



Thermal Offset Model: Derivation, Implications, and Terminology

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5 **Abstract.** Subsurface mean annual ground temperatures in frozen-ground regions typically differ from those at the ground surface because the thermal conductivity of the seasonal freeze–thaw layer changes between its thawed and frozen states. Mean annual ground temperature at the base of the freeze–thaw layer is therefore a more representative measure of the thermal state of frozen ground than mean annual ground surface temperature. A widely used analytical model for estimating the mean annual ground temperature at the base of the freeze–thaw layer from ground surface thawing and freezing indices, hereafter referred to as the thermal offset model, was introduced over three decades ago. However, its physical basis, assumptions, boundary conditions, and limits of applicability have remained poorly understood because previous publications have either presented the model without a derivation, derived it empirically, or provided mathematically complicated and cumbersome analytical derivations. This paper provides a new and simple analytical derivation of the model using the Stefan equation for seasonal thawing and freezing. This derivation is more physically intuitive and easier to understand and explains why the model takes different forms under permafrost and seasonally frozen ground conditions, clarifies how it is controlled by the ratio of effective thawed to frozen ground thermal conductivities, and reveals how it relates to the thickness of the freeze–thaw layer and latent heat. It also shows that, under sinusoidal ground surface temperature variations, the model can be reformulated in terms of the mean annual ground surface temperature and annual ground surface temperature amplitude. Lastly, it addresses several terminological ambiguities that have persisted since the model was introduced. Overall, the paper shows that the thermal offset model is physically based but strongly simplified, clarifies its behaviour and limits of applicability, and provides a basis for its further development.

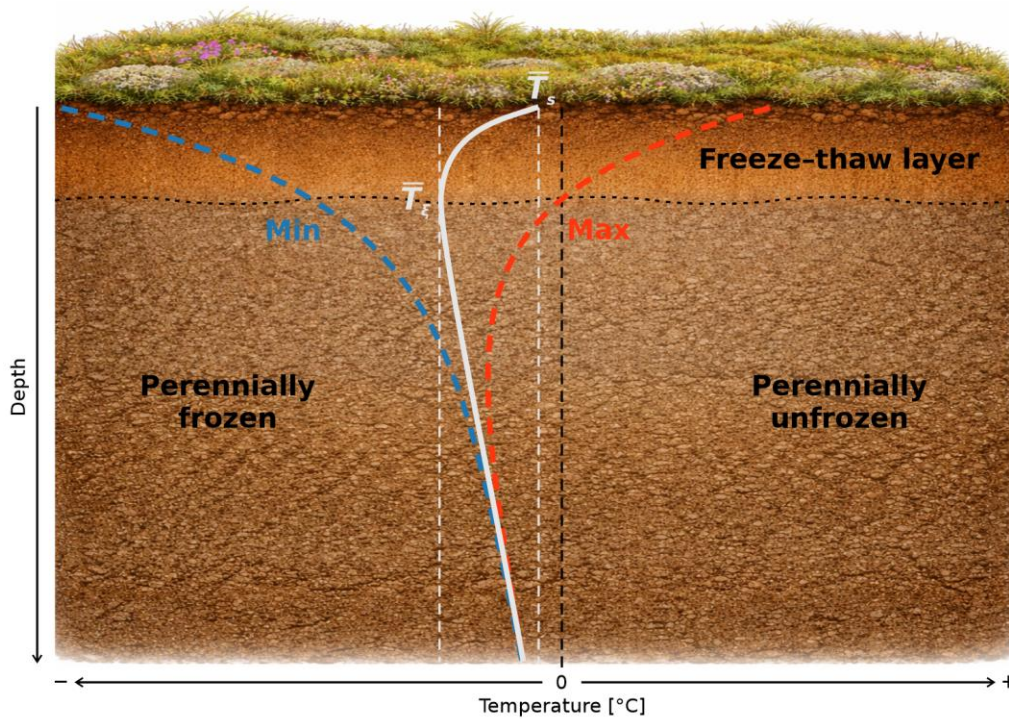
1 Introduction

It has been over three decades since an analytical model for the mean annual ground temperature at the base of the seasonal freeze–thaw layer defined in terms of ground surface thawing and freezing indices was first introduced to an international readership (Romanovsky and Osterkamp, 1995). A year later, this solution was popularized as the so-called *temperature at the top of permafrost*, or *TTOP*, model (Smith and Riseborough, 1996). However, it is hereafter referred to as the thermal offset model. Since it is an ingeniously simple representation of how the frozen-ground thermal regime relates to ground surface temperature conditions that requires only a few inputs, the model soon became a widely used tool especially for estimating ground temperatures in permafrost regions and for mapping permafrost extent at various spatial and temporal scales (e.g.,



30 Juliussen and Humlum, 2007; Gislén et al., 2013, 2017; Westermann et al., 2015; Way and Lewkowicz, 2018; Obu et al., 2019, 2020; Garibaldi et al., 2021; Kaplan Pastiriková et al., 2023; Vegter et al., 2024; Li et al., 2025; Zhao et al., 2025). Simple and elegant, the model relates the mean annual ground temperature at the base of the freeze–thaw layer $\bar{T}_\xi$ [°C] to the mean annual ground surface temperature $\bar{T}_s$ [°C] by incorporating the thermal offset $\Delta\bar{T}_\xi$ [°C] (Romanovsky and Osterkamp, 1995), which is defined as the difference between the two temperatures (Fig. 1):

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$$\Delta\bar{T}_\xi = \bar{T}_\xi - \bar{T}_s. \quad (1)$$



40 **Figure 1: Schematic illustration showing the minimum (blue), maximum (orange), and mean (white) annual ground temperature profiles under permafrost conditions with an annually periodic steady-state (equilibrium) thermal regime. Note that the mean annual ground temperature at the base of the seasonal freeze–thaw layer ($\bar{T}_\xi$) differs from the mean annual ground surface temperature ($\bar{T}_s$), which defines the thermal offset (white vertical dashed lines). The thermal offset is typically negative, as shown here, which arises when the effective thawed thermal conductivity is lower than the effective frozen thermal conductivity. The geothermal gradient shown below the freeze–thaw layer is only illustrative, as geothermal heat flux is neglected in the thermal offset model.**

45 It has been stated that the thermal offset mostly depends on the contrast between the effective thawed and frozen thermal conductivities (effective thermal conductivity ratio) of the freeze–thaw layer, which is closely related to its water/ice content (Romanovsky and Osterkamp, 1995; Smith and Riseborough, 1996), as well as on the thickness of the freeze–thaw layer that acts as a thermal buffer (Osterkamp and Romanovsky, 1998). However, the model has either been published without derivation or derived purely empirically (Kudryavtsev, 1981; Smith and Riseborough, 1996, 2002; Riseborough and Smith, 1998),



50 whereas available analytical derivations (Romanovsky and Osterkamp, 1995; Westermann et al., 2015) have been too complicated and difficult to follow for a broad readership and typical model users, who are mainly geographers (see Appendices A and B). Despite its simplicity, the model has therefore remained somewhat puzzling, especially regarding its differing behaviour under permafrost and seasonally frozen ground conditions and its relationship to the thickness of the freeze–thaw layer and latent heat associated with water–ice phase transitions, neither of which is explicitly represented in the model equations (e.g., Osterkamp and Romanovsky, 1998; Smith and Riseborough, 1998).

55 Here, I provide a new and simple analytical derivation of the model for the mean annual ground temperature at the base of the freeze–thaw layer using the Stefan (1891) equation for seasonal thawing and freezing. This derivation is intended to be more physically intuitive and digestible for typical model users than the previous derivations (Romanovsky and Osterkamp, 1995; Westermann et al., 2015). On this basis, I elucidate the model physics, assumptions, and boundary conditions, as well as its links to the effective thermal conductivity ratio, thickness of the freeze–thaw layer, and latent heat. Under sinusoidal ground surface temperature variations, I further demonstrate its previously unrecognized equivalence to the solution introduced by Garagulya (1990), which is formulated in terms of the mean annual ground surface temperature and annual ground surface temperature amplitude. Lastly, I address some terminology issues that have persisted since the model was first published and may have contributed to questionable applications of the model under permafrost and seasonally frozen ground conditions.

