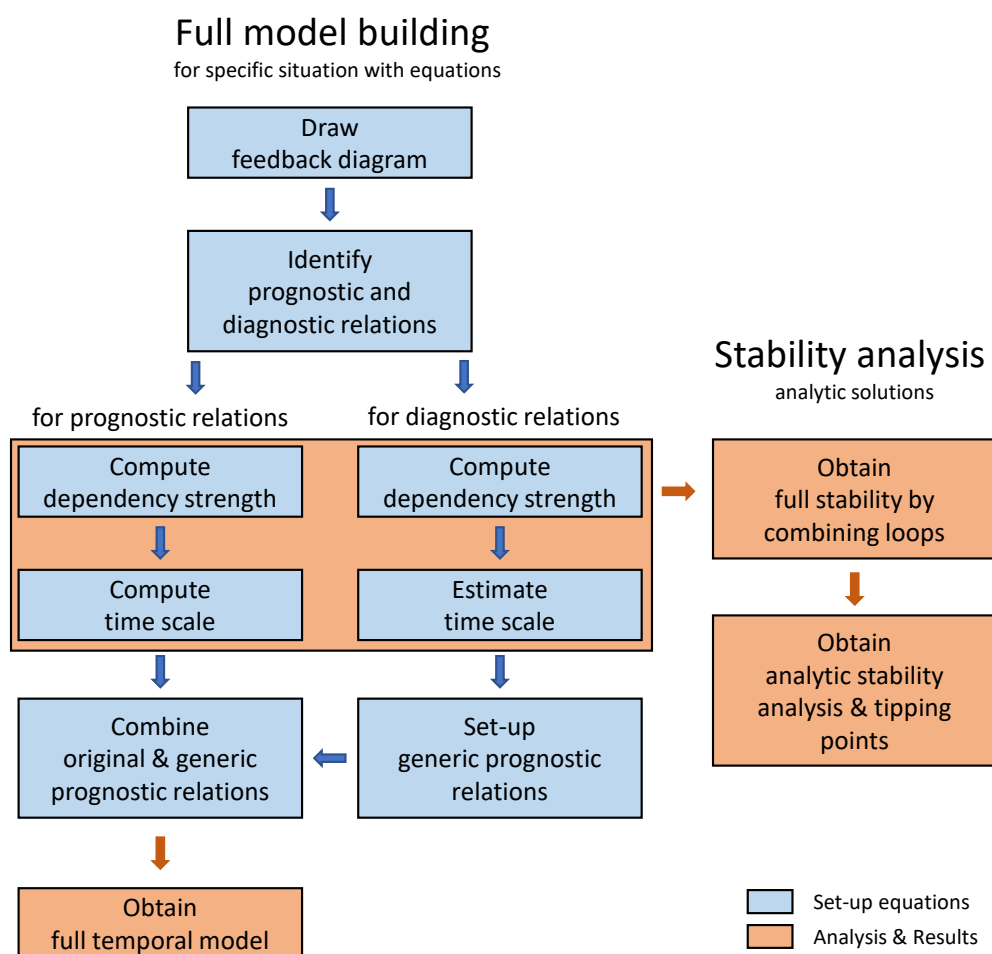


Figure 12. Cookbook for building a full temporal model, with or without fundamental equations (Sect. 7.5): once prognostic and diagnostic relations are identified, dependency strengths and timescales are obtained for each (computed directly where equations are known, estimated otherwise, Sect. 2.1); combining the original prognostic relations with the generic relaxation form of diagnostic ones (left) yields a full temporal model, while combining the same loops (right) yields the analytic stability analysis and tipping points of Fig. 6.

920

case the recipe is built for, consider the feedback between wages and prices familiar from macroeconomics: firms pass part of



an increase in labour costs on to consumers as higher prices, and workers and unions negotiate wage increases that compensate, at least in part, for a higher cost of living. Two choices decide whether the resulting model is useful for an economic analysis. First, the variables have to be the *rates of change* of wages and prices, i.e. wage inflation π_W and price inflation π_P (both in % yr^{-1}), rather than their levels. The policy debate about wage–price spirals concerns whether inflation persists or accelerates (Blanchard, 1986; Lorenzoni and Werning, 2023); moreover, a formulation in levels with dependency strengths below one would imply a permanent loss of real wages after every price shock without any economic reason for it. Second, the loop has to be closed by whatever anchors inflation in the long run. We therefore build two models with the same recipe: the wage–price loop on its own (model A, Sect. 7.5.1), and the same loop as one petal of a daisy whose second petal represents aggregate demand and monetary policy (model B, Sect. 7.5.2).

7.5.1 Model A: the wage–price loop

There is no equation of motion for π_W or π_P comparable to Eq. (113); all we have is a diagram $\pi_W \rightarrow \pi_P \rightarrow \pi_W$ – and a qualitative sense of how strongly each link acts (Fig. 13).

1. *Diagram*: two variables, π_W and π_P , each depending on the other – the same topology as Fig. 3.
2. *Knowledge level*: KL3; no governing equations are available.
3. *Dependency strengths* (illustrative, not fitted to any particular economy): a cost pass-through of $\phi_{P \leftarrow W} = 0.5$ (half of an increase in wage growth appears in price inflation, the remainder being absorbed by profit margins or diluted by other cost components) and an indexation of $\phi_{W \leftarrow P} = 0.6$ (60 % of an increase in price inflation is recovered in wage growth). As ratios of two rates, both are dimensionless.
4. *Timescales*: $\tau_W \approx 1$ yr (the typical interval between wage renegotiations) and $\tau_P \approx 0.5$ yr (the time firms take to adjust prices after a cost shock).
5. *Stability*: $S = \phi_{P \leftarrow W} \phi_{W \leftarrow P} = 0.3 < 1$ and $\tau = \tau_W + \tau_P = 1.5$ yr. The linear estimate of Eq. (46), $\lambda \approx (S - 1)/\tau \approx -0.47 \text{ yr}^{-1}$, is accurate only close to $S = 1$; the exact roots, Eq. (42), are $\lambda_+ = -0.58 \text{ yr}^{-1}$ and $\lambda_- = -2.42 \text{ yr}^{-1}$. A temporary cost shock, e.g. an energy-price spike that raises the price-inflation target π_P^* by a transient amount $\varepsilon(t)$, therefore raises both inflation rates only temporarily, and the excess inflation decays with an e -folding time of $1/|\lambda_+| \approx 1.7$ years. A persistent shock, i.e. a lasting shift $\Delta\pi_P^*$, instead moves inflation to a new steady state that exceeds the direct effect by the gain factor of Sect. 4.5, $\Delta\pi_P = \Delta\pi_P^*/(1 - S) \approx 1.43 \Delta\pi_P^*$: the wage–price loop adds 43 % in “second-round effects”.
6. *Temporal model*: with Eq. (135), the full, simulate-able model is

