



A practical quantile approach to nonlinear uncertainty propagation in radiometric dating

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Radiometric ages are commonly reported with a symmetric uncertainty ($\pm u$) obtained by first-order propagation. This is adequate when the age uncertainty is small relative to the characteristic timescale of the chronometer, but becomes unreliable when a measured isotope or activity ratio is transformed through a strongly nonlinear age equation. Equal positive and negative ratio deviations then produce unequal age deviations and asymmetric coverage intervals. Where the measured ratio approaches a physical bound, a finite upper age limit may cease to exist. Monte Carlo propagation can account for these effects, but is unnecessary for monotonic dating equations when the uncertainty budget can be adequately represented by a single equivalent perturbation of the measured ratio with standard deviation σ_{eq} . The degree of nonlinearity is then described by a single dimensionless parameter, $\beta = u_{lin}/\tau$, which compares the linearly propagated age uncertainty with the chronometer timescale ($\tau = 109\text{--}158\text{ ka}$ for $^{230}\text{Th}/^{238}\text{U}$, 8.2 ka for ^{14}C). Symmetric propagation is generally adequate for $|\beta| \lesssim 0.1$. At larger $|\beta|$, asymmetric coverage limits follow from solving the dating equation at the measured ratio $\pm k\sigma_{eq}$. Above $|\beta| \approx 1/k$, the ratio quantile near the physical bound reaches the limit of the age equation, so no finite upper coverage limit exists and the result must be reported as a lower age limit. The approach is developed for $^{230}\text{Th}/^{238}\text{U}$ dating and validated against Monte Carlo simulation. It is then extended to three families of monotonic chronometers: decaying parents, daughter/parent activity ratios approaching secular equilibrium, and accumulating daughter/parent atom ratios. Only the first two families have a dating limit, whereas classical geochronometers such as K–Ar, Rb–Sr, and U–Pb generally do not. Requiring only standard uncertainty budget parameters, the method can be implemented directly in a spreadsheet or data-reduction software.

Keywords: Radiometric dating; Uncertainty propagation; Chronometers; $^{230}\text{Th}/^{238}\text{U}$; Coverage interval; Nonlinearity; Dating limit; Quantile transformation



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2 1 **Introduction**

3 Reporting a radiometric age as a central estimate with a symmetric uncertainty, $\pm u$, derived by first-
4 order propagation of the measurement uncertainty budget, is standard practice. It follows the *Guide to the*
5 *expression of uncertainty in measurement* (Joint Committee for Guides in Metrology, 2008a), and was
6 recently compared with the Kragten method in ^{234}U – ^{230}Th radiochronometry (Macsik and Pommé, 2026).
7 This convention is appropriate when the relationship between the measured quantity and age is effectively
8 linear across the range sampled by the uncertainty budget. Many radiometric chronometers, however,
9 require inversion of a nonlinear decay or ingrowth relation. In those cases, equal positive and negative
10 perturbations of a measured isotope or activity ratio do not always correspond to equal changes in age. The
11 resulting uncertainty interval can be asymmetric even when the underlying measurement uncertainty is
12 symmetric.

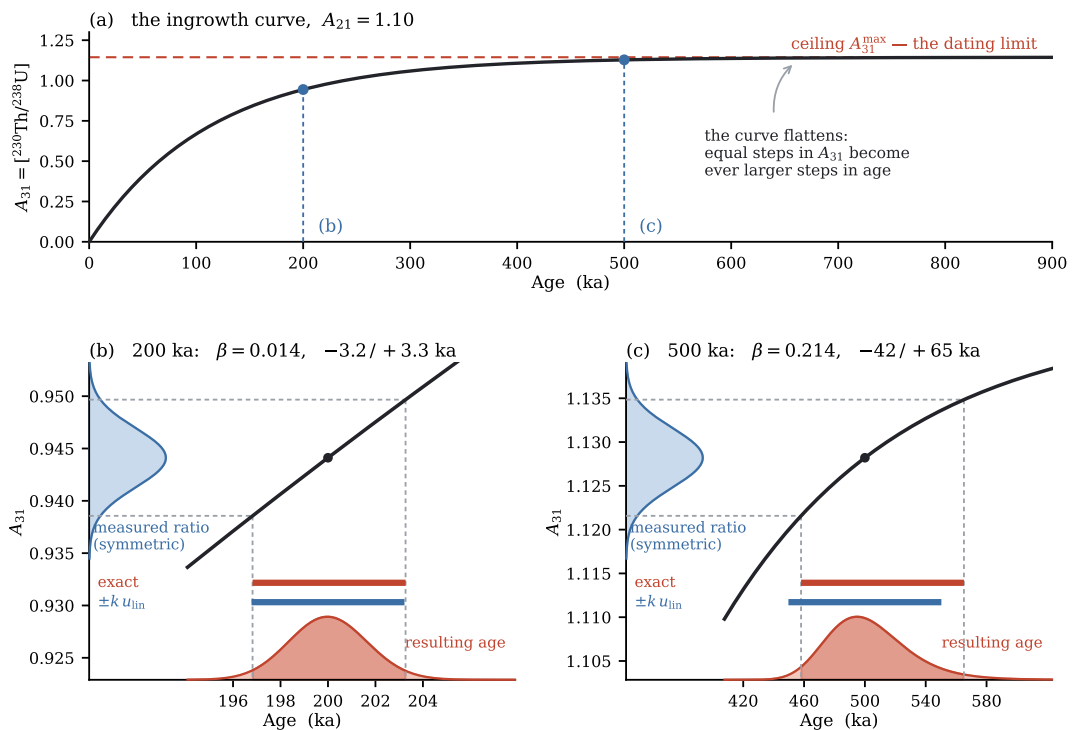
13 The nonlinearity problem, visualised in Figure 1, is familiar in $^{230}\text{Th}/^{238}\text{U}$ geochronology, where age
14 uncertainties have long been propagated with linearised (first-order) error equations (Ludwig and
15 Titterton, 1994). As a carbonate approaches secular equilibrium, the $^{230}\text{Th}/^{238}\text{U}$ ingrowth curve becomes
16 progressively flatter. A small change in activity ratio then corresponds to a larger change in calculated age
17 than it would for a younger sample. The practical consequence is that linear propagation can increasingly
18 misrepresent both the location and shape of the age interval. Similar nonlinear behaviour occurs in other
19 chronometers that estimate age by inverting a monotonic relation between time and a measured quantity,
20 including decaying-parent systems such as ^{14}C and ^{210}Pb , daughter/parent activity-ratio chronometers
21 approaching equilibrium, and accumulating parent–daughter clocks used also in nuclear forensics. While
22 the recent Amendment 1 to the GUM proposes using higher-order terms of the Taylor expansion to
23 propagate uncertainty in nonlinear measurement models (Joint Committee for Guides in Metrology, 2026),
24 this method would increase complexity while still suffering from a limited range of applicability.

25 Widely used geochronological software propagates uncertainties by linearisation (Ludwig, 2012;
26 Vermeesch, 2018), whereas Bayesian inference (Vermeesch et al., 2025) and Monte Carlo simulation serve
27 as alternatives where linear approximation fails. MC propagation is a flexible solution and is widely used
28 when nonlinearity, non-Gaussian inputs, correlations, or complex corrections must be accounted for. It is
29 applied in $^{230}\text{Th}/^{238}\text{U}$ geochronology (Ludwig, 2003), including for detrital-thorium corrections (Hellstrom,
30 2006), and in other nonlinear geochronometers such as (U–Th)/He to obtain asymmetric uncertainty
31 intervals that linear propagation cannot provide (Martin, Metcalf and Flowers, 2023). GUM Supplement 1



likewise recommends propagation of distributions by MC when first-order linearisation is inadequate (Joint Committee for Guides in Metrology, 2008b). However, MC simulation does not by itself provide a simple criterion for deciding when linear propagation has become inadequate, nor is it always necessary for calculating coverage limits once a suitable one-dimensional representation of the uncertainty budget has been established.

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Figure 1: The origin of asymmetric age uncertainties in $^{230}\text{Th}/^{238}\text{U}$ dating. The same 0.3 % uncertainty in the measured activity ratio has very different consequences at different ages. **(a)** As the $^{230}\text{Th}/^{238}\text{U}$ ingrowth function A_{31} approaches its physical ceiling, it becomes progressively flatter, so equal steps in the measured activity ratio correspond to ever larger steps in age. **(b)** At 200 ka, the symmetric Gaussian on A_{31} produces an essentially symmetric age distribution and agrees with the conventional linear uncertainty. **(c)** At 500 ka, the same measurement precision spans a bent section of the curve. The age distribution is skewed, giving a 95 % interval of $-42/+65$ ka. The conventional symmetric interval is too narrow and misplaced. The single number β captures how much the curve bends over the width of the measurement: 0.014 at 200 ka and 0.214 at 500 ka.