65 2 Derivation

At the outset, the mean annual ground surface temperature can be calculated as

$$\bar{T}_s = \frac{I_t - I_f}{P}, \quad (2)$$

70 where I_t [°C d] and I_f [°C d] are the ground surface thawing and freezing indices, respectively, and P [365 d] is the duration of one year. Here, the thawing and freezing indices represent the time integrals of positive ground surface temperatures and the absolute values of negative ground surface temperatures, respectively, and are therefore both expressed as positive quantities:

$$I_t = \int_0^P \max[T_s(\tau), 0] d\tau, \quad (3)$$

$$I_f = \int_0^P \max[-T_s(\tau), 0] d\tau, \quad (4)$$

75 where $T_s(\tau)$ [°C] is the ground surface temperature at time τ . The ground surface thawing and freezing indices are frequently estimated from the air thawing and freezing indices through multiplying them by the so-called thawing and freezing n -factors, respectively (Smith and Riseborough, 1996, 2002). The n -factors are semi-empirical scaling coefficients that account for offsets between ground surface and air temperatures, especially those arising from the buffering effects of vegetation and snow (Lunardini, 1978). However, these surface offsets are beyond the scope of this paper, which hereafter addresses only ground temperatures. The physical quantities used in the main text are summarized in Table 1.



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Table 1: Physical quantities used in the main text. Typical values or ranges are provided where applicable.

Symbol	Physical meaning	Unit	Typical value or range
$\bar{T}_s$	Mean annual ground surface temperature	°C	–
$\bar{T}_\xi$	Mean annual ground temperature at the base of the freeze–thaw layer	°C	–
$\Delta\bar{T}_\xi$	Thermal offset	°C	–
I_t	Ground surface thawing index	°C d	$I_t \geq 0$
I_f	Ground surface freezing index	°C d	$I_f \geq 0$
P	Duration of one year	d	365
$T_s(\tau)$	Ground surface temperature at time τ	°C	–
τ	Time within the annual period	d	$0 \leq \tau \leq P$
ΔI_t	Thawing index consumed in thawing of the freeze–thaw layer	°C d	$0 \leq \Delta I_t \leq I_t$
ΔI_f	Freezing index consumed in freezing of the freeze–thaw layer	°C d	$0 \leq \Delta I_f \leq I_f$
ξ_t	Thaw depth	m	$\xi_t \geq 0$
ξ_f	Freeze depth	m	$\xi_f \geq 0$
k_t	Effective thawed thermal conductivity	W m ^{−1} K ^{−1}	$k_t > 0$
k_f	Effective frozen thermal conductivity	W m ^{−1} K ^{−1}	$k_f > 0$
k_t/k_f	Ratio of effective thawed to frozen thermal conductivity	–	$k_t/k_f > 0$
L	Specific latent heat of the water–ice phase transition	J kg ^{−1}	334 000
ϕ_w	Volumetric water content	–	0–1
ϕ_i	Volumetric ice content	–	0–1
ρ_w	Density of water	kg m ^{−3}	1000
ρ_i	Density of ice	kg m ^{−3}	917
z	Depth below the ground surface within the freeze–thaw layer	m	$0 \leq z \leq \xi_t = \xi_f$
$\Delta I_t(z)$	Thawing index consumed between depths z and ξ_t	°C d	$0 \leq \Delta I_t(z) \leq \Delta I_t$
$\Delta I_f(z)$	Freezing index consumed between depths z and ξ_f	°C d	$0 \leq \Delta I_f(z) \leq \Delta I_f$
A_s	Annual ground surface temperature amplitude	°C	$A_s \geq 0$

Seasonal thawing and freezing take place at the ground surface and in the shallow subsurface, whereas at and below the base of this freeze–thaw layer, there is no thawing under permafrost conditions and no freezing under seasonally frozen ground conditions. A simple thought exercise therefore suggests that thawing and freezing indices decrease from the ground surface towards the base of the freeze–thaw layer. Using these premises, Eq. (2) can be extended to describe the mean annual ground temperature at the base of the freeze–thaw layer as follows



$$\bar{T}_{\xi} = \frac{(I_t - \Delta I_t) - (I_f - \Delta I_f)}{P}, \quad (5)$$

where ΔI_t [°C d] and ΔI_f [°C d] are the thawing and freezing indices, respectively, that are consumed in thawing and freezing of the freeze–thaw layer from the ground surface downwards.

The thickness of the freeze–thaw layer can be estimated using the simplified Stefan (1891) equation. It assumes that the temperature is linearly distributed between the upper boundary and the moving freeze–thaw front, which is maintained at 0 °C, and that heat conducted to or from the freeze–thaw front is fully balanced by latent heat absorbed or released during the water–ice phase transition. Sensible heat storage and heat conduction beneath the freeze–thaw front are neglected (Kurylyk, 2015).

The Stefan equations for the thaw ξ_t [m] and freeze ξ_f [m] depths in homogeneous ground with constant physical properties are (Lunardini, 1981)

$$\xi_t = \sqrt{\frac{2k_t \Delta I_t}{L \phi_w \rho_w}}, \quad (6)$$

$$\xi_f = \sqrt{\frac{2k_f \Delta I_f}{L \phi_i \rho_i}}, \quad (7)$$

where k_t [W m⁻¹ K⁻¹] and k_f [W m⁻¹ K⁻¹] are the effective thawed and frozen thermal conductivities, respectively, L [334000 J kg⁻¹] is the specific latent heat of the water–ice phase transition, ϕ_w [–] and ϕ_i [–] are the volumetric water and ice contents, respectively, and ρ_w [1000 kg m⁻³] and ρ_i [917 kg m⁻³] are the densities of water and ice, respectively. Note that the thawing and freezing indices in Eqs. (6) and (7) must be multiplied by the conversion factor 86 400 s d⁻¹ to ensure dimensional consistency. The denominators in Eqs. (6) and (7) are equal when the volume change associated with the water–ice phase transition at 0 °C is considered, while freeze–thaw-induced ground deformation and water migration are neglected. This follows from mass conservation, which requires that

$$\phi_w \rho_w = \phi_i \rho_i. \quad (8)$$

Since recurrent seasonal thawing and freezing of the freeze–thaw layer under an annually periodic steady-state thermal regime imply equal thaw and freeze depths, Eqs. (6) and (7) can be combined to give

$$\sqrt{\frac{2k_t \Delta I_t}{L \phi_w \rho_w}} = \sqrt{\frac{2k_f \Delta I_f}{L \phi_i \rho_i}}. \quad (9)$$

Using the mass-conservation relationship in Eq. (8), Eq. (9) simplifies to

$$k_t \Delta I_t = k_f \Delta I_f. \quad (10)$$

Eq. (10) shows that the thawing and freezing indices consumed in thawing and freezing of the freeze–thaw layer are directly proportional and can therefore be expressed in terms of each other as follows

$$\Delta I_t = \frac{k_f}{k_t} \Delta I_f, \quad (11)$$

$$\Delta I_f = \frac{k_t}{k_f} \Delta I_t. \quad (12)$$



Under permafrost conditions, the thawing index is zero at the base of the freeze–thaw layer (Fig. 1), because it is fully consumed in thawing of the freeze–thaw layer and the associated absorption of latent heat. By contrast, the freezing index at the base of the freeze–thaw layer is generally non-zero (Fig. 1), because only a portion of it is consumed in freezing of the freeze–thaw layer and the associated release of latent heat, except at the boundary between permafrost and seasonally frozen ground, where it is fully consumed. Under seasonally frozen ground conditions, the opposite situation exists and the freezing index is zero at the base of the freeze–thaw layer (Fig. 1), because it is fully consumed in freezing of the freeze–thaw layer and the associated release of latent heat. By contrast, the thawing index at the base of the freeze–thaw layer is non-zero (Fig. 1), because only a portion of it is consumed in thawing of the freeze–thaw layer and the associated absorption of latent heat. This implies that

$$\Delta I_t = I_t, \Delta I_f \leq I_f, \bar{T}_\xi \leq 0, \quad (13)$$

$$\Delta I_t < I_t, \Delta I_f = I_f, \bar{T}_\xi > 0. \quad (14)$$

Using the considerations given by Eqs. (13) and (14), Eq. (5) therefore becomes

$$\bar{T}_\xi = \begin{cases} \frac{\Delta I_f - I_f}{P}, \bar{T}_\xi \leq 0 \\ \frac{I_t - \Delta I_t}{P}, \bar{T}_\xi > 0 \end{cases}. \quad (15)$$