$$\frac{d\pi_W}{dt} = \frac{\pi_W^* + \phi_{W \leftarrow P}(\pi_P - \pi_P^*) - \pi_W}{\tau_W}, \quad (136)$$

$$\frac{d\pi_P}{dt} = \frac{\pi_P^* + \varepsilon(t) + \phi_{P \leftarrow W}(\pi_W - \pi_W^*) - \pi_P}{\tau_P}, \quad (137)$$



which can now be driven by any time-dependent shock $\varepsilon(t)$ and integrated forward in time, exactly as for the physical examples above, even though Eqs. (136)–(137) were never derived from first principles.

Economic theory singles out one particular value of S . If price setting passes increases in wage growth fully on to prices in the long run (constant mark-up, labour as the only cost) and wage setting fully compensates for price inflation (no money illusion), then $\phi_{P \leftarrow W} = \phi_{W \leftarrow P} = 1$ and $S = 1$ exactly. This long-run homogeneity places the wage–price loop precisely at the tipping point of Sect. 3.2: $\lambda_+ = 0$, every state with $\pi_W - \pi_W^* = \pi_P - \pi_P^*$ is a steady state, and a temporary shock shifts the rate of inflation permanently – the accelerationist property of the expectations-augmented Phillips curve. In terms of the diagram, the wage–price loop alone cannot determine the rate of inflation; inflation has to be anchored by a process outside this loop, which is the purpose of model B.

A self-sustaining spiral, $S > 1$, requires that at least one link over-compensates, for instance wage settlements that add a catch-up for past real-wage losses to current inflation, or price setting that restores previously compressed margins (Blanchard, 1986; Lorenzoni and Werning, 2023). For $\phi_{P \leftarrow W} = 1$ and $\phi_{W \leftarrow P} = 1.2$, $S = 1.2$ and $\lambda_+ \approx (S - 1)/\tau \approx +0.13 \text{ yr}^{-1}$ (exact value $+0.128 \text{ yr}^{-1}$): any deviation of inflation grows with an e -folding time of about 8 years, i.e. inflation accelerates. The diagram thus separates two phenomena that share the name ‘wage–price spiral’: for S slightly below 1 a shock is *prolonged*, since the recovery time $\tau/(1 - S)$ diverges as $S \rightarrow 1$ (critical slowing down), whereas only for $S > 1$ is it *amplified*, with inflation accelerating. Historical episodes of simultaneously rising price and wage inflation in advanced economies were mostly followed by a stabilization, not an acceleration, of inflation and wage growth (Alvarez et al., 2024), which is consistent with an effective feedback strength below one. Model A, however, could represent the influence of monetary policy only by assuming that policy alters the wage- and price-setting coefficients $\phi_{W \leftarrow P}$ and $\phi_{P \leftarrow W}$ themselves, whereas in practice policy acts on aggregate demand.

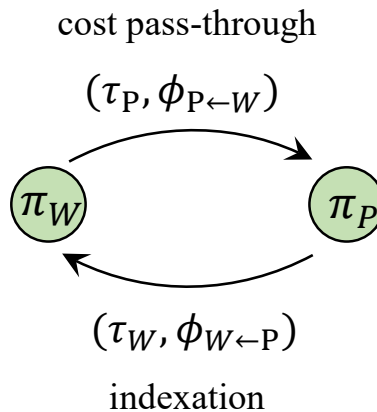


Figure 13. A wage–price feedback as a two-variable loop between wage inflation π_W and price inflation π_P (model A, Sect. 7.5.1): a classic KL3 example, with no governing equation for either variable, yet fully analysable – and simulable – using only estimated dependency strengths and timescales. Long-run homogeneity of wage and price setting places this loop exactly at its tipping point, $S = 1$.



7.5.2 Model B: wage–price petal and demand–policy petal

Model B keeps the wage–price loop of model A unchanged and adds the process that anchors inflation, aggregate demand and monetary policy. Price inflation π_P thereby becomes the centre of a two-petal daisy (Sect. 4, Fig. 14). The first petal is the wage–price loop of strength $S_{WP} = \phi_{P \leftarrow W} \phi_{W \leftarrow P}$. The second petal is the output gap Y , i.e. the deviation of output from its potential (in %). The central bank raises its nominal interest rate by θ percentage points per percentage point of excess inflation (a Taylor rule, Taylor, 1993), so that the real interest rate rises by $\theta - 1$ percentage points; a higher real rate lowers the output gap with a sensitivity $b > 0$; and a positive output gap raises price inflation with the slope $\phi_{P \leftarrow Y} > 0$ of the Phillips curve. If the policy rate is set without delay and the real rate is approximated by the nominal rate minus current inflation, which corresponds to adaptive expectations, the dependency strength of the policy link is $\phi_{Y \leftarrow P} = -b(\theta - 1)$, and the petal strength is

$$S_Y = \phi_{P \leftarrow Y} \phi_{Y \leftarrow P} = -b \phi_{P \leftarrow Y} (\theta - 1). \quad (138)$$