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2 Here, a practical approach for monotonic radiometric age equations is developed. The method
 3 separates the problem into two steps. First, the conventional first-order uncertainty budget is expressed as
 4 an equivalent perturbation of the measured ratio. Second, the quantiles of that ratio distribution are mapped
 5 directly through the inverse dating equation. Because quantiles are preserved under monotonic
 6 transformations, this second step provides the appropriate age quantiles for the specified equivalent-ratio
 7 model. The approximation in the overall workflow lies in reducing a multi-input uncertainty budget to a
 8 single equivalent perturbation, not in the subsequent quantile transformation.

9 The $^{230}\text{Th}/^{238}\text{U}$ system provides a useful example, both because it is the workhorse chronometer for
 10 speleothems, corals and other late-Quaternary carbonates, including the records that constrain past climate
 11 and sea level (Cheng et al., 2013; Dumitru et al., 2023; Dumitru, Polyak, Asmerom and Onac, 2021;
 12 Edwards, Chen and Wasserburg, 1987; Richards and Dorale, 2003), and because its ingrowth relation is
 13 bounded by secular equilibrium and contains two relevant decay constants. The practical question is: when
 14 does the familiar symmetric interval adequately represent an age, and when should an asymmetric interval
 15 or a one-sided age limit be reported instead? The method is intended for monotonic age equations when the
 16 dominant uncertainty contributions can be represented adequately by a single equivalent perturbation of the
 17 measured atom or activity ratio. The direct quantile transformation is straightforward within this reduced
 18 model. Full multivariate propagation remains preferable when additive corrections, strong correlations,
 19 non-Gaussian input distributions, or other effects cannot be represented reliably by a single perturbation
 20 direction.

21 The resulting workflow is summarised below and set out as a reporting procedure in Section 9. The
 22 intermediate Sections 3–8 develop the underlying framework. Every step draws on quantities already
 23 present in a conventional uncertainty budget, and the procedure as a whole adds only two age solutions to
 24 the calculation:

- 25 1. **Compute the age and its conventional uncertainty.** Solve the ingrowth function $\mathcal{G}(T) = R$ for
 26 the age T , and build the usual GUM-style uncertainty budget to obtain the linearly propagated
 27 combined standard uncertainty u_{lin} . This is the standard calculation, unchanged.
- 28 2. **Calculate the nonlinearity diagnostic** $\beta = u_{\text{lin}}/\tau$, where τ is a characteristic timescale of the
 29 chronometer, bounded between 109 and 158 ka for $^{230}\text{Th}/^{238}\text{U}$ and equal to 8.2 ka for ^{14}C (see
 30 Table 7). In other words, β is the age uncertainty expressed as a fraction of the chronometer's
 31 own curvature timescale, a measure of how far into the nonlinear regime the measurement sits.



Throughout the paper, β of Section 3 plays a diagnostic role: it decides whether a symmetric interval is adequate, while the quantile transformation of Section 4 is the method that supplies the correct asymmetric interval; the two are complementary. The workflow above can be implemented with the spreadsheet-based tools already available to the analyst, without simulation or additional specialised software. Readers interested primarily in practical application rather than the mathematical development of Sections 3–8 may proceed directly to the reporting procedure in Section 9.

Symbol	Meaning
R, A_{31}	the measured ratio; for $^{230}\text{Th}/^{238}\text{U}$ dating the activity ratio A_{31}



Symbol	Meaning
$\mathcal{G}(T)$	the ingrowth function: the ratio that a sample of age T would show
$\mathcal{G}', \mathcal{G}''$	its first and second derivatives with respect to T
u_{lin}	the combined standard uncertainty of the age from the ordinary GUM budget
$\sigma_{\text{eq}} = \mathcal{G}' u_{\text{lin}}$	the same uncertainty expressed as a perturbation of the ratio
$\mu = -\mathcal{G}''/\mathcal{G}', \tau = 1/\mu$	the curvature scale of the chronometer, in a^{-1} and a
$\beta = \mu u_{\text{lin}} = u_{\text{lin}}/\tau$	the dimensionless nonlinearity criterion
$k = z_p$	coverage factor; e.g. 1.96 for 95 % under a Gaussian model

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2 Terminology follows the GUM. The combined standard uncertainty u_{lin} is the age uncertainty
 3 obtained by first-order propagation, and the expanded uncertainty is k times that value. A coverage interval
 4 is the interval bounded by specified quantiles of the distribution assigned to the input quantity, mapped
 5 through the dating equation. Its endpoints are coverage limits, and when only one of them is finite, an upper
 6 or lower age limit is reported. The dating limit is the point at which a requested coverage limit falls outside
 7 the finite domain of the inverse age equation.

8 Section 2 sets out the dating model and Section 3 introduces the curvature parameter β and derives
 9 its practical bounds. Section 4 presents the coverage interval obtained from the monotonic age–ratio
 10 relationship, given the equivalent perturbation of Section 3. Section 5 defines the dating limit
 11 probabilistically and explains why the unconditional mean and standard uncertainty of the age may not
 12 exist when the uncertainty distribution extends beyond the physical limit of the dating equation. Section 6
 13 validates the method against Monte Carlo simulation for three synthetic U–Th test cases. Section 7
 14 generalises the approach to monotonic chronometers, Section 8 tests the criterion against real
 15 measurements, and Section 9 sets out a reporting procedure for laboratories. Section 10 sets out the
 16 limitations of the approach, and Section 11 presents the conclusions.

17 **2 Radiometric dating model**

18 Radiometric dating of a closed system via $^{230}\text{Th}/^{238}\text{U}$ ingrowth follows from the Bateman equations,
 19 which describe how activity passes down a decay chain (Bateman, 1910; Pommé et al., 1996). Along the
 20 relevant segment, $^{238}\text{U} \rightarrow ^{234}\text{U} \rightarrow ^{230}\text{Th}$, three simplifications apply over the timescales considered here.
 21 First, ^{238}U ($T_{1/2} = 4.468 \text{ Ga}$) decays negligibly because of its long half-life, so its activity is effectively



1 constant. Second, the short-lived intermediates ^{234}Th and $^{234\text{m}}\text{Pa}$ remain in secular equilibrium with their
2 parent and need not be tracked. Third, the system is assumed to have remained closed since formation and
3 to have incorporated no ^{230}Th at $t = 0$; where detrital thorium is present, a detrital correction must be applied.
4 Under these conditions the Bateman solution, normalised to the ^{238}U activity, reduces to a two-exponential
5 expression in which only the ^{230}Th and ^{234}U decay constants appear. Chain members are indexed by their
6 position: $1 = ^{238}\text{U}$, $2 = ^{234}\text{U}$, $3 = ^{230}\text{Th}$, so that λ_1 , λ_2 and λ_3 are their decay constants.

7 For a closed system with no initial unsupported ^{230}Th , the age T is the value that makes the predicted
8 ratio equal the measured one. Writing the ingrowth function for the normalized activity ratio $A_{31} =$
9 $[^{230}\text{Th}/^{238}\text{U}]$ at the time of measurement, T is the root of:

$$10 \quad \mathcal{G}(T; A_{21}) = 1 - e^{-\lambda_3 T} + (A_{21} - 1) \kappa (1 - e^{-(\lambda_3 - \lambda_2)T}) = A_{31}, \quad (1)$$

11 with $\kappa = \frac{\lambda_3}{\lambda_3 - \lambda_2}$, and $A_{21} = [^{234}\text{U}/^{238}\text{U}]$ the measured, present-day activity ratio. A detrital correction modifies
12 A_{31} and A_{21} . It leaves the monotonic structure of Eq. (1) intact, but, unlike other budget contributors, it
13 does not enter multiplicatively (see Section 6 for a numerical example). (The ^{234}U – ^{230}Th model age uses
14 the 2→3 segment of the same chain with the ratio taken relative to member 2; the normalising nuclide
15 therefore depends on the chronometer.)