Eq. (15) shows that, under permafrost conditions, the mean annual ground temperature at the base of the freeze–thaw layer equals the negative of the ground surface freezing index reduced by the portion consumed in freezing the seasonally thawed layer, divided by the duration of one year. Conversely, under seasonally frozen ground conditions, it equals the ground surface thawing index reduced by the portion consumed in thawing the seasonally frozen layer, divided by the duration of one year. Unfortunately, Eq. (15) is impractical because calculating the thawing and freezing indices consumed in thawing and freezing the freeze–thaw layer would require accounting for the thickness of the freeze–thaw layer, effective thawed or frozen thermal conductivity, and volumetric water or ice content (see Eqs. 6 and 7). Substituting Eqs. (11) and (12) into Eq. (15), however, yields

$$\bar{T}_\xi = \begin{cases} \frac{\frac{k_t}{k_f} I_t - I_f}{P}, \bar{T}_\xi \leq 0 \\ \frac{I_t - \frac{k_f}{k_t} I_f}{P}, \bar{T}_\xi > 0 \end{cases}, \quad (16)$$

which is identical to the equations published previously by Romanovsky and Osterkamp (1995) and Riseborough and Smith (1998), but is derived in a different manner (see Appendix A). Eq. (16) has the advantage of requiring only the ground surface thawing and freezing indices and the effective thawed and frozen thermal conductivities or the effective thermal conductivity ratio as input variables.

Subsequently, the thermal offset can be obtained by subtracting Eq. (2) from Eq. (16) as follows



$$\Delta \bar{T}_\xi = \begin{cases} \left(\frac{k_t - 1}{k_f} \right) \frac{I_t}{P}, \bar{T}_\xi \leq 0, \\ \left(1 - \frac{k_f}{k_t} \right) \frac{I_f}{P}, \bar{T}_\xi > 0 \end{cases}, \quad (17)$$

which is also identical to the thermal offset equation shown by Romanovsky and Osterkamp (1995), but is derived in a different manner (see Appendix A).

Notably, the conditions in Eqs. (13) to (17) can also equivalently be expressed in terms of the potential thaw and freeze depths calculated using the ground surface thawing and freezing indices. Under permafrost conditions, the freeze depth must equal or exceed the thaw depth, whereas under seasonally frozen ground conditions, the thaw depth must exceed the freeze depth:

$$\sqrt{\frac{2k_t I_t}{L\phi_w \rho_w}} \leq \sqrt{\frac{2k_f I_f}{L\phi_i \rho_i}}, \bar{T}_\xi \leq 0, \quad (18)$$

$$\sqrt{\frac{2k_t I_t}{L\phi_w \rho_w}} > \sqrt{\frac{2k_f I_f}{L\phi_i \rho_i}}, \bar{T}_\xi > 0. \quad (19)$$

Recalling the mass-conservation relationship in Eq. (8), Eqs. (18) and (19) reduce to

$$k_t I_t \leq k_f I_f \Leftrightarrow \bar{T}_\xi \leq 0, \quad (20)$$

$$k_t I_t > k_f I_f \Leftrightarrow \bar{T}_\xi > 0. \quad (21)$$

These equivalences explain why previous studies have used the index- and temperature-based conditions interchangeably to distinguish between permafrost and seasonally frozen ground conditions (e.g., Romanovsky and Osterkamp, 1995; Riseborough, 2002, 2007; Gislén et al., 2013, 2017; Westermann et al., 2015, 2023; Way and Lewkowicz, 2018; Obu et al., 2019, 2020; Vegter et al., 2024; Zhao et al., 2025).

3 Implications

3.1 Model behaviour and its physical controls

Unlike Eq. (15), Eq. (16) is somewhat counterintuitive because the thawing index reappears in the permafrost variant of the model and the freezing index in the seasonally frozen ground variant without representing their actual values. This is because at and below the base of the freeze–thaw layer the thawing index is zero under permafrost conditions and the freezing index is zero under seasonally frozen ground conditions. Since the thawing and freezing indices consumed to thaw and freeze the freeze–thaw layer are directly proportional (see Eqs. 11 and 12), under permafrost conditions, the ground surface thawing index scaled by the effective thermal conductivity ratio represents the portion of the freezing index consumed in freezing of the seasonally thawed layer and the associated release of latent heat. Conversely, under seasonally frozen ground conditions, the ground surface freezing index scaled by the inverse effective thermal conductivity ratio represents the portion of the thawing index consumed in thawing of the seasonally frozen layer and the associated absorption of latent heat. These scaled ground surface thawing and freezing indices therefore serve only as auxiliary terms in Eq. (16) that determine how much the



170 ground surface freezing and thawing indices are reduced under permafrost and seasonally frozen ground conditions, respectively.

Notably, the derivation of Eq. (17) shows that the thermal offset is implicit in Eq. (16). Eq. (17) also implies that the magnitude of the thermal offset is primarily controlled by the effective thermal conductivity ratio (Smith and Riseborough, 2002). When the ratio equals one, the thermal offset is zero (Smith and Riseborough, 2002). Otherwise, the magnitude of the thermal offset increases linearly as the thawing and freezing indices increase under permafrost and seasonally frozen ground conditions, respectively (Figs. 2c and 2d). This further suggests that the thermal offset can attenuate or amplify multi-year temperature trends at the base of the freeze–thaw layer relative to those at the ground surface. Since the thermal conductivity of water is nearly four times lower than the thermal conductivity of ice, the effective thermal conductivity ratio is typically less than one (Smith and Riseborough, 2002). This produces a negative thermal offset, whereby the mean annual ground temperature at the base of the freeze–thaw layer is lower than the mean annual ground surface temperature (Figs. 2a and 2c). Under these conditions, permafrost can occur even where the mean annual ground surface temperature is positive. In some environments, such as arid and semi-arid regions, however, the effective thermal conductivity ratio frequently exceeds one (Uxa et al., 2026). This results in a positive thermal offset, whereby the mean annual ground temperature at the base of the freeze–thaw layer is higher than the mean annual ground surface temperature (Figs. 2b and 2d). Under these conditions, permafrost can be absent even where the mean annual ground surface temperature would suggest otherwise. The magnitude of the thermal offset becomes more pronounced as the effective thermal conductivity ratio deviates from unity (Figs. 2c and 2d). Consequently, mean annual ground surface temperature alone may not reliably indicate the presence of permafrost or seasonally frozen ground (Smith and Riseborough, 2002).

While not explicitly involved in Eqs. (16) and (17), the mean annual ground temperature at the base of the freeze–thaw layer and especially the thermal offset are inherently related to the thickness of the freeze–thaw layer and the effects of latent heat (Osterkamp and Romanovsky, 1998). This was initially overlooked (Romanovsky and Osterkamp, 1995; Smith and Riseborough, 1996) and has not been highlighted much since (Osterkamp and Romanovsky, 1998; Smith and Riseborough, 1998). However, as shown previously, Eqs. (16) and (17) can be derived using the Stefan equation, which assumes that the conductive heat flux above the freeze–thaw front is equal to the rate at which latent heat is absorbed or released as the freeze–thaw front propagates downwards. In addition, it implies that the total amount of latent heat absorbed and released during thawing and freezing, respectively, is directly proportional to the thickness of the freeze–thaw layer. Consequently, the thermal offset and the thickness of the freeze–thaw layer exhibit consistent behaviour (Figs. 2c–f). At the same time, however, there is no specific thickness of the freeze–thaw layer corresponding to a given mean annual ground temperature at its base (Smith and Riseborough, 1998). Eq. (16) therefore provides the mean annual ground temperature at the base of the freeze–thaw layer, the thickness of which is actually unknown.