1. *Diagram*: a daisy with centre π_P and the two petals π_W and Y – the topology of Fig. 4 with two petals.
2. *Knowledge level*: KL3; no governing equations are available.
3. *Dependency strengths* (illustrative): the wage–price petal at the homogeneous benchmark, $\phi_{P \leftarrow W} = \phi_{W \leftarrow P} = 1$, i.e. $S_{WP} = 1$; for the demand–policy petal a Phillips-curve slope $\phi_{P \leftarrow Y} = 0.3$ (% yr⁻¹ of inflation per % of output gap), $b = 1$ (% of output gap per percentage point of real interest rate) and $\theta = 1.5$ (Taylor, 1993), so that $\phi_{Y \leftarrow P} = -0.5$ and $S_Y = -0.15$. The product of each petal is dimensionless, as required.
4. *Timescales*: $\tau_P \approx 0.5$ yr and $\tau_W \approx 1$ yr as in model A, and $\tau_Y \approx 1.5$ yr for the transmission of a change in the real interest rate to aggregate demand.
5. *Stability*: by the sum rule, Eq. (68), $S = S_{WP} + S_Y = 0.85 < 1$: the system is stable, although its wage–price petal on its own sits exactly at the tipping point. The effective timescale, Eq. (69), is $\tau = 0.5 + 1 \cdot (1 - 0.85 + 1) + 1.5 \cdot (1 - 0.85 - 0.15) = 1.65$ yr, so that $\lambda \approx (S - 1)/\tau \approx -0.09$ yr⁻¹; the exact dominant root of Eq. (64) is -0.11 yr⁻¹. Inflation is therefore anchored but very persistent, with an e -folding time of about 9 years. A stronger response, $\theta = 3$, gives $S_Y = -0.6$ and $S = 0.4$; the dominant roots then form the complex pair $-0.46 \pm 0.29i$ yr⁻¹, i.e. inflation returns to target with an e -folding time of about 2 years and a slight overshoot (the oscillation period of about 22 years is much longer than the decay time).

6. *Temporal model*: with Eq. (135), the full, simulate-able model is

$$\frac{d\pi_P}{dt} = \frac{\pi_P^* + \varepsilon(t) + \phi_{P \leftarrow W}(\pi_W - \pi_W^*) + \phi_{P \leftarrow Y}(Y - Y^*) - \pi_P}{\tau_P}, \quad (139)$$

$$\frac{d\pi_W}{dt} = \frac{\pi_W^* + \phi_{W \leftarrow P}(\pi_P - \pi_P^*) - \pi_W}{\tau_W}, \quad (140)$$

$$\frac{dY}{dt} = \frac{Y^* + \phi_{Y \leftarrow P}(\pi_P - \pi_P^*) - Y}{\tau_Y}, \quad (141)$$



in which π_P^* is now the inflation target of the central bank and $Y^* = 0$.

Model B has four advantages over model A, all of which follow directly from the daisy sum rule. First, it respects long-run homogeneity: $S_{WP} = 1$ is no longer a tipping point of the whole system, because stability is supplied by a separate, negative petal. Second, the tipping condition $S < 1$ becomes $S_{WP} - b\phi_{P \leftarrow Y}(\theta - 1) < 1$, or

$$\theta > \theta_c \equiv 1 + \frac{S_{WP} - 1}{b\phi_{P \leftarrow Y}}. \quad (142)$$

For $S_{WP} = 1$ this is $\theta > 1$, the Taylor principle: the nominal interest rate has to rise more than one-for-one with inflation. The daisy recovers this known result in one line from the sum rule, just as Sect. 7.2 recovered the classical fold of the Stommel model. Third, Eq. (142) generalizes the principle. If the wage–price petal over-compensates, $S_{WP} > 1$, the required response exceeds one: for $S_{WP} = 1.2$ and the parameters above, $\theta_c \approx 1.67$, so that $\theta = 1.5$, which stabilizes the homogeneous economy, now fails ($S = 1.05$, $\lambda = +0.035 \text{ yr}^{-1}$), whereas $\theta = 3$ restores stability ($S = 0.6$). A flat Phillips curve, i.e. a small $\phi_{P \leftarrow Y}$, weakens the policy petal, raises θ_c whenever $S_{WP} > 1$, and prolongs inflation through critical slowing down. Fourth, policy enters the diagram as what it is – a separate process acting through demand – rather than as an assumed change of wage- and price-setting behaviour as in model A, and the strengths of the two petals can be estimated independently and then simply added. The gain factor of Sect. 4.5 adds a further, quantitative statement: for $S_{WP} = 1$ a persistent cost shock $\Delta\pi_P^*$ raises inflation permanently by $\Delta\pi_P^*/(1 - S) = \Delta\pi_P^*/[b\phi_{P \leftarrow Y}(\theta - 1)]$, i.e. a policy rule that responds only in proportion to the current deviation leaves a lasting deviation from target. Removing it requires the policy rate to accumulate past deviations, i.e. integral control (Phillips, 1954), which in the diagram corresponds to a variable without self-restoring dynamics of the kind discussed in Sect. 4.4.

Model B is deliberately minimal. Demand acts on price inflation only; a wage Phillips curve would add a direct link $Y \rightarrow \pi_W$, which connects the two petals without passing through the centre and thereby turns the daisy into a rose topology (part II). Inflation expectations are adaptive and enter only through the timescales; under rational expectations, the Taylor principle is the condition for a unique stable equilibrium (determinacy) rather than for stability, and under adaptive learning it reappears as the condition for expectational stability (Bullard and Mitra, 2002). Finally, a two-petal daisy can wobble only if one petal is strongly negative and the other, destabilizing petal is much faster (Sect. 4.3, Appendix D1); condition (RH-3) excludes this for all parameter sets above. The oscillatory over-correction by delayed stabilization policy that Phillips (1954) identified requires the transmission to be resolved into at least three processes within one loop – the mechanism discussed in the next section.