16 Two properties of $\mathcal{G}$ are essential. First, it increases steadily with age: it is strictly increasing in T .
17 Second, it approaches a finite upper bound:

$$18 \quad \mathcal{G}(T; A_{21}) < A_{31}^{\max} = 1 + (A_{21} - 1)\kappa \text{ for all } T, \quad (2)$$

19 with $A_{31}^{\max} - \mathcal{G}(T) = e^{-\lambda_3 T} + (A_{21} - 1)\kappa e^{-(\lambda_3 - \lambda_2)T}$. This ceiling is what ultimately sets the dating limit:
20 as a sample ages, the measured ratio creeps toward it but never reaches it, so the oldest ages sit on an ever-
21 flatter part of the curve. The first and second derivatives of $\mathcal{G}$ (the slope and its rate of change) quantify that
22 flattening:

$$23 \quad \begin{aligned} \mathcal{G}' &= \lambda_3 e^{-\lambda_3 T} + (A_{21} - 1)\lambda_3 e^{-(\lambda_3 - \lambda_2)T}, \\ \mathcal{G}'' &= -\lambda_3^2 e^{-\lambda_3 T} - (A_{21} - 1)\lambda_3(\lambda_3 - \lambda_2)e^{-(\lambda_3 - \lambda_2)T}, \\ \mathcal{G}''' &= \lambda_3^3 e^{-\lambda_3 T} + (A_{21} - 1)\lambda_3(\lambda_3 - \lambda_2)^2 e^{-(\lambda_3 - \lambda_2)T}, \end{aligned} \quad (3)$$

24 Derivatives of the age with respect to the measured ratio, $T^{(n)} = \partial^n T / \partial A_{31}^n$, are by the inverse-
25 function rule:

$$26 \quad T^{(1)} = \frac{1}{\mathcal{G}'}, \quad T^{(2)} = -\frac{\mathcal{G}''}{\mathcal{G}'^3}, \quad T^{(3)} = \frac{3\mathcal{G}''^2 - \mathcal{G}'\mathcal{G}'''}{\mathcal{G}'^5}. \quad (4)$$



1 **3 The curvature parameter β**

2 **3.1 Equivalent perturbation of the measured ratio**

3 Calculating β involves two conceptual steps. First, the many contributors to the age uncertainty are
 4 combined into a single equivalent perturbation of the measured ratio; this perturbation represents the ratio
 5 change that would produce the same first-order age uncertainty as the full uncertainty budget. Second, the
 6 size of this perturbation is compared with the local curvature of the age–ratio relation at the sample's
 7 measured age. Their ratio defines β , which quantifies the extent to which the nonlinear transformation
 8 affects the age uncertainty. The remainder of this section develops these two steps.

9 A full uncertainty budget for a mass-spectrometric $^{230}\text{Th}/^{238}\text{U}$ age comprises numerous contributors,
 10 including sample and spike masses, spike calibration, measured isotope ratios, mass fractionation, hydride
 11 and tailing corrections, blank subtraction, decay constants, detrital parameters and sample heterogeneity.
 12 These contributors enter the age equation in different algebraic forms: most act multiplicatively on the
 13 measured ratio, in the form $A_{31} \propto x_1 x_2 / x_3$ (and similarly for A_{21}), whereas the decay constants act through
 14 the ingrowth function itself. What matters here is that they affect the calculated age principally through
 15 their first-order influence on the measured activity ratios, so their combined first-order effect can be
 16 represented locally as an equivalent perturbation of the primary daughter ratio A_{31} , evaluated at fixed A_{21} .
 17 The principal exception is the detrital correction, whose effect on A_{31} and A_{21} is not adequately represented
 18 by a single first-order perturbation, as discussed in Section 6. Eq. (5) defines the equivalent ratio uncertainty
 19 as

$$20 \quad \sigma_{\text{eq}} = |\mathcal{G}'(T)| u_{\text{lin}}(T), \quad (5)$$

21 where u_{lin} is the combined standard uncertainty of the age from the conventional first-order budget.
 22 Conceptually, Eq. (5) expresses the age uncertainty in ratio units: the local slope $\mathcal{G}'$ tells how much the
 23 measured ratio changes for a small change in age, so multiplying the age uncertainty by this slope converts
 24 it into an equivalent perturbation of the ratio. This equivalent perturbation is generally larger than the
 25 experimental uncertainty of the measured ratio itself, because it also carries the contributions of all other
 26 multiplicative inputs. The formulation applies to first order in every input by construction. Appendix A
 27 derives the small second-order residue introduced when the complete uncertainty budget is replaced by this
 28 single perturbation.



3.2 Definition and interpretation

The amount of nonlinearity depends on both the age and the analytical precision; the age alone cannot be the deciding variable, because the same chronometer can be measured with different levels of precision. This is already visible in existing literature (Ludwig, 2003): two thresholds are advised for the same chronometer, about 250 ka for alpha-spectrometric analyses and about 350 ka for thermal-ionisation mass spectrometry, because the two techniques achieve different precisions on the measured ratio. Two thresholds for one chronometer show that the relevant criterion is not an age at all, but a joint property of the age and the measurement precision. The parameter defined below makes that statement quantitative:

$$\beta = \frac{T^{(2)} \sigma_{\text{eq}}}{T^{(1)}} = -\frac{G'' \sigma_{\text{eq}}}{G'^2} \quad (6)$$

the second derivative of the inverse transformation measured in units of the first, over one standard deviation of the equivalent ratio perturbation. A small value of β means that the uncertainty samples an almost linear part of the transformation; a larger value means that it spans appreciable curvature. The magnitude $|\beta|$ measures how much curvature the uncertainty spans; the sign of β indicates the direction of the asymmetry and which coverage limit is at risk. Positive values skew the age distribution toward older ages and threaten the upper limit, negative values skew it toward younger ages and threaten the lower one. Substituting (5) and (4), the sensitivity coefficients cancel identically and

$$\beta = -\frac{G''}{G'} u_{\text{lin}} \equiv \mu(T) u_{\text{lin}}. \quad (7)$$

The factor $\mu = -G''/G'$ has the dimensions of a decay constant, so its reciprocal $\tau = 1/\mu$ is a timescale that sets the pace of the flattening. Writing $w_a = e^{-\lambda_3 T}$ and $w_b = (A_{21} - 1)e^{-(\lambda_3 - \lambda_2)T}$,

$$\mu = \frac{\lambda_3 w_a + (\lambda_3 - \lambda_2) w_b}{w_a + w_b} = \frac{\lambda_3 + (\lambda_3 - \lambda_2)(A_{21}^0 - 1)}{A_{21}^0}, \quad (8)$$

which is simply a weighted mean of the two decay rates λ_3 and $\lambda_3 - \lambda_2$, with weights that stay positive whenever $A_{21} \geq 1$. The second form follows by writing the weights in terms of the initial activity ratio A_{21}^0 , using $A_{21} - 1 = (A_{21}^0 - 1)e^{-\lambda_2 T}$: both weights then share the factor $e^{-\lambda_3 T}$, so μ and τ are independent of age and set by the initial ratio alone. Hence

$$\lambda_3 - \lambda_2 < \mu \leq \lambda_3 \quad (A_{21} \geq 1), \quad (9)$$

which has been verified numerically to a relative 10^{-9} over $A_{21} \in [1.001, 3]$ and $T \in [1 \text{ ka}, 1.5 \text{ Ma}]$. For uranium-depleted samples ($A_{21} < 1$) the weight w_b is negative and μ slightly exceeds λ_3 , by less than 15 % for a mild depletion of $A_{21} \geq 0.9$. The broader lesson is not restricted to this particular decay chain:



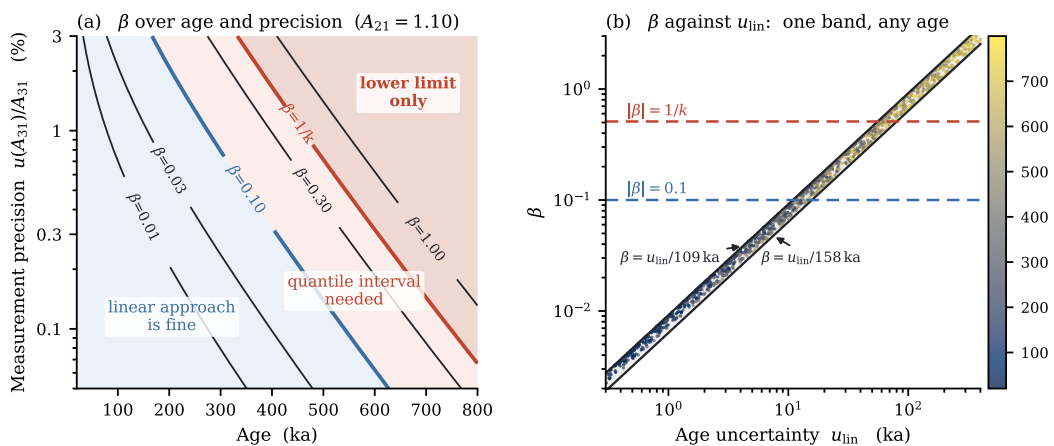
1 whenever the slope of the ingrowth function is a sum of decaying exponentials with positive coefficients,
 2 μ is a weighted mean of their rates and therefore lies between the slowest and the fastest rate. For a single-
 3 exponential chronometer, only one decay rate is present and μ reduces to that rate.

4 Equations (7) and (9) define the diagnostic used in this paper: for $^{230}\text{Th}/^{238}\text{U}$ dating, the degree of
 5 nonlinearity is, to within a factor $\kappa = 1.44$, the age uncertainty expressed in units of the time constant that
 6 controls the flattening of the ingrowth curve. Since $1/\lambda_3 = 109.0$ ka and $1/(\lambda_3 - \lambda_2) = 157.5$ ka, β is
 7 pinned to a narrow band,

$$8 \quad \frac{u_{\text{lin}}}{157.5 \text{ ka}} \leq \beta \leq \frac{u_{\text{lin}}}{109.0 \text{ ka}}. \quad (10)$$

9 In practice, an estimate of β can therefore be obtained without explicitly calculating μ : divide the
 10 reported linear age uncertainty by a characteristic timescale between approximately 109 and 158 ka. This
 11 estimate is sufficiently accurate to identify the relevant propagation regime. Figure 2 maps β across a range
 12 of ages and measurement precisions, and Figure 3 illustrates its geometric meaning. In essence, β compares
 13 the local curvature of the age–ratio relation with the width of the measurement uncertainty and therefore
 14 quantifies how strongly the nonlinear transformation distorts an otherwise symmetric measurement
 15 uncertainty. For a single-exponential chronometer the resulting response depends only on β .