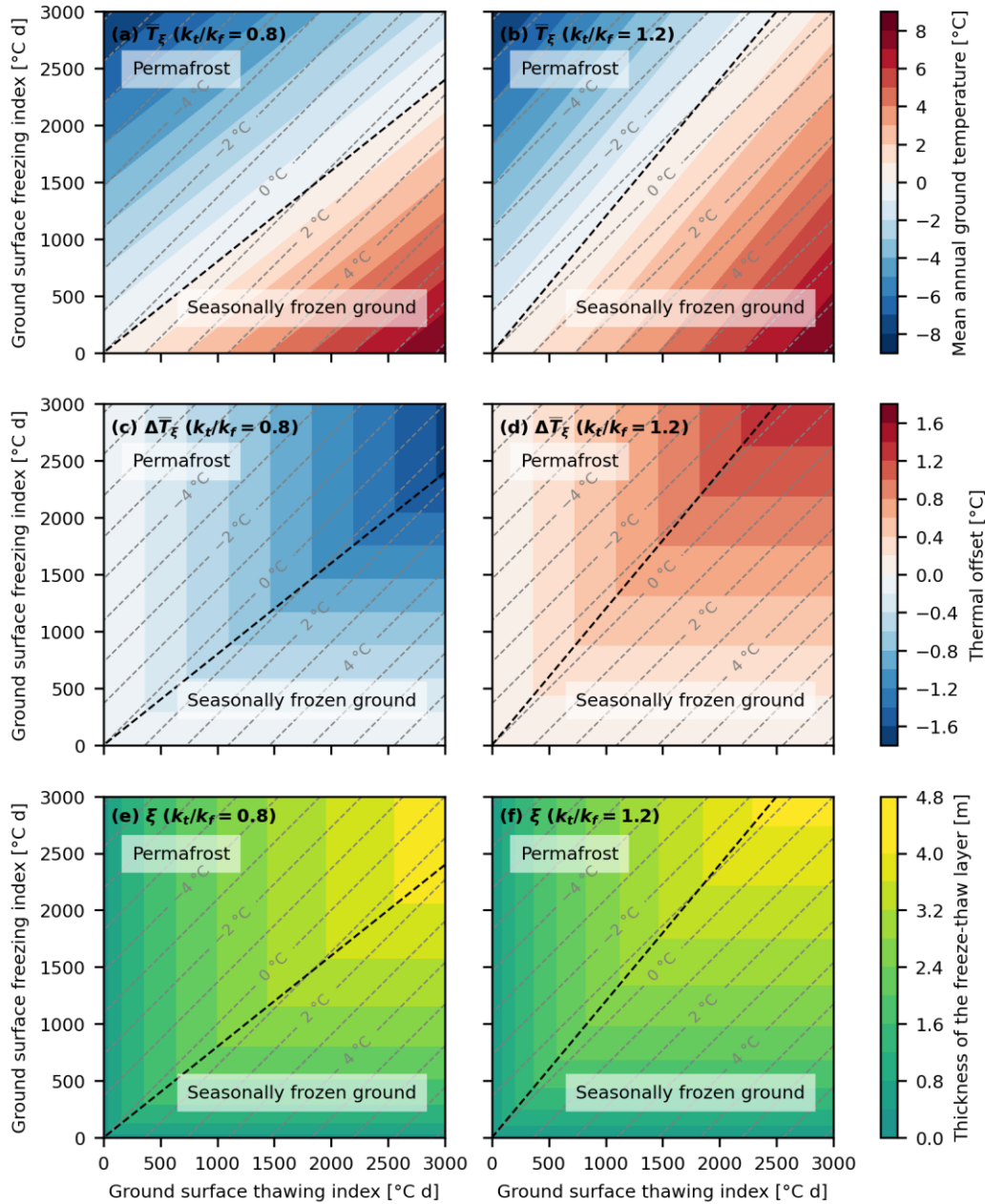


Figure 2: (a–b) Mean annual ground temperature at the base of the seasonal freeze–thaw layer calculated using Eq. (16), (c–d) thermal offset calculated using Eq. (17), and (e–f) thickness of the seasonal freeze–thaw layer calculated using the Stefan equation (Eqs. 6 and 7) across a range of ground surface thawing and freezing indices and two effective thermal conductivity ratios. The black dashed diagonal line represents the boundary between permafrost and seasonally frozen ground. The grey dashed diagonal lines show the mean annual ground surface temperatures. An annually periodic steady-state (equilibrium) thermal regime is assumed. See Appendix C for details on the effective thawed and frozen thermal conductivities and volumetric water and ice contents used to estimate the thickness of the freeze–thaw layer.



210 Considering that the thawing and freezing indices consumed in seasonal thawing and freezing of the freeze–thaw layer are directly proportional (see Eqs. 11 and 12), the same mean annual ground temperature at the base of the freeze–thaw layer can be obtained using thawing and freezing indices from any depth within the freeze–thaw layer (Uxa et al., 2026). This can be demonstrated by rewriting Eq. (9) as follows

$$z + \sqrt{\frac{2k_t \Delta I_t(z)}{L \phi_w \rho_w}} = z + \sqrt{\frac{2k_f \Delta I_f(z)}{L \phi_i \rho_i}}, \quad (22)$$

215 where $\Delta I_t(z)$ [°C d] and $\Delta I_f(z)$ [°C d] are the thawing and freezing indices, respectively, that are consumed in thawing and freezing of the freeze–thaw layer from depth z [m] downwards. Using the mass-conservation relationship in Eq. (8), Eq. (22) simplifies to

$$k_t \Delta I_t(z) = k_f \Delta I_f(z), \quad (23)$$

which has the same form as Eq. (10) but is valid for any depth within the freeze–thaw layer. Using thawing and freezing indices measured at any depth within the freeze–thaw layer is highly advantageous because ground surface temperature time series are limited in availability, as ground surface temperatures are difficult to measure accurately owing to radiative and convective energy fluxes and uncertainties in sensor placement (Riseborough, 2003). The linear relationship between the thawing and freezing indices within the freeze–thaw layer (Eq. 23) is further advantageous in that thawing and freezing indices measured at two depth levels within the freeze–thaw layer can be used to determine the mean annual ground temperature at the base of the freeze–thaw layer without knowing the effective thermal conductivity ratio, as was recently shown by Uxa et al. (2026). The considerations above suggest that Eq. (16) is essentially a piecewise bivariate linear model (Uxa et al., 2026), in which division by the duration of one year converts degree-days to degrees Celsius. Under permafrost conditions, the ground surface thawing index acts as the independent variable, while the negative ground surface freezing index represents the intercept term and the effective thermal conductivity ratio controls the slope. Under seasonally frozen ground conditions, the negative ground surface freezing index is the independent variable, while the ground surface thawing index acts as the intercept term and the inverse effective thermal conductivity ratio controls the slope. Hence, the model has a physical basis, but its form and behaviour also allow it to be interpreted statistically (Uxa et al., 2026). Nevertheless, it is by no means an empirical model.

3.2 Model assumptions and limitations

235 Eqs. (16) and (17) represent strongly simplified, physically based analytical solutions designed for a homogeneous freeze–thaw layer with constant effective thawed and frozen thermal conductivities and an annually periodic steady-state (equilibrium) thermal regime. In this context, homogeneity does not require all ground properties to be spatially uniform but implies that the entire freeze–thaw layer can be adequately represented by a single pair of effective thawed and frozen thermal conductivities or a single effective thermal conductivity ratio. This approximation is most appropriate where ground texture, mineral



240 composition, porosity, and water and ice contents vary little or gradually with depth, but becomes less suitable in layered ground or across sharp transitions between materials.

Using constant effective thawed and frozen thermal conductivities, or a constant effective thermal conductivity ratio, does not require water and ice contents or actual thermal conductivity to remain unchanged throughout the year. Instead, the two effective conductivities can be interpreted as representative of the entire thawing and freezing periods, respectively, and their ratio as representative of the entire year. The seasonal contrast between thermal conductivities in the thawed and frozen states associated with the water–ice phase transition is represented explicitly, whereas shorter-term changes in thermal conductivity caused by infiltration, drainage, evaporation, water migration, or ice sublimation are not resolved. This approximation is most appropriate when the seasonal cycle recurs similarly from year to year. Climatically or hydrologically induced changes that alter the effective thermal conductivities or their ratio between years may affect both the estimated mean annual ground temperature and its temporal trend. As indicated by Eq. (17), sensitivity to changes in the effective thermal conductivity ratio increases with the ground surface thawing index under permafrost conditions and with the ground surface freezing index under seasonally frozen ground conditions.

Besides these assumptions, the model considers only one-dimensional vertical heat conduction and neglects lateral heat conduction, advective heat transfer, and geothermal heat flux. It is therefore most applicable to relatively level terrain with spatially uniform upper-boundary forcing and negligible groundwater flow. It additionally assumes that temperature is linearly distributed between the upper boundary and the moving freeze–thaw front, which is maintained at 0 °C, and that conductive heat transfer at the front is entirely associated with latent heat absorbed or released during the water–ice phase transition, while sensible heat storage and heat conduction beneath the freeze–thaw front are neglected.