7.6 2nd example of model building: the hog cycle as an oscillatory feedback

Both models of Sect. 7.5 are built from two-variable loops. The single loop of model A can only tip, never wobble (Sects. 3.3 and 3.6), and the two-petal daisy of model B could wobble only for strongly asymmetric petals (Sect. 4.3). As a complementary KL3 example with a genuine oscillatory mode, consider the hog cycle (German: *Schweinezyklus*), the regular rise and fall of pig prices and pig supply first documented by Hanau (1928) and a classical example of endogenous cycles in economics. Its mechanism is a negative feedback that acts through biological delays: high prices lead farmers to enlarge their breeding herds, the additional pigs reach the market about a year later, the increased supply depresses the price, and the herds are reduced

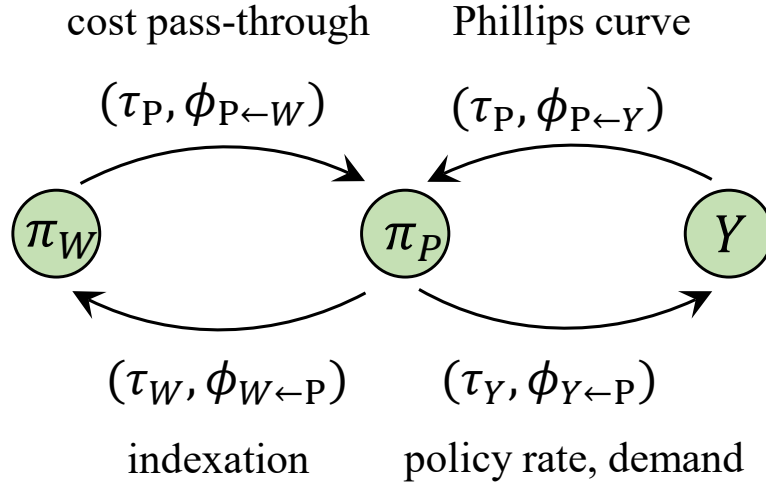


Figure 14. Wage-price feedback anchored by monetary policy as a two-petal daisy (model B, Sect. 7.5.2): price inflation π_P is the centre; the wage-price loop of Fig. 13 is one petal ($S_{WP} > 0$), and the output gap Y , driven by the policy-controlled real interest rate, is a second, stabilizing petal ($S_Y < 0$). The sum rule $S_{WP} + S_Y < 1$ reproduces the Taylor principle, Eq. (142).

- again. Unlike in macroeconomic business-cycle models, which combine several interlocking loops such as the multiplier and the accelerator (Samuelson, 1939), every link of this single loop can be identified directly, and its timescales are largely set by biology. All variables are measured as relative (logarithmic) deviations from their steady state, so that the dependency strengths are elasticities; the feedback strength S does not depend on this choice, since every variable enters the product around the loop once in a numerator and once in a denominator.
1. *Diagram*: four variables – the market price P , the price expected by producers E , the breeding herd B (number of sows) and the slaughter supply H – connected as $P \rightarrow E \rightarrow B \rightarrow H \rightarrow P$ (Fig. 15); the pilea topology of Fig. 1 with $N = 4$.
 2. *Knowledge level*: KL3; no governing equations, only estimated elasticities and the durations of the biological processes.
 3. *Dependency strengths* (illustrative): $\phi_{E \leftarrow P} = 1$ (adaptive expectations: the expected price follows the market price, Nerlove, 1958); $\phi_{B \leftarrow E} = \varepsilon_s = 1.4$ (long-run elasticity of the breeding herd with respect to the expected price); $\phi_{H \leftarrow B} = 1$ (slaughter supply proportional to the breeding herd); and $\phi_{P \leftarrow H} = -1/\varepsilon_d \approx -1.43$, the inverse of a price elasticity of demand $\varepsilon_d = 0.7$. The negative sign of this last link makes the loop a negative feedback and thereby permits oscillations (Sect. 3.3).
 4. *Timescales*: the market price clears supply within weeks, $\tau_P \ll \tau_E, \tau_B, \tau_H$; producers form their expected price from roughly the past year of prices, $\tau_E \approx 1$ yr; adjusting the number of farrowing sows, including a gestation of about four months, takes $\tau_B \approx 0.5$ yr; and fattening the piglets to slaughter weight takes $\tau_H \approx 0.5$ yr. The production lag between a breeding decision and the resulting supply is thus $\tau_B + \tau_H \approx 1$ yr. Because τ_P is negligible, the factor $(1 + \lambda\tau_P)$ in



Eq. (15) is unity, and P can be eliminated exactly: the loop behaves as a three-process loop of E , B and H . Unlike the contraction of Sect. 3.6, which merges two processes with finite timescales, eliminating a process with vanishing timescale removes no oscillatory mode.

- 1055 5. *Stability and oscillation*: $S = \phi_{E \leftarrow P} \phi_{B \leftarrow E} \phi_{H \leftarrow B} \phi_{P \leftarrow H} = -\varepsilon_s / \varepsilon_d = -2$. Since $|S| > 1$, the simple $|S| < 1$ criterion of Sect. 3.2 does not directly apply, and the relevant comparison is against the two nearest crossing points: the saddle-node threshold at $S = 1$ and the wobble threshold S_0^- of Eq. (25). Equation (32) gives the frequency at which the oscillatory mode crosses the imaginary axis,

$$\omega^2 = \frac{\tau_E + \tau_B + \tau_H}{\tau_E \tau_B \tau_H} = \frac{1 + 0.5 + 0.5}{1 \cdot 0.5 \cdot 0.5} \text{yr}^{-2} = 8 \text{yr}^{-2}, \quad (143)$$

- 1060 and Eq. (25) gives the corresponding wobble threshold,

$$S_0^- = -\sqrt{1 + \tau_E^2 \omega^2} \sqrt{1 + \tau_B^2 \omega^2} \sqrt{1 + \tau_H^2 \omega^2} = -3 \cdot \sqrt{3} \cdot \sqrt{3} = -9. \quad (144)$$

- 1065 Since $-9 < S = -2 < 1$, the loop lies inside the stable interval bounded by both thresholds: a self-sustained cycle would require $\varepsilon_s / \varepsilon_d > 9$. Note that Eq. (143) is the frequency *at* the threshold, with a period of $2\pi / \omega \approx 2.2$ yr. At $S = -2$ itself, the exact roots of Eq. (15) are $\lambda = -0.64 \pm 1.68i \text{yr}^{-1}$ and $\lambda = -3.7 \text{yr}^{-1}$: perturbations oscillate with a period of 3.7 yr and decay with an e -folding time of 1.6 yr, so that the amplitude falls to about one tenth within one cycle. A persistent cycle thus has to be re-excited repeatedly by external shocks such as changes in feed prices, animal diseases or demand – the impulse–propagation view of economic fluctuations (Frisch, 1933).