16



17

18 **Figure 2:** β over age and measurement precision. **(a)** Contours of constant β for $A_{21} = 1.10$. They run
 19 diagonally, so neither axis alone decides the regime: the value $\beta = 0.1$ reached at 232 ka with 1.5 %
 20 precision is reached again at 351 ka with 0.5 % precision. The shaded bands are the three regimes of the



recipe: linear propagation adequate, quantile interval required, and lower limit only. **(b)** Each dot is one randomly drawn case (age 20–800 ka, activity ratio precision 0.05–3 %, A_{21} between 1.02 and 2.00), coloured by age and plotted at the u_{lin} and β it produces. All fall in the band between $u_{lin}/157.5$ ka and $u_{lin}/109.0$ ka, with the colours mixed throughout, as Eq. (10) requires.

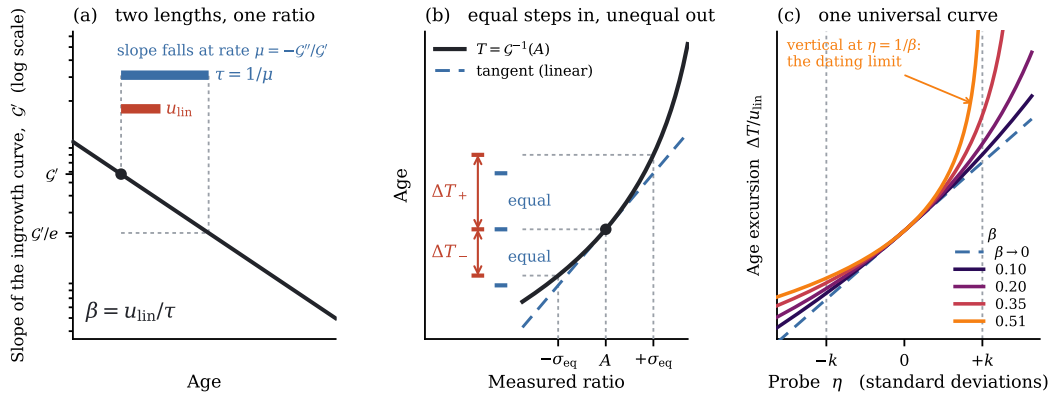


Figure 3: Geometric meaning of β . The curvature is exaggerated for clarity, and the axes are schematic rather than to scale. **(a)** The slope of the ingrowth curve decreases with age at a rate $\mu = -G''/G'$, so $\tau = 1/\mu$ is a length on the age axis, i.e. the distance over which the slope falls by a factor e . The age uncertainty u_{lin} is a second length, and their ratio is $\beta = u_{lin}/\tau$. **(b)** This ratio determines how equal ratio uncertainties are transformed into age uncertainties. The ratio values $A - \sigma_{eq}$ and $A + \sigma_{eq}$ map through $T = G^{-1}(A)$ to unequal age excursions, $\Delta T_+ > \Delta T_-$. The tangent, which represents the linear approximation, returns equal excursions. Because G^{-1} is convex, the true curve lies above the tangent, and the asymmetric age interval moves toward older ages. **(c)** For a single exponential chronometer, the age excursion depends only on β : $\Delta T/u_{lin} = -\ln(1 - \eta\beta)/\beta$. The family of curves therefore collapses onto a universal relation. The upper branch becomes vertical at $\eta = 1/\beta$, marking the dating limit.



3.3 Practical interpretation of β

The parameter β also unifies Ludwig's two empirical thresholds: at representative precisions of 1.5 % and 0.5 % for alpha and mass spectrometry, the two limits correspond to $\beta = 0.12$ and $\beta = 0.10$, respectively. They seem to represent the same underlying criterion reached at different measurement precisions. Figure 2a shows why: the contours of constant β run diagonally across the age–precision plane, so the value reached at 250 ka with alpha-spectrometric precision is reached again at 350 ka with mass-spectrometric precision. The four-step workflow of Section 1 can therefore be expressed as thresholds on the reported linear standard uncertainty of the $^{230}\text{Th}/^{238}\text{U}$ age: symmetric propagation is generally adequate in the linearity region $|\beta| \lesssim 0.1$, i.e. $u_{\text{lin}} < 11\text{--}16$ ka; an asymmetric coverage interval is needed when $0.1 \lesssim |\beta| \lesssim 1/z_p$, i.e. $11\text{--}16$ ka $< u_{\text{lin}} < 56\text{--}80$ ka; and for $|\beta| \gtrsim 1/z_p$, i.e. $u_{\text{lin}} > 56\text{--}80$ ka, no finite upper coverage limit exists at the specified probability level and the result should instead be reported as a lower age limit.

The value of β provides more than a yes-or-no test for nonlinearity: it also describes how the conventional linear treatment fails. To see this, expand the age transformation $T(A_{31} + \delta)$ to second order about $\delta = 0$, with $\delta \sim N(0, \sigma_{\text{eq}}^2)$. The resulting expressions describe three departures from the Gaussian distribution that linear propagation assumes: a shift in the centre, a change in the variance, and increasing asymmetry:

$$\begin{aligned} E[T] - T &= \frac{1}{2} \beta u_{\text{lin}}, \\ \text{Var}[T] &\approx u_{\text{lin}}^2 \left(1 + \frac{5}{2} \beta^2\right), \\ \gamma_1 &= 3\beta + \beta^3. \end{aligned} \quad (11)$$

A single computed value of β therefore provides approximate information about the bias, the skewness and the variance error at once. Appendix B records the moment calculation behind Eq. (11). Table 2 compares the series with the accurate values obtained by Gauss–Hermite quadrature (Sections 4–5). The bias approximation remains accurate to 2 % at $\beta = 0.1$ and to 8 % at $\beta = 0.21$, but the skewness series is already 23 % low at $\beta = 0.21$ and wrong by a factor of 1.8 at $\beta = 0.31$. The skewness fails first because it is a third-moment quantity dominated by the tail of the distribution, which is where the ingrowth curve bends most strongly. The series should therefore be treated as a diagnostic rather than as a correction: when more accurate coverage limits are required, the preferred approach is to solve the age equation directly at $A_{31} \pm k\sigma_{\text{eq}}$, rather than to approximate its inverse $\mathcal{G}^{-1}$ through an asymptotic series. Both the variance coefficient in Eq. (11) and the logarithmic form in Eq. (15) of Section 4.3 assume a constant μ , which holds for a single-exponential ingrowth function and to good approximation for Eq. (1).



1

2 **Table 2:** Second-order series ($A_{21} = 1.10$) against reference values obtained by Gauss–Hermite
 3 quadrature on the transformation (Sections 4–5), conditional on a finite age; they agree with a 1.2×10^5 -
 4 trial Monte Carlo to the digits shown.

T (ka)	$u(A_{31})/A_{31}$	β	bias/ u_{lin}		skewness γ_1	
			$\beta/2$	Ref.	$3\beta + \beta^3$	Ref.
100	0.3 %	0.0043	0.0021	0.0021	0.013	0.013
200	0.3 %	0.0144	0.0072	0.0072	0.043	0.043
300	0.3 %	0.0380	0.0190	0.0190	0.114	0.115
400	0.3 %	0.0922	0.0461	0.0467	0.277	0.285
500	0.3 %	0.2143	0.1071	0.1157	0.653	0.848
300	1.0 %	0.1267	0.0633	0.0649	0.382	0.404
400	1.0 %	0.3074	0.1537	0.1830	0.951	1.721

5 Finally, the reduction of the full uncertainty budget to a single equivalent perturbation is itself valid
 6 only to first order in each input. Appendix A derives its second-order residue in closed form and bounds it
 7 by a quantity requiring only numbers already on the report sheet; it is negligible for most practical purposes.
 8 Appendix A also gives a fixed-point refinement of the equivalent perturbation that removes most of the
 9 residue at the cost of one further Newton pass.