Finally, the annually periodic steady-state thermal regime implies equal recurrent thaw and freeze depths and no net year-to-year change in ground heat storage. The model may therefore be less reliable under rapid warming, permafrost degradation, and talik development, when transient thermal effects become important.

While the above assumptions are frequently violated to some extent, the model generally reproduces observed mean annual ground temperatures at the base of the freeze–thaw layer well, provided that representative input values are used (Garibaldi et al., 2026; Uxa et al., 2026). Nevertheless, it should be regarded as an equilibrium representation based on effective bulk properties that does not explicitly resolve coupled heat and water transfer.

3.3 Sinusoidal temperature variations

Garagulya (1990) also introduced an analytical model for the mean annual ground temperature at the base of the freeze–thaw layer defined in terms of the mean annual ground surface temperature and annual ground surface temperature amplitude. Hrbáček and Uxa (2020) noted that this model produces the same results as Eq. (16) under sinusoidal ground surface temperature variations. However, Garagulya (1990) provided no derivation, and the exact mathematical relationship between the two models has remained unknown.



Suppose that ground surface temperature varies sinusoidally throughout the year as

$$T_s(\tau) = \bar{T}_s + A_s \sin\left(\frac{2\pi\tau}{P}\right), \quad (24)$$

275 where A_s [°C] is the annual ground surface temperature amplitude. Under this assumption, the ground surface thawing and freezing indices defined by Eqs. (3) and (4), respectively, can be evaluated analytically as follows

$$I_t = P \left\{ \frac{\bar{T}_s}{2} + \frac{A_s}{\pi} \left[\frac{\bar{T}_s}{A_s} \arcsin\left(\frac{\bar{T}_s}{A_s}\right) + \sqrt{1 - \left(\frac{\bar{T}_s}{A_s}\right)^2} \right] \right\}, \quad (25)$$

$$I_f = P \left\{ -\frac{\bar{T}_s}{2} + \frac{A_s}{\pi} \left[\frac{\bar{T}_s}{A_s} \arcsin\left(\frac{\bar{T}_s}{A_s}\right) + \sqrt{1 - \left(\frac{\bar{T}_s}{A_s}\right)^2} \right] \right\}. \quad (26)$$

Eqs. (25) and (26) are valid for $|\bar{T}_s| \leq A_s$. Outside this range, when temperatures remain either positive or negative throughout the year, $I_t = P\bar{T}_s$ and $I_f = 0$ for $\bar{T}_s > A_s$, whereas $I_t = 0$ and $I_f = -P\bar{T}_s$ for $-\bar{T}_s > A_s$.
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Substituting Eqs. (25) and (26) into Eq. (16) yields the solution introduced by Garagulya (1990) for the mean annual ground temperature at the base of the freeze–thaw layer. Using the notation defined here, this solution can be written as follows

$$\bar{T}_\xi = \begin{cases} \frac{\frac{\bar{T}_s}{2}(k_t + k_f) + \frac{A_s}{\pi}(k_t - k_f) \left[\frac{\bar{T}_s}{A_s} \arcsin\left(\frac{\bar{T}_s}{A_s}\right) + \sqrt{1 - \left(\frac{\bar{T}_s}{A_s}\right)^2} \right]}{k_f}, & \bar{T}_\xi \leq 0 \\ \frac{\frac{\bar{T}_s}{2}(k_t + k_f) + \frac{A_s}{\pi}(k_t - k_f) \left[\frac{\bar{T}_s}{A_s} \arcsin\left(\frac{\bar{T}_s}{A_s}\right) + \sqrt{1 - \left(\frac{\bar{T}_s}{A_s}\right)^2} \right]}{k_t}, & \bar{T}_\xi > 0 \end{cases}. \quad (27)$$

Eq. (27) has commonly been referred to as semi-empirical (Anisimov et al., 1997; Shiklomanov and Nelson, 1999; Hrbáček and Uxa, 2020). However, the derivation above demonstrates that Eq. (27) is mathematically equivalent to Eq. (16) under sinusoidal ground surface temperature variations and therefore both have a physical basis. Using the effective thermal conductivity ratio, Eq. (27) becomes
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$$\bar{T}_\xi = \begin{cases} \left[\frac{\bar{T}_s}{2} \left(1 + \frac{k_t}{k_f} \right) + \frac{A_s}{\pi} \left(\frac{k_t}{k_f} - 1 \right) \left[\frac{\bar{T}_s}{A_s} \arcsin\left(\frac{\bar{T}_s}{A_s}\right) + \sqrt{1 - \left(\frac{\bar{T}_s}{A_s}\right)^2} \right] \right], & \bar{T}_\xi \leq 0 \\ \left[\frac{\bar{T}_s}{2} \left(1 + \frac{k_f}{k_t} \right) + \frac{A_s}{\pi} \left(1 - \frac{k_f}{k_t} \right) \left[\frac{\bar{T}_s}{A_s} \arcsin\left(\frac{\bar{T}_s}{A_s}\right) + \sqrt{1 - \left(\frac{\bar{T}_s}{A_s}\right)^2} \right] \right], & \bar{T}_\xi > 0 \end{cases}. \quad (28)$$

Similarly, substituting Eqs. (25) and (26) into Eq. (17) yields the corresponding solution for the thermal offset:

$$\Delta\bar{T}_\xi = \begin{cases} \left(\frac{k_t}{k_f} - 1 \right) \left\{ \frac{\bar{T}_s}{2} + \frac{A_s}{\pi} \left[\frac{\bar{T}_s}{A_s} \arcsin\left(\frac{\bar{T}_s}{A_s}\right) + \sqrt{1 - \left(\frac{\bar{T}_s}{A_s}\right)^2} \right] \right\}, & \bar{T}_\xi \leq 0 \\ \left(1 - \frac{k_f}{k_t} \right) \left\{ -\frac{\bar{T}_s}{2} + \frac{A_s}{\pi} \left[\frac{\bar{T}_s}{A_s} \arcsin\left(\frac{\bar{T}_s}{A_s}\right) + \sqrt{1 - \left(\frac{\bar{T}_s}{A_s}\right)^2} \right] \right\}, & \bar{T}_\xi > 0 \end{cases}. \quad (29)$$

Unlike Eq. (27), Eq. (29) does not appear to have been published previously. Eqs. (27), (28), and (29) are valid for $|\bar{T}_s| \leq A_s$. Outside this range, the thermal offset is zero, and the mean annual ground temperature at the base of the freeze–thaw layer therefore equals the mean annual ground surface temperature.



These derivations demonstrate that Eqs. (27) and (29) are closed-form representations of Eqs. (16) and (17), respectively, under sinusoidal ground surface temperature variations. If ground surface temperature variations are not sinusoidal, however, the ground surface thawing and freezing indices cannot generally be estimated accurately from the mean annual ground surface temperature and annual ground surface temperature amplitude, and the two pairs of equations may therefore produce different outputs. Since annual ground surface temperature cycles are rarely purely sinusoidal, Eqs. (27) and (29) are less generally applicable than Eqs. (16) and (17). They are also algebraically more complex and therefore less convenient.

Nevertheless, the mean annual ground temperature at the base of the freeze–thaw layer calculated using Eq. (27) has commonly served as an input to the Kudryavtsev equation (Kudryavtsev et al., 1977) for estimating the thickness of the freeze–thaw layer (e.g., Shiklomanov and Nelson, 1999; Hrbáček and Uxa, 2020; Wang et al., 2020). This is because the two equations are both defined in terms of the mean annual ground surface temperature and annual ground surface temperature amplitude ((Kudryavtsev et al., 1977; Garagulya, 1990).

Originally, Kudryavtsev et al. (1977) calculated the mean annual ground temperature at the base of the freeze–thaw layer using an implicit transcendental equation (their Eq. 5.2.1). However, this equation requires numerous inputs, including the thickness of the freeze–thaw layer, volumetric latent heat, volumetric heat capacity, effective thawed and frozen thermal conductivities, and annual ground temperature amplitude within the freeze–thaw layer, and must be solved iteratively.