6. *Temporal model*: with Eq. (135), the price as an instantaneous diagnostic relation,

$$P = P^* + \phi_{P \leftarrow H}(H - H^*), \quad (145)$$

- 1070 and the three relaxing processes

$$\frac{dE}{dt} = \frac{E^* + \phi_{E \leftarrow P}(P - P^*) - E}{\tau_E}, \quad (146)$$

$$\frac{dB}{dt} = \frac{B^* + \phi_{B \leftarrow E}(E - E^*) - B}{\tau_B}, \quad (147)$$

$$\frac{dH}{dt} = \frac{H^* + \phi_{H \leftarrow B}(B - B^*) - H}{\tau_H}, \quad (148)$$

- 1075 form the full, simulate-able model, which can be driven by external shocks (e.g. a jump in P^* from a demand shock, or in B^* from a change in feed costs) and integrated forward in time to reproduce a damped cycle of about four years.

The expectation stage determines the length of the cycle. If producers responded to the current price directly (naive expectations, $\tau_E \rightarrow 0$), the loop would reduce to the two processes B and H , which cannot wobble (Sect. 3.3); at $S = -2$ its roots are $\lambda = -2 \pm 2.83i \text{yr}^{-1}$, with a period of 2.2 yr, i.e. about twice the production lag. This is the cobweb result that Coase and Fowler (1935) found incompatible with the observed British pig cycle of about four years. Adding the expectation stage



lengthens the period to 3.7 yr. In the diagram, adaptive expectations are simply one more relaxing node in the loop, which is the extension that Nerlove (1958) introduced into the cobweb model.

The classical cobweb theorem (Ezekiel, 1938) is itself a limit of the pilea formulas. It assumes naive expectations and a fixed production lag L , and predicts oscillations with a period $2L$ that decay if and only if $\varepsilon_s/\varepsilon_d < 1$, i.e. $S > -1$. A fixed lag can be represented by N consecutive stages of equal duration, $\tau_j = L/N$, a chain that approaches a fixed delay as N grows. The phase condition, Eq. (24), then gives $\omega_0^- L/N = \tan(\pi/N)$, and Eq. (25) gives

$$T_0 = \frac{2\pi L}{N \tan(\pi/N)} \xrightarrow{N \rightarrow \infty} 2L, \quad S_0^- = - \left[\cos\left(\frac{\pi}{N}\right) \right]^{-N} \xrightarrow{N \rightarrow \infty} -1, \quad (149)$$

with $S_0^- = -8$ for $N = 3$ (Eq. (34)), -1.65 for $N = 10$ and -1.05 for $N = 100$. The threshold $S = -1$ of the cobweb theorem, the ‘folk value’ discussed in Sect. 3.3, is therefore the limit of a sharply defined delay, and its period of $2L$ is that of such a delay. Distributed lags, represented by few stages, and adaptive expectations both move the threshold to much more negative values of S , i.e. they stabilize the market, in agreement with Nerlove (1958), and both lengthen the period. The same feedback strength $S = -2$ is stable in the three-stage model above and unstable in the cobweb limit. Whether the hog cycle is self-sustained or only re-excited by shocks thus depends on how sharply defined the biological delays are – a question for data, which the diagram turns into a single comparison of S with S_0^- .

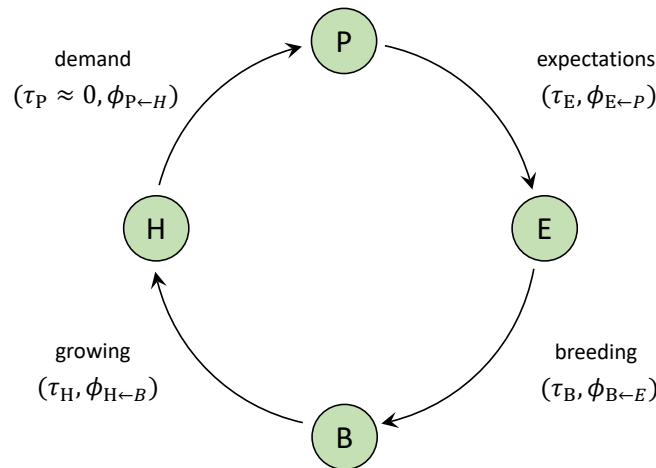


Figure 15. The hog cycle as a four-variable pilea loop (Sect. 7.6): the market price P sets the price expected by producers E , which determines the breeding herd B , which determines the slaughter supply H after the fattening period, which lowers the price through demand ($\phi_{P \leftarrow H} < 0$). Because the price adjusts almost instantaneously, the loop acts as a three-process loop with one oscillatory mode (Sect. 3.4); in the limit of naive expectations and a sharply defined production lag, its wobble threshold approaches the cobweb value $S = -1$, Eq. (149).