10 4 The coverage interval

11 4.1 Quantile transformation of the measured ratio

12 The central age is obtained by solving the dating equation $T = \mathcal{G}^{-1}$ for the measured ratio. Because
 13 the transformation is strictly monotonic, every possible ratio value corresponds to at most one age. This
 14 provides a direct way to propagate uncertainty: first determine the desired quantiles of the measured ratio,
 15 then map them through the inverse dating equation. This makes the origin of asymmetric age limits
 16 transparent. For a trial age t , the event $T \leq t$ is equivalent to the event that the corresponding ratio is no
 17 greater than $\mathcal{G}(t)$. Therefore, without further approximation beyond the reduction of the uncertainty budget
 18 to a single number in Section 3:



$$P(T \leq t) = P(\tilde{A}_{31} \leq \mathcal{G}(t)) = \Phi\left(\frac{\mathcal{G}(t) - A_{31}}{\sigma_{\text{eq}}}\right), \quad (12)$$

where $\tilde{A}_{31} \sim N(A_{31}, \sigma_{\text{eq}}^2)$ denotes the Gaussian random variable assigned to the activity ratio, centred on the measured value A_{31} . Equation (12) says that the cumulative distribution of the age can be obtained directly from that of the measured ratio. Inverting Eq. (12) gives the age corresponding to any cumulative probability p :

$$t_p = \mathcal{G}^{-1}(A_{31} + z_p \sigma_{\text{eq}}) \quad (13)$$

provided that $A_{31} + z_p \sigma_{\text{eq}} < A_{31}^{\text{max}}$. Here z_p is the standard-normal quantile corresponding to probability p ; for a central 95 % coverage interval, $z_{0.025} = -1.96$ and $z_{0.975} = +1.96$, so the two coverage limits are obtained simply by solving the age equation twice, at $A_{31} \pm 1.96 \sigma_{\text{eq}}$. The two limits are generally not equally distant from the central age, because the age–ratio relation is nonlinear. Each solution requires only six to eight Newton iterations, starting from the estimate $T_0 = -\ln(1 - x/A_{31}^{\text{max}})/\lambda_3$, where x is the ratio value being inverted. In a spreadsheet, the iterations can be performed in adjacent columns, applying the same routine already used for the central age. If an iteration step gives an impossible age, one bisection step brings the solution back into the allowed range, keeping the calculation stable near the dating limit. If $x \geq A_{31}^{\text{max}}$, the logarithm is undefined because no finite age corresponds to that ratio; this is the unbounded case treated in Section 5. The transformation itself carries no assumption about the underlying statistical distribution: Eq. (13) needs only the quantile of whatever distribution is assigned to the equivalent perturbation, so that the Gaussian quantile is replaced by the corresponding Student-t quantile when the inputs warrant it (Section 10).

4.2 Geometric interpretation

Figure 4 illustrates the transformation at 500 ka. Equally spaced ratio quantiles map to unequally spaced ages. In the example, the nearly equal quantiles at $z = -1.96, -1, 0, +1, +1.96$ produce successive age intervals of 18.7, 23.2, 28.9 and 36.3 ka. The increasingly wide intervals reflect the flattening of the ingrowth curve: the same change in ratio corresponds to a larger change in age at older ages. The asymmetric age interval is therefore not introduced by an arbitrary correction; it follows directly from the monotonic dating equation.

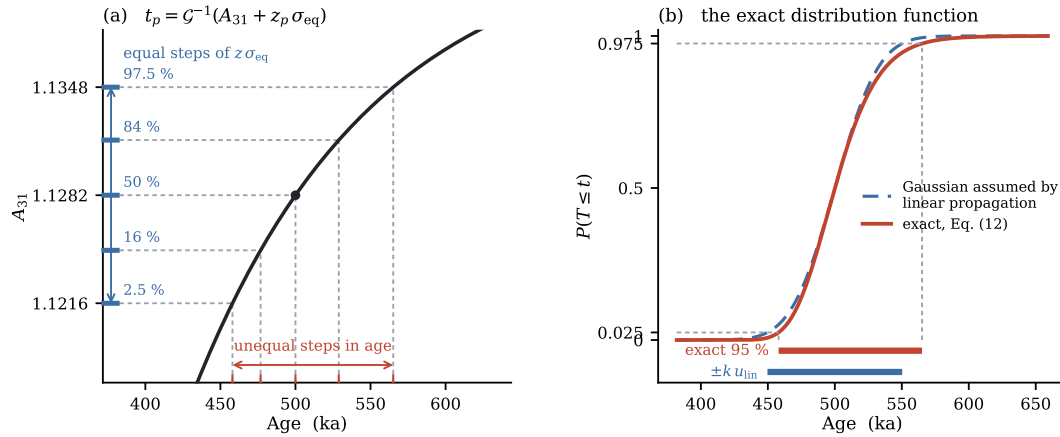


Figure 4: Construction of the coverage interval at 500 ka. **(a)** Nearly equally spaced quantiles of the measured ratio, $A_{31} + z_p \sigma_{eq}$, map through the ingrowth function to unequally spaced ages: the gaps between successive rungs are 18.7, 23.2, 28.9 and 36.3 ka. **(b)** The distribution function of the age follows from that of the ratio through Eq. (12) without approximation; the interval is read off at $p = 0.025$ and 0.975 . The Gaussian assumed by linear propagation is shown for comparison.

Equation (13) is a standard property of monotonic transformations: quantiles are preserved under a monotonic mapping. Its practical importance here is that the nonlinear age interval can be calculated directly from the ratio quantiles: once the uncertainty budget has been reduced to the equivalent perturbation of Section 3, no Monte Carlo simulation is required to obtain the asymmetric coverage limits. If age rises steadily with the measured ratio, then the age marking the 2.5th percentile is simply the age obtained by feeding the 2.5th-percentile ratio through the dating equation. Each percentile of the ratio maps to the same percentile of the age. The remaining approximation lies in the scalar reduction itself, which expresses the age as a monotonic function of a single Gaussian variable; its second-order residual error is bounded in Appendix A.

4.3 Relation to the Kragten method

Spreadsheets are commonly used in radiometric dating laboratories for data reduction and for the propagation and assessment of uncertainty budgets. One method used to propagate uncertainty is the Kragten method (Eurachem, 2012; Kragten, 1994), in which the age is recalculated after each input is shifted by its own standard uncertainty and the resulting age changes are combined in quadrature. It was



1 recently compared with analytical and GUM-Workbench propagation (Macsik and Pommé, 2026). The
2 quantile interval can be implemented as a two-sided Kragten calculation applied to the combined equivalent
3 perturbation, probing it at $\pm k\sigma_{\text{eq}}$ to obtain the two age excursions

$$4 \quad [T(A_{31} - k\sigma_{\text{eq}}), T(A_{31} + k\sigma_{\text{eq}})], \quad (14)$$

5 where k is the desired coverage factor; for a central 95 % interval under a Gaussian model, $k = 1.96$. This
6 procedure is Eq. (13) written in spreadsheet form. It requires the same two additional age solves as a
7 conventional two-sided probe, but the probe is made at k standard deviations, not at one standard deviation
8 followed by multiplication by k .

9 It may seem natural to probe at a single standard deviation $\pm\sigma_{\text{eq}}$ instead, to obtain the two age
10 excursions $u^+ = T(A_{31} + \sigma_{\text{eq}}) - T$ and $u^- = T - T(A_{31} - \sigma_{\text{eq}})$, and then to expand each by the coverage
11 factor k : $[T - k u^-, T + k u^+]$. This improves on the symmetric interval, but it does not reproduce the
12 correct coverage interval, because nonlinear transformations do not scale linearly with the size of the input
13 perturbation, as shown below.

14 To see this, expand the inverse transformation $T = \mathcal{G}^{-1}$ about the measured ratio, with the step
15 measured in standard deviations, $\delta = \eta \sigma_{\text{eq}}$. The age excursion can then be written as a series in which β
16 is the only parameter (the same expansion as Eq. (11), reorganised):

$$17 \quad \Delta T(\eta) = u_{\text{lin}} \left(\eta + \frac{1}{2}\beta\eta^2 + \frac{1}{3}\beta^2\eta^3 + \frac{1}{4}\beta^3\eta^4 + \dots \right) = -\frac{u_{\text{lin}}}{\beta} \ln(1 - \eta\beta) \quad (15)$$

18 so that the interval is $[\Delta T(-k), \Delta T(k)]$ while the expanded 1σ Kragten interval is $[k\Delta T(-1), k\Delta T(1)]$.
19 Subtracting, with the upper sign for the upper limit, yields

$$20 \quad \text{error} = ku_{\text{lin}} \left[\frac{1}{2}\beta(k-1) \pm \frac{1}{3}\beta^2(k^2-1) \right] \quad (16)$$

21 The first term is common to both limits and therefore displaces the interval without changing its width,
22 while the second has opposite signs at the two ends and therefore changes the width. To leading order, then,
23 the 1σ Kragten interval has approximately the right width but sits in the wrong place: it is displaced toward
24 younger ages by

$$25 \quad \frac{1}{2}\beta k(k-1) u_{\text{lin}} = k(k-1) \times (\text{the bias}), \quad (17)$$

26 For $k = 1.96$ this displacement is 1.88 times the bias. The second term makes the interval too narrow.
27 Expressed as fractions of the interval ku_{lin} , the displacement and the width deficit are



$$\frac{\text{displacement}}{k u_{lin}} = \frac{1}{2}(k-1)|\beta| \approx 0.48 |\beta|, \quad (18)$$

$$\frac{\text{width deficit}}{2k u_{lin}} = \frac{(k^2-1)}{3}\beta^2 \approx 0.95 \beta^2. \quad (19)$$

The improvement is incomplete: the leading error of the plain GUM interval is $\frac{1}{2}\beta k^2 u_{lin}$, while Eq. (17) gives the fraction $\frac{(k-1)}{k} = 1 - \frac{1}{k}$ of that, so Kragten at 1σ probe removes $k^{-1} \approx 51\%$ of the displacement error. For the width deficit, the error reduction is smaller; it falls from $\frac{k^2}{3}\beta^2$ (GUM) to $\frac{(k^2-1)}{3}\beta^2$, a factor $k^{-2} \approx 26\%$ only. For n^{th} order terms, the reduction is by k^{-n} . Moreover, the errors in Eqs. (18) and (19) still grow with β , with only the coefficient reduced. Two-sided Kragten probed at $\pm\sigma_{eq}$ repairs the shape of the interval but not its scaling with β , therefore it hardly extends the usable range of the expanded uncertainty propagation compared to the GUM.