4 Terminology

Although Eq. (16) can estimate the mean annual ground temperature at the base of the freeze–thaw layer under both permafrost and seasonally frozen ground conditions (Romanovsky and Osterkamp, 1995; Riseborough and Smith, 1998), its variant for permafrost conditions alone has sometimes been utilized to model temperatures under both ground thermal states (e.g., Janke et al., 2012; Zhao et al., 2017; Hrbáček et al., 2020; Kukkonen et al., 2020; Wang et al., 2020; Garibaldi et al., 2021, 2026). This can be attributed to four likely reasons. First, the pioneering and most influential publications (Romanovsky and Osterkamp, 1995; Smith and Riseborough, 1996, 2002) introduced only Eq. (16) for permafrost conditions in its entirety. Second, the differing model behaviour under permafrost and seasonally frozen ground conditions (Fig. 2) has not been highlighted much. Third, the most widely used name of the model, *temperature at the top of permafrost*, or *TTOP*, is ambiguous and obscures its applicability in seasonally frozen ground conditions. Fourth, the model has been much more widely used for permafrost conditions, which may have contributed to reduced attention to its variant for seasonally frozen ground. Whatever the reason may be, it is obvious that this practice can lead to misleading outputs under equilibrium conditions (Fig. 3). Although Way and Lewkowicz (2018) reported that the permafrost variant of the model performed better under seasonally frozen ground conditions in Labrador, Canada, this seemingly counterintuitive result likely arose because the permafrost variant tends to underestimate the mean annual ground temperature at the base of the freeze–thaw layer when applied to seasonally frozen ground with an effective thermal conductivity ratio of less than one (Fig. 3a). This underestimation may therefore have compensated for errors arising under transient warming conditions, especially during talik development, because the model



cannot capture the thermal inertia associated with degrading permafrost (Riseborough, 2007). However, it remains to be determined whether this has a general validity and, if so, what threshold temperature distinguishes between the permafrost and seasonally frozen ground variants of the model, and whether and how this threshold varies with climate trends and averaging periods.

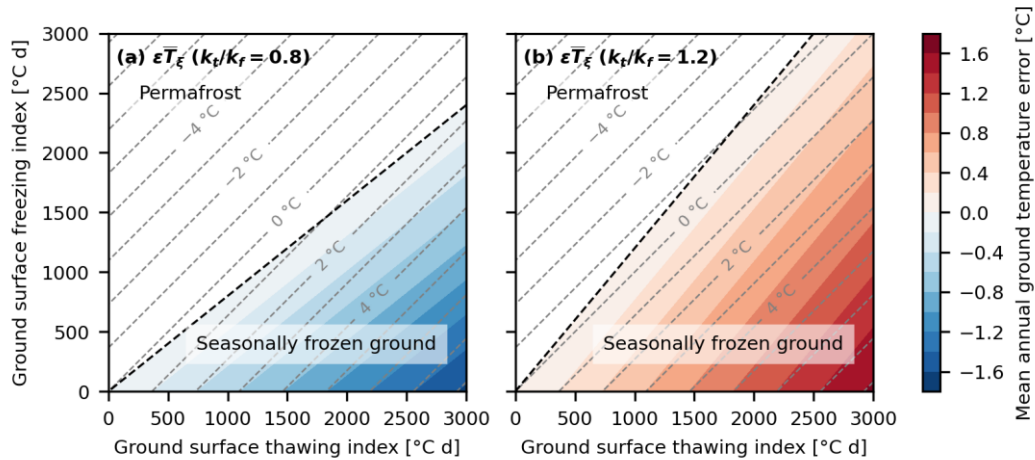


Figure 3: Errors in mean annual ground temperature at the base of the seasonal freeze–thaw layer resulting from the application of Eq. (16) for permafrost conditions to both permafrost and seasonally frozen ground across a range of ground surface thawing and freezing indices and two effective thermal conductivity ratios. The white areas show zero error. The black dashed diagonal line represents the boundary between permafrost and seasonally frozen ground. The grey dashed diagonal lines show the mean annual ground surface temperatures. An annually periodic steady-state (equilibrium) thermal regime is assumed.

A further issue concerning the most widely used name of the model, *temperature at the top of permafrost*, or *TTOP*, is that the model estimates the mean annual ground temperature at the base of the freeze–thaw layer regardless of its thickness. If the effective thermal conductivity ratio is assumed constant, which is a common practice (Obu et al., 2019, 2020; Kaplan Pastříková et al., 2023), while ground surface thawing and freezing indices for two consecutive years differ, the thickness of the freeze–thaw layer in these years will also inherently differ. However, this contradicts the definition of permafrost, which is ground that remains at or below 0 °C for at least two consecutive years (van Everdingen, 1998). Over multi-year periods involving year-to-year variability, the modelled temperatures therefore correspond neither to the top of permafrost nor to the base of seasonally frozen ground. Instead, they correspond to the transient layer, which extends over the depth range where the base of the freeze–thaw layer fluctuates from year to year and where permafrost and seasonally frozen ground alternately occur (Shur et al., 2005). In this perspective, the name of the model *temperature at the top of permafrost*, or *TTOP* (Smith and Riseborough, 1996), is imprecise and ambiguous even under permafrost conditions. Obviously, it is misleading for seasonally frozen ground conditions. Likewise, it is wrong when positive mean annual ground temperatures at the base of the freeze–thaw layer, which correspond to seasonally frozen ground, are referred to as *temperature at the top of permafrost*, or *TTOP* (e.g.,



350 Janke et al., 2012; Zhao et al., 2017; Hrbáček et al., 2020; Kukkonen et al., 2020; Wang et al., 2020; Garibaldi et al., 2021, 2026).

Considering that thermal offset is the core and only component of the model regardless of permafrost or seasonally frozen ground conditions, thickness of the freeze–thaw layer, or time scale, it would be much more sensible to refer to it as the *thermal offset model*. Consistent with this, the model outputs would be referred to as *mean annual ground temperatures at the base of the freeze–thaw layer* or, more concisely, as *mean annual ground temperatures* (e.g., Gislén et al., 2013; Westermann et al., 355 2015; Obu et al., 2019, 2020). This would resolve ambiguities related to the currently most widely used name of the model and its outputs. Additionally, it could help ensure that the model is used appropriately under both permafrost and seasonally frozen ground conditions and that its outputs are referred to in a consistent manner.

Of course, changing long-established names that are widely used by the scientific community is difficult and often unrealistic. 360 Nevertheless, if the current names are inconvenient, the issue is at least worth considering, because science is a discussion and should strive for terminology that is as clear and unambiguous as possible.

5 Conclusions

The thermal offset model, which estimates the mean annual ground temperature at the base of the seasonal freeze–thaw layer from the ground surface thawing and freezing indices, has been widely known and used by the frozen-ground community for 365 over thirty years. However, no simple and comprehensive analytical derivation of the model has been available, and some aspects of its behaviour have remained puzzling until now. This paper provides a new derivation of the model using the Stefan equation for seasonal thawing and freezing, which is intended to be easy to understand. The following implications can be drawn and highlighted:

- The model is essentially a piecewise bivariate linear model (Eq. 16). However, it has a physical basis and, although 370 strongly simplified, is by no means empirical.
- This physical basis is demonstrated by its derivation from the Stefan equation for seasonal thawing and freezing. The derivation also reveals a close relationship between the thermal offset and Stefan models, which has not yet been fully recognized.
- The model assumes one-dimensional heat conduction between the upper boundary and the moving freeze–thaw front, 375 a homogeneous freeze–thaw layer with constant effective thawed and frozen thermal conductivities, a water–ice phase transition at 0 °C with latent heat absorption and release, no freeze–thaw-induced ground deformation and water migration, an annually periodic steady-state thermal regime, and no geothermal heat flux. These assumptions limit the applicability of the model where the freeze–thaw layer is strongly heterogeneous, its total water content and thermal properties vary over time, or lateral or advective heat fluxes or transient thermal effects are considerable.
- The model can use thawing and freezing indices observed at any depth within the freeze–thaw layer as upper- 380 boundary forcing without explicitly accounting for their depth.



- Under sinusoidal ground surface temperature variations, the model can be reformulated in terms of the mean annual ground surface temperature and annual ground surface temperature amplitude, which yields the solution introduced by Garagulya (1990) and thereby reveals a mathematical equivalence that has remained unknown until now. However, this solution is algebraically more complex and therefore less convenient, while the assumption of sinusoidal ground surface temperature variations is rarely met.
- The model behaves differently under permafrost and seasonally frozen ground conditions, and the underlying equations differ for these two ground thermal states.
- The thermal offset is primarily controlled by the effective thermal conductivity ratio, which represents the contrast between the effective thawed and frozen thermal conductivities. The magnitude of the thermal offset increases as this ratio deviates from unity. The offset is negative when the ratio is less than one, positive when the ratio exceeds one, and zero when the ratio equals unity.
- The magnitude of the thermal offset also depends directly on the thawing index under permafrost conditions and on the freezing index under seasonally frozen ground conditions. It is therefore closely related to the thickness of the freeze–thaw layer and the total amount of latent heat associated with seasonal thawing and freezing.
- Despite these relationships, the model does not explicitly account for the thickness of the freeze–thaw layer and the associated latent heat. Instead, it estimates the mean annual ground temperature at the base of the freeze–thaw layer irrespective of its thickness.
- The most widely used name of the model *temperature at the top of permafrost*, or *TTOP*, is imprecise and ambiguous, even under permafrost conditions.
- The *thermal offset model* is proposed as a more suitable alternative name.