8 Discussion and conclusion

1095 9 Discussion and conclusion

This paper provides the first set of building blocks for a modular feedback analysis. Three rules suffice to reduce the diagrams considered here (Sect. 6). Processes in series around one loop multiply their dependency strengths and add their timescales (pilea topology, Sects. 3 and 3.6). Loops that share one central variable add their feedback strengths (daisy topology, Sect. 4). Two daisies coupled through their centres combine their stability margins, $1 - S$, multiplicatively (bouquet topology, Sect. 5). In all cases the result has the same form: a tipping point at $S_{\text{eff}} = 1$, which does not depend on the timescales, and a growth or decay rate near this threshold of $\lambda \approx (S_{\text{eff}} - 1)/\tau_{\text{eff}}$, which does. A second route to instability, growing oscillations through a Hopf bifurcation, exists for loops of three or more processes, but only at strongly negative feedback strengths that follow from Eq. (25). The often-quoted threshold $S = -1$ is the limit of a sharply defined delay, as in the cobweb model (Eq. (149)); for a few distributed processes the threshold lies much lower, e.g. at $S = -8$ for three equal timescales.

1105 The approach is intended for scientists who know their system well but do not want to set up and analyse a full Jacobian matrix. Its use requires three steps: drawing the diagram, estimating how strongly each variable responds to its driver, and estimating how fast it does so. The cookbook of Sect. 7 offers two uses of the same diagram. If governing equations are known (KL1, KL2), the diagram reduces a stability analysis to a few derivatives and timescales (Sects. 7.2–7.4). If they are not known (KL3), the diagram together with estimated dependency strengths and timescales yields both a stability statement (Fig. 6) and, through the generic relaxation equation, Eq. (135), a model that can be integrated in time (Fig. 12, Sects. 7.5–7.6). In each example the simple rules reproduce a known result: the fold of the Stommel model, the classical amplification of climate sensitivity by radiative feedbacks, the Taylor principle of monetary policy (Eq. (142)) and the cobweb theorem. These agreements are a test of the approach, and they show that the diagram identifies which process is responsible for a given stability property.

1115 For the Earth system, the daisy of global mean temperature (Sect. 7.4) gives a quantitative perspective on self-reinforcing warming. The radiative and carbon-cycle feedbacks add up to $S \approx 0.7$, and the carbon-cycle feedbacks of permafrost and forests contribute only about 0.04, far less than the remaining margin to the threshold. A feedback-driven runaway of global mean temperature is therefore not supported by the feedback strengths. The relevant risk lies elsewhere: individual petals, such as an ice sheet or the overturning circulation, can tip at their own thresholds, $S_i = 1$, while the sum over all petals remains well below one. Whether such local transitions trigger each other depends on direct couplings between the elements, which are outside the daisy topology and are the subject of the next part of this series.

1125 The approach has limits. First, all linear results describe only the onset of an instability and its initial rate; following a tipping event into the nonlinear regime requires the time-dependent model of the second route, integrated numerically. Second, the estimate $\lambda \approx (S - 1)/\tau$ is accurate only close to $S = 1$; further from the threshold the exact rates follow from the characteristic polynomials given in the text, which standard numerical tools solve directly. Third, every variable is assumed to relax towards a state set by its drivers (Eq. (2)); self-amplification of a single variable has to be represented by an additional variable and an explicit loop (Sect. 2.1), and a centre without its own restoring dynamics requires the modified condition of Sect. 4.4. Fourth,



the relaxation equation, Eq. (4), is exact only to linear order near a steady state, and any model built from it is only as reliable as the estimated dependency strengths and timescales. Because S is a product around a loop, the relative uncertainties of the individual dependency strengths combine (for small, independent errors, in quadrature); because petal strengths add, so do their absolute uncertainties, and the uncertainty of a daisy is dominated by its most uncertain petals. Both properties indicate directly which estimate most needs to be improved.

Subsequent parts of this series extend the set of building blocks. Part II derives general rules for arbitrary diagrams from their loop structure, treats loops that share a process rather than a variable (the rose topology), and applies these rules to epidemics, financial crises and political polarization. The aim of the series is a modular toolkit with which feedback structures can be assembled, analysed and turned into models from a small number of estimated quantities. The examples of this paper – from ice sheets, ocean circulation and global temperature to wage setting and pig farming – show that the same two numbers, S and τ , and the same threshold, $S = 1$, apply in nature and in society.

Appendix A: Determinant of the one-loop Jacobian

We show that for the $N \times N$ matrix $\mathbb{M} \equiv \mathbb{J} - \lambda \mathbb{1}$ of Sect. 3.2, with $M_{ii} = -(r_i + \lambda) \equiv -d_i$, $M_{i,i-1} = r_i \phi_i$ (cyclically, i.e. $M_{1,N} = r_1 \phi_1$), and all other entries zero,

$$\det(\mathbb{M}) = (-1)^N \left(\prod_{i=1}^N d_i - \prod_{i=1}^N r_i \phi_i \right). \quad (\text{A1})$$

Expand $\det(\mathbb{M})$ along its first row, which has only two non-zero entries, $M_{11} = -d_1$ and $M_{1N} = r_1 \phi_1$:

$$\det(\mathbb{M}) = M_{11} C_{11} + M_{1N} C_{1N}, \quad (\text{A2})$$

with cofactors $C_{11} = (+1)^{1+1} \det(\mathbb{M}_{\hat{1}\hat{1}})$ and $C_{1N} = (-1)^{1+N} \det(\mathbb{M}_{\hat{1}\hat{N}})$, where $\mathbb{M}_{\hat{1}\hat{k}}$ denotes $\mathbb{M}$ with row 1 and column k removed. Deleting row and column 1 breaks the cycle: $\mathbb{M}_{\hat{1}\hat{1}}$ is lower triangular, with diagonal entries $-d_2, \dots, -d_N$, so $\det(\mathbb{M}_{\hat{1}\hat{1}}) = \prod_{i=2}^N (-d_i)$. Deleting row 1 and column N instead removes row N 's diagonal entry (which sat in column N) while leaving every other row's two entries intact; relabelling, $\mathbb{M}_{\hat{1}\hat{N}}$ is upper triangular with diagonal entries $r_2 \phi_2, \dots, r_N \phi_N$, so $\det(\mathbb{M}_{\hat{1}\hat{N}}) = \prod_{i=2}^N r_i \phi_i$. Collecting terms,

$$\begin{aligned} \det(\mathbb{M}) &= (-d_1) \cdot (-1)^{N-1} \prod_{i=2}^N d_i + (r_1 \phi_1) \cdot (-1)^{N+1} \prod_{i=2}^N r_i \phi_i \\ &= (-1)^N \prod_{i=1}^N d_i + (-1)^{N+1} \prod_{i=1}^N r_i \phi_i = (-1)^N \left(\prod_{i=1}^N d_i - \prod_{i=1}^N r_i \phi_i \right), \end{aligned} \quad (\text{A3})$$

which is Eq. (A1). Setting $d_i = \lambda + r_i$ recovers the determinant quoted in Sect. 3.2. The same argument, with the appropriate rows and columns deleted, underlies the daisy determinant of Eq. (64).