Eq. (14) uses the same two Newton solves as the 1σ probe. Probing the combined perturbation at one standard deviation and multiplying by k is never preferable to probing it at k standard deviations. The per-input Kragten block of the uncertainty budget is a different object and should stay at $\pm u_i$, since that is the scale at which sensitivity coefficients and per-input curvatures are defined. It is only the combined probe used to build the coverage interval that should be taken at $k\sigma_{eq}$. Table 3 shows results from the three methods: linear “GUM”, “Kragten 1σ ”, and “Kragten $k\sigma$ ”. The practical recommendation is therefore straightforward: retain the ordinary per-input uncertainty budget, but calculate the final asymmetric interval by solving the age equation at the target ratio quantiles.

Table 3: Endpoint errors as a percentage of the linear half-width ku_{lin} ($k = 1.96$) at $A_{21} = 1.10$. Positive values mean the endpoint is too young. “GUM” is linear, “Kragten 1σ ” probes at $\pm\sigma_{eq}$ and scales by k , and “Kragten $k\sigma$ ” probes at $\pm k\sigma_{eq}$.

T	$u(A_{31})/A_{31}$	β	GUM		Kragten 1σ		Kragten $k\sigma$		Eq. (18)
(ka)			lower	upper	lower	upper	lower	upper	0.48β
200	0.3 %	0.0144	1.4	1.4	0.7	0.7	0.0	0.0	0.7
300	0.3 %	0.0380	3.6	3.9	1.7	2.0	0.0	0.0	1.8
400	0.3 %	0.0922	8.1	10.3	3.7	5.4	0.0	0.0	4.4
500	0.3 %	0.2143	16.5	29.5	7.1	17.0	0.0	0.0	10.3



T	$u(A_{31})/A_{31}$	β	GUM		Kragten 1σ		Kragten $k\sigma$		Eq. (18)
300	1.0 %	0.1267	10.7	14.9	4.8	8.0	0.0	0.0	6.1
400	1.0 %	0.3074	21.8	52.5	9.0	33.1	0.0	0.0	14.8

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2 **5 A probabilistic dating limit**

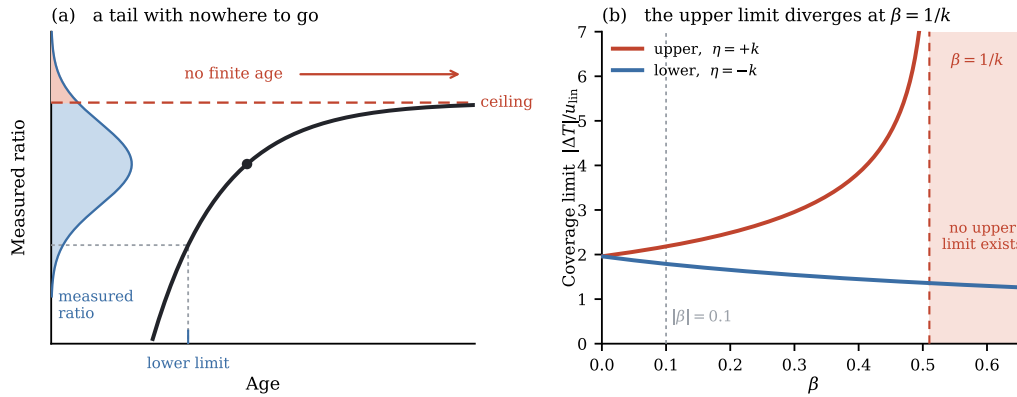
3 The dating limit described in this section is a joint property of the chronometer, the precision achieved
4 and the requested coverage probability; it is not a new instrumental or analytical limit. Its origin is simple:
5 the measured ratio can only take values below a physical ceiling, but the Gaussian placed on its uncertainty
6 has tails that run to infinity. Formally, the Gaussian variable $\tilde{A}_{31}$ assigned to the measured ratio has
7 unbounded range, whereas the inverse of the ingrowth function $\mathcal{G}^{-1}$ exists only below $A_{31}^{\max}$. A random
8 draw from that Gaussian can therefore land above the ceiling, where no finite age exists. This can happen
9 *at any age*, not only close to the dating limit. Figure 5 shows the mechanism and its consequence. The
10 probability of such a draw is

$$11 \quad P(\tilde{A}_{31} \geq A_{31}^{\max}) = 1 - \Phi\left(\frac{A_{31}^{\max} - A_{31}}{\sigma_{\text{eq}}}\right) > 0 \quad (20)$$

12 This probability is never zero, though it becomes vanishingly small when the sample sits well below the
13 ceiling. Consequently, whenever a drawn ratio exceeds the ceiling, the age T is undefined, so the age
14 distribution always carries a small probability of “no finite age”.

15 Two probability constructions must be distinguished: the unconditional equivalent-ratio model, which
16 keeps those draws, and the conditional finite-age model, which excludes them and renormalises. The two
17 agree while the excluded probability is negligible, and diverge as it grows. Strictly speaking, therefore,
18 neither the ordinary (unconditional) mean $E[T]$ nor the standard uncertainty $u(T)$ exists, because the
19 distribution extends over ratio values where T is undefined. This observation also clarifies the status of the
20 series in Eq. (11): they are asymptotic descriptions of finite-age behaviour rather than moments of the
21 unconditional age distributions. They may provide excellent approximations for small β and yet cannot be
22 made convergent, since the quantity they approximate does not exist. If a simulation is asked for the mean
23 age, the number it returns depends on what it did with the draws that yielded no age at all.

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Figure 5. The dating limit. (a) Schematic; the curvature is exaggerated and the axes carry no scale. The Gaussian assigned to the measured ratio extends above the ceiling of the ingrowth function, such that the shaded tail does not correspond to a finite age. The lower quantile still maps to a finite age, so a lower limit survives. (b) Evaluating Eq. (15) at $\eta = \pm k$ gives the two coverage limits in units of u_{lin} . The lower limit contracts as β grows, whereas the upper diverges at $\beta = 1/k$, beyond which no finite upper limit exists at any age.

The problem is not specific to a Gaussian, as it arises for any input distribution with non-zero probability above the ceiling. The divergence of T at the ceiling is only logarithmic, so *conditional* moments given a finite age can still exist and be well behaved. However, conditioning on $T < \infty$ is a modelling choice, not a property of the measurand. A Monte Carlo that silently discards draws above the ceiling is therefore estimating $E[T | T < \infty]$ and the corresponding conditional quantiles, and its answer drifts as the discarded fraction grows. By contrast, the quantile in Eq. (13) imposes no such conditioning. The difference therefore arises from the conditioning convention rather than from an error in either numerical procedure.

Although the unconditional moments of T do not exist because the Gaussian distribution assigns non-zero probability above the physical ceiling, moments of the finite-age distribution can still be useful as diagnostics, and they need no Monte Carlo simulation. Writing $T(z) = \mathcal{G}^{-1}(A_{31} + z\sigma_{\text{eq}})$, each moment is a one-dimensional Gaussian integral $\int T(z)^n \varphi(z) dz$ normalised over the finite-age domain and computed by Gauss-Hermite quadrature, a standard numerical scheme that replaces the integral with a weighted sum over a few well-chosen points. Forty such points reproduce a 1.2×10^5 -trial Monte Carlo calculation to six digits in the mean and four digits in the standard deviation. The skewness converges more slowly, to about



two digits, because $T(z)$ diverges logarithmically as $A_{31} + z\sigma_{eq} \rightarrow A_{31}^{\max}$, so the third moment is dominated by the outermost node on the upper side. This slow convergence is a property of the underlying distribution, not a deficiency of the quadrature. The moments can in turn be converted into approximate quantiles through a Cornish–Fisher expansion, $t_p \approx \bar{T} + \sigma[z_p + (\gamma_1/6)(z_p^2 - 1)]$, with $\bar{T}$, σ , and γ_1 the conditional mean, standard deviation and skewness, which improves the interval compared with a Gaussian interval built from the same mean and standard deviation. It remains, however, an expansion in β and degrades in the same way as the series of Eq. (11).