While simple, the thermal offset model has a relatively wide range of applications and provides useful insights into the fundamental physics of frozen ground and its relationship to ground surface temperature conditions, provided that its assumptions and limitations are recognized. Additionally, it is much less computationally demanding and requires fewer inputs than complex numerical models, which makes it a useful tool for modelling ground thermal regimes and permafrost and seasonally frozen ground extents across spatial and temporal scales.

I believe that the paper will be useful for the broad community of model users because it is important to understand the model physics, its behaviour under permafrost and seasonally frozen ground conditions, as well as its relationship to the thickness of the freeze–thaw layer and the associated latent heat. Finally, I am convinced that this paves the way for novel developments because understanding how models work can ultimately help us improve them further.

Appendix A: Derivation by Romanovsky and Osterkamp (1995, pp. 325–326)

Romanovsky and Osterkamp (1995) derived the equation for the mean annual ground temperature at the base of the freeze–thaw layer (Eq. 16) in detail only for permafrost conditions, whereas they derived the equation for the thermal offset (Eq. 17)



for both permafrost and seasonally frozen ground conditions. The following text recasts their derivation using the notation and explanations consistent with the present paper.

Romanovsky and Osterkamp (1995) assumed constant and spatially uniform effective thawed and frozen thermal conductivities and volumetric heat capacities, a water–ice phase transition at 0 °C, a periodic steady-state thermal regime with a period of one year, and no geothermal heat flux. Using these assumptions, they started their derivation from the one-dimensional heat diffusion equation:

$$C(T) \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial z} \left(k(T) \frac{\partial T}{\partial z} \right), \quad (A1)$$

where C [$\text{J m}^{-3} \text{K}^{-1}$] is the volumetric heat capacity, T [°C] is the temperature, and k [$\text{W m}^{-1} \text{K}^{-1}$] is the thermal conductivity. Subsequently, they modified Eq. (A1) for the seasonally thawed layer by introducing a new variable T' [°C] (Carslaw and Jaeger, 1959) as follows

$$T' = \begin{cases} T, & T \leq 0 \\ \frac{k_t}{k_f} T, & T > 0 \end{cases} \quad (A2)$$

Using Eq. (A2), the Stefan thawing problem can be transformed into

$$C_t \frac{k_f}{k_t} \frac{\partial T'}{\partial \tau} = k_f \frac{\partial^2 T'}{\partial z^2}, \quad z < \xi_t, \quad (A3)$$

$$C_f \frac{\partial T'}{\partial \tau} = k_f \frac{\partial^2 T'}{\partial z^2}, \quad z \geq \xi_t, \quad (A4)$$

$$T'(\xi_t^-, \tau) = T'(\xi_t^+, \tau) = 0 \text{ °C}, \quad (A5)$$

$$k_f \frac{\partial T'}{\partial z} \Big|_{\xi_t^-} - k_f \frac{\partial T'}{\partial z} \Big|_{\xi_t^+} = -L \phi_w \rho_w \frac{d\xi_t}{d\tau}, \quad (A6)$$

where C_t [$\text{J m}^{-3} \text{K}^{-1}$] and C_f [$\text{J m}^{-3} \text{K}^{-1}$] are the thawed and frozen volumetric heat capacities, respectively. The quantities ξ_t^- and ξ_t^+ denote the upper and lower sides of the thaw front (phase boundary), respectively. Eqs. (A3) to (A6) show that the transformation of Eq. (A2) produces a homogeneous half-space with the thermal conductivity k_f in both the thawed and frozen states and the volumetric heat capacity $C_t(k_f/k_t)$ in the thawed state and C_f in the frozen state.

As shown previously, the thermal offset is the difference between the mean annual ground temperature at the base of the freeze–thaw layer and the mean annual ground surface temperature (Eq. 1) and arises from the contrast between the thawed and frozen thermal conductivities. Under a periodic steady-state thermal regime and in the absence of geothermal heat flux, no thermal offset would exist if the thermal conductivities in the thawed and frozen states were the same (Romanovsky and Osterkamp, 1995), so that

$$\bar{T}'_\xi = \bar{T}'_s, \quad (A7)$$



where $\bar{T}_\xi'$ [°C] and $\bar{T}_s'$ [°C] denote the transformed mean annual ground temperature at the base of the freeze–thaw layer and the transformed mean annual ground surface temperature, respectively, both based on T' . The transformed mean annual ground surface temperature can then be written as

$$445 \quad \bar{T}_s' = \frac{I_t' - I_f'}{P}, \quad (A8)$$

where I_t' [°C d] and I_f' [°C d] are the transformed ground surface thawing and freezing indices, respectively, based on T' . Since $T' = T$ for $T \leq 0^\circ\text{C}$, the transformed and non-transformed ground surface freezing indices are identical:

$$I_f' = I_f. \quad (A9)$$

Using Eq. (A2), however, the transformed ground surface thawing index is related to the non-transformed ground surface
 450 thawing index as follows

$$I_t' = \frac{k_t}{k_f} I_t. \quad (A10)$$

Combining Eqs. (A8) to (A10) then gives

$$\bar{T}_s' = \frac{\frac{k_t}{k_f} I_t - I_f}{P}. \quad (A11)$$

Therefore, it follows from Eqs. (A7) and (A11) that

$$455 \quad \bar{T}_\xi' = \frac{\frac{k_t}{k_f} I_t - I_f}{P}. \quad (A12)$$

At the base of the freeze–thaw layer, where $T \leq 0^\circ\text{C}$, Eq. (A2) implies that $\bar{T}_\xi' = \bar{T}_\xi$. Using Eqs. (2) and (A12), the thermal offset is then

$$\Delta\bar{T}_\xi = \frac{I_t}{P} \left(\frac{k_t}{k_f} - 1 \right). \quad (A13)$$

Eq. (A13) was derived for a seasonally thawed layer under permafrost conditions. Using Eq. (A12)

$$460 \quad \bar{T}_\xi = \frac{k_t I_t - k_f I_f}{k_f P} \leq 0, \quad (A14)$$

which is equivalent to

$$k_t I_t \leq k_f I_f. \quad (A15)$$

Under seasonally frozen ground conditions, when a seasonally frozen layer exists with a thawed layer underneath, the following condition applies

$$465 \quad k_t I_t > k_f I_f. \quad (A16)$$

The thermal offset for this case can be derived analogously, except that a homogeneous thawed half-space must be used instead of a homogeneous frozen half-space. The final result is



$$\Delta \bar{T}_{\xi} = \begin{cases} \frac{I_t}{P} \left(\frac{k_t}{k_f} - 1 \right), k_t I_t \leq k_f I_f \\ \frac{I_f}{P} \left(1 - \frac{k_f}{k_t} \right), k_t I_t > k_f I_f \end{cases} \quad (\text{A17})$$

Note that Eq. (A14) is equivalent to Eq. (16) for permafrost conditions, whereas Eq. (A17) is equivalent to Eq. (17).

Overall, the temperature transformation introduced in Eq. (A2) is well motivated because it recasts the effect of contrasting thawed and frozen thermal conductivities into an equivalent form with a single thermal conductivity. However, the derivation is somewhat convoluted and less physically intuitive than the solution based on thawing and freezing indices and their consumption within the freeze–thaw layer (Sect. 2) because the thermal conductivity ratio is introduced through a change of variables rather than through an explicit energy balance between seasonal thawing and freezing. Although the derivation includes a moving phase boundary and latent heat in Eqs. (A3) to (A6), their roles in the subsequent derivation of the equations for the mean annual ground temperature at the base of the freeze–thaw layer and the thermal offset remain implicit. Consequently, the effects of the thickness of the freeze–thaw layer and latent heat cannot be inferred directly.