Appendix B: List of symbols

- 1155 A_i a generic variable of the system.
- C the central variable of a daisy topology.
- C_p the effective heat capacity per unit area of the climate system in the greenhouse example, Eq. (123); not to be confused with the daisy centre C .
- κ_i, κ_P the radiative feedback parameter of petal i and the Planck response of the radiation budget (both in $\text{W m}^{-2}\text{K}^{-1}$),
 1160 Sect. 7.4; $S_i = \kappa_i / \kappa_P$. Denoted λ_i in the climate-feedback literature, a symbol reserved here for Lyapunov exponents.
- F_i the (KL1) prognostic time-evolution function of variable A_i , Eq. (1).
- f_i the (KL2) diagnostic steady-state function of variable A_i , Eq. (3).
- τ_i the intrinsic relaxation (restoring) timescale of variable A_i , Eq. (2).
- $r_i \equiv 1/\tau_i$ the corresponding relaxation rate.
- 1165 $\phi_{i \leftarrow j} \equiv \partial f_i / \partial A_j$ the dependency strength of A_j on A_i .
- S the feedback strength of a loop or daisy (dimensionless); product over a loop (Eq. (50)) or sum over petals (Eq. (68)).
- S_i the feedback strength of petal i of a daisy.
- S_{eff} the effective combined feedback strength of a nested or bouquet topology, Eq. (98).
- s_i the feedback strength of petal i of a centre-manifold daisy, normalized by the centre's (arbitrary) reference timescale
 1170 (Sect. 4.4); carries units of inverse time.
- τ the effective (total) timescale of a loop or daisy, Eqs. (51), (69).
- λ a Lyapunov exponent: the exponential growth ($\lambda > 0$) or decay ($\lambda < 0$) rate of a small perturbation.
- λ^* the dominant Lyapunov exponent near the saddle-node (tipping) threshold, $\lambda^* \approx (S - 1)/\tau$.
- ω an oscillation (angular) frequency; $\omega_k^\pm, \omega_H$ denote specific Hopf/wobble frequencies.
- 1175 g_i the gain factor of petal i (Sect. 4.5); numerically equal to S_i .
- KL1, KL2, KL3** knowledge levels: a full prognostic equation, a steady-state relation plus an estimated timescale, and a purely qualitative estimate of both, respectively (Sect. 2.1).



Appendix C: Glossary of notation

- Dependency strength** The partial derivative of one process's steady-state output with respect to a variable it depends on,
 1180 $\phi_{i \leftarrow j} = \partial f_i / \partial A_j$; dimensionless once multiplied by the appropriate timescale ratio, but not necessarily so on its own.
- Feedback strength (S)** The product (around a single loop) or sum (across a daisy's petals) of the dependency strengths involved; a dimensionless number whose value of 1 marks the saddle-node tipping point of the system.
- Tipping point / saddle-node bifurcation** The threshold $S = 1$ at which a real Lyapunov exponent crosses zero from below; beyond it, a perturbation no longer decays but grows.
- 1185 **Wobble instability / Hopf bifurcation** An oscillatory instability arising when a complex-conjugate pair of Lyapunov exponents crosses the imaginary axis; for a single loop this occurs at specific, generally strongly negative values of S (Sect. 3.3), not simply at $S = -1$.
- Lyapunov exponent** The exponential rate, λ , at which a small perturbation from the steady state grows ($\lambda > 0$) or decays ($\lambda < 0$) in time.
- 1190 **Pilea topology** A single closed feedback loop of $N \geq 2$ variables (Sect. 3).
- Daisy topology** Several two-variable loops ('petals') sharing one central variable (Sect. 4).
- Nested / bouquet daisy** Two or more daisies coupled by replacing one daisy's petal with a second daisy's centre, or by a direct link between two centres (Sects. 5–6).
- Centre manifold (here)** A central variable with no restoring dynamics of its own, driven entirely by its petals (Sect. 4.4).
- 1195 **Gain factor** The equilibrium amplification of an external forcing contributed by one feedback petal; numerically equal to that petal's feedback strength S_i (Sect. 4.5).
- Knowledge level (KL1/KL2/KL3)** How much is known about a variable's governing dynamics: a full time-dependent equation (KL1), a steady-state relation plus an estimated timescale (KL2), or only a qualitative, estimated dependency strength and timescale (KL3); Sect. 2.1.

1200 Appendix D: Additional calculations for the oscillatory modes

D1 Hopf bifurcation in a two-petal daisy without tipping

Section 4.3 showed that, for petals of comparable timescale, the Hopf boundary (RH-3) always lies beyond the saddle-node boundary (RH-1): tipping precedes wobbling. We now show, with an explicit, verified example, that sufficiently asymmetric timescales can reverse this order.