Note that the upper limit is not lost as soon as some probability crosses the ceiling. The 97.5 % quantile of T ceases to exist only when more than 2.5 % of the Gaussian probability lies above the ceiling. For example, at $T = 600$ ka, $A_{21} = 1.10$, $u(A_{31})/A_{31} = 0.3$ %, the probability above the ceiling is 1.65 % and the upper limit does exist, at 930 ka. A Monte Carlo procedure that discards the 1.65 % of draws above the ceiling and renormalises over the surviving draws reports 819 ka, because its 97.5 % quantile is the $0.975 \times 0.9835 = 0.958$ quantile of the original, unconditional distribution. The discrepancy is therefore a consequence of the renormalisation.

The reporting rule is simple. The age solution should be reported without an ad hoc shift, together with the asymmetric coverage interval obtained from Eq. (13). The bias term $1/2 \beta u_{lin}$ is best regarded as a diagnostic of nonlinearity rather than a correction: there is nothing to correct toward. Once the upper coverage limit ceases to exist, the result should instead be reported as a lower age limit (Feldman and Cousins, 1998).

Concretely, an upper limit to a $100p$ % coverage interval exists if and only if $A_{31} + z_p \sigma_{eq} < A_{31}^{\max}$, or equivalently

$$A_{31}^{\max} - \mathcal{G}(T) > z_p \mathcal{G}'(T) u_{lin}. \quad (21)$$

Division by $\mathcal{G}'$ and using Eqs. (2) and (3) yields $(w_a + \kappa w_b)/[\lambda_3(w_a + w_b)]$ on the left-hand side, which is bounded between $1/\lambda_3$ and $1/(\lambda_3 - \lambda_2)$ by the same argument as in Eq. (9), and exceeds $1/\mu$ by not more than a few percent over a useful range. Replacing it by the bound gives the conservative practical criterion

$$u_{lin} < \frac{1}{z_p \mu} \Leftrightarrow |\beta| < \frac{1}{z_p} \quad (22)$$

The practical form is deliberately conservative: it may declare the upper limit lost marginally before the criterion of Eq. (21) is met. For a 95 % coverage interval, it corresponds to u_{lin} being below approximately 56–80 ka. Beyond that precision threshold, the correct statement of the result is a lower limit on the age,



because no finite upper coverage limit exists. Eq. (22) provides an operational definition of the dating limit and, it is not a fixed age built into the chronometer, but a joint property of the chronometer and the measurement precision: the very same chronometer may or may not yield a finite upper coverage limit, depending only on how precisely the ratio was measured.

That an upper limit must sometimes be infinite is not a flaw of the method: it follows mathematically from demanding correct coverage when a bounded measurement is inverted through a nonlinear relation. It has been shown for this class of models that a confidence procedure with correct coverage must produce an unbounded interval with positive probability (Gleser and Hwang, 1987). Avoiding this conclusion using an asymptotic approximation does not solve the problem; it merely hides it. Linear propagation “avoids” the unbounded interval by construction: it simply assigns a finite symmetric uncertainty even when the transformed distribution has no finite upper limit.

6 Validation

The validation evaluates two elements of the proposed workflow: (i) the adequacy of representing a multi-input uncertainty budget by a single equivalent ratio perturbation, and (ii) the accuracy of the subsequent monotonic quantile transformation under that reduced model. Table 4 contains three synthetic U–Th cases: LIN-1, a mid-Pleistocene speleothem with near-linear propagation; DIRTY-1, a case with its uncertainty dominated by a detrital correction; and OLD-1, an old sample near the dating limit. The detrital parameters are bulk-earth assumptions in all three cases. The simulation samples every input of a multifaceted uncertainty budget independently, discarding draws above the ceiling, whereas the direct method reduces the uncertainty budget to a single combined perturbation σ_{eq} and retains the unconditional quantile interpretation. These are worked examples rather than an exhaustive test for any arbitrary uncertainty budget.

Table 4: Direct quantile intervals under the equivalent-perturbation model compared with Monte Carlo results for three synthetic U–Th cases; LIN-1 represents a linear case, whereas DIRTY-1 and OLD-1 exercise the detrital correction and the dating limit, respectively. Although u_{MC} strictly denotes the



1 standard deviation of the finite-age distribution, the discarded fraction was only 0.096 % for OLD-1 and
 2 negligible for the other two cases. Intervals are evaluated at $k = 1.96$ using MC with 2 million trials.
 3

Sample	T	u_{lin}	u_{MC}	2.5 % (ka)		97.5 % (ka)	
	(ka)	(ka)	(ka)	MC	Eq. (13)	MC	Eq. (13)
LIN-1	157.996	1.640	1.640	154.83	154.83	161.26	161.26
DIRTY-1	115.955	2.114	2.113	111.89	111.89	120.18	120.18
OLD-1	500.117	37.217	45.811	443.84	443.80	614.93	616.51

4 For LIN-1 ($\beta = 0.014$), linear propagation is appropriate and the quantile and Monte Carlo treatments
 5 agree to the digits shown. For OLD-1 ($\beta = 0.33$), the age distribution is strongly right-skewed: the upper
 6 excursion is twice the lower, and 68 % of the variance stems from ages above the mean. The interval of Eq.
 7 (13) is strongly asymmetric, $-56/+116$ ka, and the symmetric half-width is correspondingly 16 % narrower
 8 than its mean half-width, 86 ka. The lower limit from the quantile method agrees with the Monte Carlo
 9 result to 0.04 ka. The upper limits differ by 1.6 ka, because 0.096 % of the MC draws yield no finite age
 10 and are discarded, making its 97.5 % point a conditional quantile whereas Eq. (13) gives the unconditional
 11 one. That discarded fraction predicts a difference of 1.6 ka, which is what is observed.

12 DIRTY-1 ($\beta = +0.019$) illustrates a different limitation: here β and the bias diverge. The coverage
 13 interval is robust, landing within 0.13 ka of the MC benchmark, as is the Kragten central difference. But β
 14 does not correctly predict the bias: it predicts +20 a (years) where the simulation gives -11 ± 5 a. The reason
 15 is that the dominant detrital parameter a_0 , which carries 81 % of the sample's variance, contributes second-
 16 order curvature of its own that a single perturbation of A_{31} cannot represent, so that it falls outside the scalar
 17 reduction of Section 3 and β , which measures only the curvature of the $A_{31} \rightarrow T$ map, cannot see it. It is
 18 not the second-order residue of Appendix A: only 3.6 % of the variance is carried by A_{21} , for which Eq.
 19 (A.6) predicts 4 a against an observed 130 a. Implicit differentiation instead supplies the missing term in
 20 closed form, $\frac{1}{2}T_{a_0 a_0} u^2(a_0) = -17.3$ a, which, added to the A_{31} and A_{21} contributions scaled to their true
 21 shares of the variance (+2.8 a and +1.8 a), totals -12.7 a and matches the two-sided Kragten sum. This is
 22 not a failure of the interval method but a statement of the criterion's scope: β diagnoses the curvature of the
 23 ratio-to-age transformation along the equivalent-perturbation direction, and is not in general a predictor of
 24 the total bias generated by arbitrary input parameters. The gap between β and the true bias leaves the
 25 coverage interval unaffected, since the Kragten block probed at $A_{31} \pm k\sigma_{\text{eq}}$ still predicts the uncertainty
 26 intervals accurately.



To test the method more stringently than by comparing interval endpoints, we ran a direct coverage test. The two-input model, with the daughter ratio at the stated precision and the parent ratio at 0.2 %, was sampled 5×10^6 times. Each draw was solved for an age, and the fraction of the resulting age distribution contained in each candidate interval collected. Over ages from 200 to 450 ka and ratio precisions from 0.3 % to 1.0 %, spanning β from 0.018 to 0.49, the interval of Eq. (13) achieves 95.0 % coverage to within the sampling noise throughout. By contrast, the symmetric GUM interval falls to 92.5 % at $\beta = 0.32$ and to 91.3 % at $\beta = 0.49$, and the one-standard-deviation Kragten interval to 94.0 %. Because the simulation perturbs both inputs separately, whereas the interval is constructed from their single equivalent perturbation, this test also probes the error introduced by the reduction (Section 3), not only the transformation. The resulting discrepancy remains below the sampling noise up to $\beta \approx 0.3$. At $\beta = 0.49$, with the upper limit close to diverging, the Monte Carlo discards the 1.8 % of draws that have no finite age, whereas Eq. (13) conditions on nothing. Since every discarded draw lies above the upper limit, the interval still holds 95 % of the unconditional probability but $0.95/(1 - 0.018) = 96.7$ % of the conditional one, which is what the simulation reports.