Appendix B: Derivation by Westermann et al. (2015, p. 1316)

Westermann et al. (2015) derived the equation for the mean annual ground temperature at the base of the freeze–thaw layer (Eq. 16) in detail only for permafrost conditions. The following text recasts their derivation using the notation and explanations consistent with the present paper.

Westermann et al. (2015) assumed constant and spatially uniform effective thawed and frozen thermal conductivities, a water–ice phase transition at 0 °C, annual thermal equilibrium between the thawing and freezing seasons, no geothermal heat flux, and a constant thickness of the freeze–thaw layer ξ [m] throughout both the thawing and freezing seasons.

During the thawing season, they expressed the conductive heat flux $q_t(\tau)$ [W m⁻²] between the ground surface and the base of the freeze–thaw layer, whose temperature was assumed to be constant at 0 °C, as follows

$$q_t(\tau) = -k_t \frac{T_s(\tau) - 0 \text{ °C}}{\xi}. \quad (\text{B1})$$

During the freezing season, they expressed the conductive heat flux $q_f(\tau)$ [W m⁻²] between the ground surface and the base of the freeze–thaw layer, whose temperature was assumed to be constant at T_{ξ} [°C], as follows

$$q_f(\tau) = -k_f \frac{T_s(\tau) - T_{\xi}}{\xi}. \quad (\text{B2})$$

Under annual thermal equilibrium between the thawing and freezing seasons and in the absence of geothermal heat flux, the seasonally integrated conductive heat fluxes must balance each other:

$$\int_{\text{thawing}} q_t(\tau) d\tau + \int_{\text{freezing}} q_f(\tau) d\tau = 0. \quad (\text{B3})$$

Substituting Eqs. (B1) and (B2) into Eq. (B3) and rearranging yields the freezing-season equilibrium ground temperature at the base of the freeze–thaw layer as



$$P_f T_{\xi} = \frac{k_t}{k_f} \int_{\text{thawing}} T_s(\tau) d\tau + \int_{\text{freezing}} T_s(\tau) d\tau, \quad (\text{B4})$$

where P_f [d] is the duration of the freezing season. Note that the integrals over the thawing and freezing seasons in Eq. (B4) correspond to the ground surface thawing index and the negative of the ground surface freezing index, respectively. With P_t [d] being the duration of the thawing season, one can define the mean annual ground temperature at the base of the freeze–thaw layer as the time-weighted average of thawing- and freezing-season temperatures at the base of the freeze–thaw layer as

$$\bar{T}_{\xi} := \frac{P_f T_{\xi} + P_t 0 \text{ }^{\circ}\text{C}}{P}. \quad (\text{B5})$$

Inserting Eq. (B4) into Eq. (B5), one obtains Eq. (16) for permafrost conditions. Under seasonally frozen ground conditions, the derivation is analogous, but the thawing and freezing seasons change roles because the freezing-season temperature at the base of the frozen layer is assumed to be constant at 0 °C in this case.

Overall, this derivation is somewhat cumbersome and less physically intuitive than the solution based on thawing and freezing indices and their consumption within the freeze–thaw layer (Sect. 2). Additionally, it relies on strong simplifying assumptions, including seasonally constant temperatures at the base of the freeze–thaw layer and conductive heat fluxes calculated across the full thickness of the freeze–thaw layer throughout both the thawing and freezing seasons. Consequently, the movement of the freeze–thaw front and the associated effects of latent heat are not considered, and the influence of latent heat on the thermal offset cannot be inferred from this formulation. Lastly, Eq. (B5) assumes that the temperature at the base of the freeze–thaw layer is constant at 0 °C throughout the thawing season. Under permafrost conditions, however, the temperature would generally reach 0 °C only near the time of maximum thaw (see Fig. 1).

Appendix C: Thermal conductivity

Along with the thawing and freezing indices, the Stefan equations for the thaw and freeze depths (Eqs. 6 and 7) also require the effective thawed and frozen thermal conductivities, respectively, and the volumetric water and ice contents. In order to keep these properties mutually consistent and physically meaningful, the effective thawed and frozen thermal conductivities were calculated using the Johansen (1977) model for coarse ground, which also allows for the effective thermal conductivity ratio to exceed one when the volumetric water/ice content is low. The main equations for the effective thawed and frozen thermal conductivities are as follows (Johansen, 1977)

$$k_t = k_{dry} + (k_{sat} - k_{dry})(0.7 \log S + 1) \quad (\text{C1})$$

$$k_f = k_{dry} + (k_{sat} - k_{dry})S \quad (\text{C2})$$

where k_{dry} [W m⁻¹ K⁻¹] is the dry thermal conductivity, k_{sat} [W m⁻¹ K⁻¹] is the saturated thermal conductivity, and S [–] is the degree of saturation.

The dry thermal conductivity was calculated as follows (Johansen, 1977):



$$k_{dry} = \frac{0.135\rho + 64.7}{2700 - 0.947\rho} \quad (C3)$$

where ρ [kg m^{-3}] denotes the dry bulk density and 2700 kg m^{-3} is the typical density of solid ground particles, which was also used in the other equations.

The saturated thermal conductivity in the frozen ($T \leq 0^\circ\text{C}$) and thawed ($T > 0^\circ\text{C}$) state was calculated as follows (Johansen, 1977):

$$k_{sat} = \begin{cases} k_s^{1-n} k_i^n, & T \leq 0, \\ k_s^{1-n} k_w^n, & T > 0 \end{cases} \quad (C4)$$

where k_s [$\text{W m}^{-1} \text{K}^{-1}$], k_i [$2.22 \text{ W m}^{-1} \text{K}^{-1}$], and k_w [$0.57 \text{ W m}^{-1} \text{K}^{-1}$] denote the thermal conductivities of solid ground particles, ice, and water, respectively, and n [–] is the ground porosity. The thermal conductivity of solids was computed as

$$k_s = k_q^q k_o^{1-q}, \quad (C5)$$

where k_q [$7.7 \text{ W m}^{-1} \text{K}^{-1}$] and k_o [$2 \text{ W m}^{-1} \text{K}^{-1}$] are the thermal conductivities of quartz and other minerals, respectively, while q [–] denotes the content of quartz. The porosity was expressed as

$$n = 1 - \frac{\rho}{2700}. \quad (C6)$$

Lastly, the degree of saturation was calculated as follows

$$S = \frac{\phi}{n}. \quad (C7)$$

Subsequently, the effective thawed and frozen thermal conductivities were estimated using the dry bulk density of 1755 kg m^{-3} , which corresponds to the porosity of 35 %, and the quartz content of 50 %. Under these assumptions, the effective thermal conductivity ratios of 0.8 and 1.2, as used in Figs. 2 and 3, correspond to the volumetric water/ice contents of ~23 % and ~10%, respectively (Fig. C1).

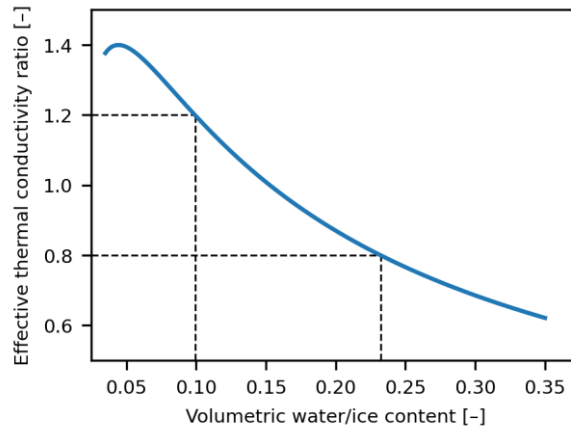


Figure C1: Effective thermal conductivity ratio as a function of volumetric water/ice content calculated using the Johansen (1977) thermal conductivity model for coarse ground with dry bulk density of 1755 kg m^{-3} and the quartz content of 50 %.



Competing interests

I declare that I have no competing interests.

Acknowledgements

550 Work on this paper began more than five years ago, but the manuscript was subsequently set aside. Only recently did I find the motivation to revisit and completely rework it. ChatGPT-5 was used to improve the rigour and readability of the text, correct spelling and grammar, check the consistency of the equations, and assist with figure preparation. All AI-assisted outputs were reviewed and verified by the author.

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555 Data availability

Data sharing is not applicable to this paper, as no observational or external datasets were used. All synthetic results are presented in the figures and can be reproduced using the equations and parameter values provided in the paper.

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