1205 Take $\tau_C = \tau_0 = 1$ (in arbitrary time units, so $r_C = r_0 = 1$) and a petal 1 that is much faster, $r_1 = 50$. Fix $S_0 = -6$ (petal 0 strongly stabilizing) and increase S_1 from a very negative value. Solving the characteristic polynomial, Eq. (54), numerically for these parameters shows that the dominant real part of the eigenvalues crosses zero at $S_1 \approx 2.045$, where the crossing eigenvalues are a genuine complex-conjugate pair, $\lambda \approx \pm 2.18i$ – a Hopf bifurcation, exactly where the Routh–Hurwitz boundary (RH-3) predicts it, $S_1 = [(r_0 + r_1)/r_C - r_0 S_0 / (r_C + r_1)] (r_C + r_0) / r_1 \approx 2.045$. Yet at this point $S_0 + S_1 \approx -3.96$, far inside
 1210 the stable region of condition (RH-1), $S_0 + S_1 < 1$; tipping, by contrast, would require $S_1 = 1 - S_0 = 7$. This daisy therefore wobbles at $S_1 \approx 2.05$, long before it could ever tip at $S_1 = 7$.

The mechanism is the same one identified for the single loop in Sect. 3.3: a wobble instability is fundamentally a negative-feedback phenomenon, requiring at least one sufficiently negative S_i , combined here with a strong separation between the petals’ timescales ($r_1/r_0 = 50$ above). Petals of similar strength and comparable timescale, by contrast, simply tip before they
 1215 can wobble (Sect. 4.3).

D2 Oscillatory modes in the general daisy topology

The same approach generalizes to a daisy with any number of petals. Setting $\lambda = i\omega$ in the characteristic equation, Eq. (64), and separating real and imaginary parts gives two simultaneous real conditions relating ω^2 to the feedback strengths S_i and rates r_i ; a genuine Hopf bifurcation occurs only where both are satisfied at once (just as the single boundary RH-3 for $N = 2$ petals
 1220 in Sect. 4.3 arose from combining both the real and the imaginary conditions, with ω^2 eliminated between them). Carrying this elimination out explicitly becomes algebraically more involved as the number of petals grows, but the qualitative lesson of Appendix D1 persists at any N : a daisy wobbles before it tips only where at least one petal contributes a sufficiently negative feedback strength together with a sufficiently different timescale from the rest, and otherwise tips first. That a negative petal is necessary holds for any number of petals: without negative loops, the Lyapunov exponent with the largest real part is real
 1225 (part II, Appendix A2, part f).

Appendix E: The runaway greenhouse limit in more detail

Section 7.4 noted that increasing optical depth weakens the Planck response in the simple grey-gas model of Eq. (124). We now discuss more carefully why a genuine upper limit on outgoing longwave radiation – rather than merely a region of slowed growth – arises once the full vertical structure of a moist atmosphere is taken into account, following Nakajima et al. (1992)
 1230 and the earlier argument of Simpson (1928).

The key simplification of Eq. (124) is to let the outgoing flux depend on the *surface* temperature T directly, through a single bulk optical depth $\chi(T)$. In reality, the flux escaping to space originates from the level in the atmosphere where the optical depth looking upward from space first reaches order unity – the *emission level* – and it is the temperature *at that level*, T_e , not the surface temperature, that sets the outgoing flux through $R_{\text{LW}} \sim \sigma T_e^4$. For a dry atmosphere, T_e rises together with
 1235 the surface temperature, and R_{LW} increases without bound as the surface warms, exactly as the Planck feedback of Sect. 7.4 assumes. For a moist atmosphere whose dominant greenhouse gas is water vapour, however, the water-vapour path above the



emission level is set by the *saturation* vapour pressure at the local temperature T_e itself, via the Clausius–Clapeyron relation – so that the emission level’s own temperature becomes, self-consistently, almost independent of the (much warmer) surface temperature below it, once the atmosphere is warm and moist enough that this self-consistent balance is reached throughout the troposphere. The outgoing flux then approaches a finite limit,

$$R_{\text{LW}}^{\text{max}} \sim \sigma T_e^4 \Big|_{\text{self-consistent}}, \quad (\text{E1})$$

essentially independent of surface temperature – the Simpson–Nakajima limit. If the absorbed shortwave radiation, $R_{\text{SW}} = (1 - \alpha)R_{\text{Sun}}$, exceeds $R_{\text{LW}}^{\text{max}}$, Eq. (123) has no steady state at all: the surface warms without bound (in this idealized picture, until the ocean has entirely evaporated), a true runaway greenhouse instability, qualitatively distinct from the ordinary water-vapour feedback instability of Sect. 7.4, which merely requires $S > 1$ but still admits a (hotter) steady state.

Quantitative radiative-transfer calculations place $R_{\text{LW}}^{\text{max}}$ at roughly $280\text{--}310 \text{ W m}^{-2}$ for Earth-like atmospheres (Nakajima et al., 1992), comfortably above the roughly 240 W m^{-2} of absorbed solar radiation Earth receives today. A dedicated assessment of whether increasing CO_2 alone could close this gap concludes that it could not: the total carbon available in Earth’s fossil-fuel reserves is far too small to trigger a true runaway transition, even though it is more than enough to drive a conventional, bounded warming through the ordinary greenhouse effect (Goldblatt et al., 2013). The distinction matters for the framework of this paper: the loop strength S of Sect. 7.4 governs the ordinary, bounded water-vapour feedback and its associated tipping behaviour, while the runaway limit of Eq. (E1) is reached along the other route to the same threshold identified in Sect. 7.4: not through a growing petal strength κ_i , but through the saturation of the Planck response itself, $\kappa_P \rightarrow 0$, at which the centre’s restoring timescale $\tau_T = C_p/\kappa_P$ diverges and every $S_i = \kappa_i/\kappa_P$ with it.

Code and data availability. No code or model output was generated for this theoretical study; all results follow from the analytical derivations given in the text.

Competing interests. The author declares that he has no conflict of interest.

Acknowledgements. This paper is supported by the EconX project of the Max Planck Institute of Geoanthropology in Jena. The EconX project (2023 FWF 0002) is funded by the ”FTI-Thüringen-Forschung” programme as part of the umbrella brand ”Thüringen MOTIVation”. I have used an LLM to check equations, calculations and text.



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