15

Table 5: Empirical coverage of the three intervals under the full two-input model, in contrast to the single-input endpoint errors of Table 3. Each row is 5×10^6 draws of A_{31} at the stated precision and A_{21} at 0.2 %, each solved for an age. Note that β is larger than in Table 3 for the same scenario because $u(A_{21}) = 0.2$ % is propagated here as well. The entries are the percentage of the resulting age distribution contained in the nominal 95 % interval. “GUM” is the symmetric interval, “Kragten 1σ ” probes at $\pm\sigma_{eq}$ and scales by k , and Kragten $k\sigma$ from Eq. (13) probes at $\pm k\sigma_{eq}$. Sampling noise is about 0.01 %. The excess coverage of the Kragten $k\sigma$ interval at the highest β values reflects the fraction f of simulation draws discarded above the ceiling, resulting in a conditional coverage of $0.95/(1 - f)$.

23

T (ka)	$u(A_{31})/A_{31}$	β	GUM	Kragten 1σ	Kragten $k\sigma$
200	0.3 %	0.018	94.99	95.00	95.00
300	0.3 %	0.050	94.92	94.96	95.00
400	0.3 %	0.123	94.56	94.82	94.99
500	0.3 %	0.288	92.81	94.13	95.01
300	1.0 %	0.131	94.52	94.82	95.00
400	1.0 %	0.318	92.49	94.02	95.06
450	1.0 %	0.487	91.27	94.25	96.72



1 The validation therefore supports three conclusions: (a) the scalar reduction is sufficiently accurate
 2 for the tested cases; (b) the direct quantile transformation provides the correct asymmetric interval; and (c)
 3 Monte Carlo results must be interpreted carefully when non-solvable draws are discarded.

4 7 Generalisation to monotonic chronometers

5 7.1 Monotonic chronometers

6 Although the method has been developed with the $^{230}\text{Th}/^{238}\text{U}$ chronometer in a central role, the
 7 mathematical argument applies unchanged to any monotonic chronometer, of which radiometric dating
 8 offers a wide variety across geochronology and nuclear forensics (Dickin, 2018; Kristo et al., 2016; Mayer,
 9 Wallenius and Varga, 2013; Pommé, 2015; Pommé and Collins, 2014; Pommé, Collins, Harms and Jerome,
 10 2016; Pommé, Jerome and Venchiarutti, 2014; Reiners et al., 2018; Williams, Gaffney and Kristo, 2016).
 11 Nothing in Sections 3–6 used the specific form of Eq. (1). The quantile transformation requires only that
 12 the measured ratio be a strictly monotonic function of the age over the interval of interest. The scalar
 13 reduction to a single equivalent perturbation and the resulting criterion β additionally rely on the
 14 uncertainty-budget assumptions of Section 3, while the dating limit requires the measured quantity to
 15 approach a finite asymptotic bound. Under those conditions, the chronometer enters only through μ . Let a
 16 chronometer be defined by a strictly monotonic ingrowth or decay function $R = \mathcal{G}(T)$ relating a measured
 17 ratio to the age. Then $\beta = -(\mathcal{G}''/\mathcal{G}') u_{\text{lin}} = \mu u_{\text{lin}}$, the coverage interval is $\mathcal{G}^{-1}(R \pm z_p \sigma_{\text{eq}})$, and the upper
 18 limit is unbounded when $|\beta| > 1/z_p$. When $\mathcal{G}$ is a single exponential, the result is sharper still, because μ
 19 is then a decay constant and Eq. (10) becomes an equality. The discussion below therefore concerns the
 20 nonlinear transformation of idealised monotonic ratio–age relations; it does not encompass the geologic
 21 complexity or model choices of a full data-reduction workflow.

22 7.2 Three families

23 In practice, the monotonic chronometers fall into three families, set out in Table 6; Table 7 provides
 24 the resulting thresholds. The distinction that matters is whether the measured quantity is bounded. In family
 25 A the measured quantity is a surviving parent that decreases toward zero; in family B it is a daughter/parent
 26 activity ratio that increases toward secular equilibrium; in family C it is a daughter/parent atom ratio that
 27 accumulates without bound (Figure 6a). Besides being inferred from atom ratios measured by mass
 28 spectrometry, activity ratios can also be determined directly by radioactivity measurements traceable to the
 29 SI (Pommé, 2022). Families A and B have a finite physical bound, so part of the input uncertainty
 30 distribution can correspond to “no finite age”, producing a probabilistic dating limit. For family A the bound



1 acts from below: once the lower ratio quantile reaches the floor at zero, the upper age solve has no root.
 2 This is the mirror image of the ceiling case in family B. For family C the age–ratio relation has the opposite
 3 curvature: β is negative, the age distribution is skewed toward *younger* values rather than older, and the
 4 measured ratio grows without bound, so no upper coverage limit can fail (Figure 6b). Family C is therefore
 5 exempt from the upper dating limit, since there is no ceiling for the input distribution to cross. It is not
 6 exempt from nonlinearity in principle, but the bound derived below shows that the criterion cannot be
 7 approached at any achievable precision.

8

9 **Table 6:** The three families of monotonic chronometers. Symbols used are $\Delta\lambda = \lambda_D - \lambda_P$ and $\kappa = \lambda_D/\Delta\lambda$,
 10 where λ_P and λ_D are the parent and daughter decay constants, $\kappa' = \lambda_P/\Delta\lambda = \kappa - 1$ is the atom-ratio
 11 counterpart of κ , and $\partial T/\partial R$ is the age sensitivity to R . Since $\mathcal{G}'$ is affine in R , the second derivative is
 12 $\mathcal{G}'' = -\mu\mathcal{G}'$ and $\mu = -\mathcal{G}''/\mathcal{G}'$ is constant. The bound R_∞ is the ratio at which $\mathcal{G}' = 0$; for families A and B
 13 this is a physical bound, whereas for family C it lies outside the physical range. Families B and C differ in
 14 the ratio measured and in the sign of $\Delta\lambda$: the daughter is shorter-lived in B, so the activity ratio saturates at
 15 κ , and longer-lived in C, so $\kappa' < 0$ and the atom ratio can grow without bound. For most family C
 16 members in Table 7, $\mu \approx -\lambda_P$ to good approximation, except for ^{243}Am – ^{239}Pu .

Family	Measured quantity	$\mathcal{G}(T)$	$\partial T/\partial R = 1/\mathcal{G}'$	$\mu = -\mathcal{G}''/\mathcal{G}'$	R_∞
A	surviving parent N_P	$e^{-\lambda_P T}$	$-(\lambda_P R)^{-1}$	$\lambda_P > 0$	$R = 0$
B	activity ratio A_D/A_P	$\kappa(1 - e^{-\Delta\lambda T})$	$(\lambda_D - \Delta\lambda R)^{-1}$	$\Delta\lambda > 0$	$R = \kappa$
C	atom ratio N_D/N_P	$\kappa'(1 - e^{-\Delta\lambda T})$	$(\lambda_P - \Delta\lambda R)^{-1}$	$\Delta\lambda < 0$	$R = \kappa'$

17

18 **Table 7:** Thresholds for common chronometers; the family column refers to Table 6 and $\tau = 1/\mu$. The
 19 linear regime is $u_{\text{lin}} \lesssim 0.1\tau$ and the upper coverage limit is unbounded for $u_{\text{lin}} \gtrsim \tau/z_p$ ($z_{0.975} = 1.96$); the last
 20 column does not apply to family C, whose only failure mode is at the young end, for very large
 21 uncertainties. For $^{230}\text{Th}/^{238}\text{U}$ ($A_{21} > 1$), the range covers A_{21} from 1 upward, Eqs. (1) and (2). Half-lives are
 22 consistent with the DDEP-recommended values (Bé et al., 2004–2016), with two exceptions following
 23 geochronological convention: ^{230}Th and ^{234}U (Cheng et al., 2013), and ^{40}K and ^{81}Kr , taken from
 24 NUBASE2020 (Kondev et al., 2021); ^{81}Kr lies outside the DDEP evaluation set and its half-life carries a
 25 5 % uncertainty.



chronometer	family	unit	τ	0.1τ	τ/z_p
$^{140}\text{Ba}-^{140}\text{La}$	B	d	2.788	0.279	1.422
$^{227}\text{Th}-^{223}\text{Ra}$	B	d	42.49	4.249	21.68
$^{95}\text{Zr}-^{95}\text{Nb}$	B	d	111.3	11.13	56.79
$^{241}\text{Pu}-^{241}\text{Am}$	C	a	21.38	2.138	—
^{210}Pb	A	a	32.03	3.203	16.34
$^{238}\text{Pu}-^{234}\text{U}$	C	a	126.6	12.66	—
$^{241}\text{Am}-^{237}\text{Np}$	C	a	624.2	62.42	—
^{14}C	A	ka	8.223	0.822	4.196
$^{240}\text{Pu}-^{236}\text{U}$	C	ka	9.468	0.947	—
$^{243}\text{Am}-^{239}\text{Pu}$	C	ka	15.30	1.530	—
$^{239}\text{Pu}-^{235}\text{U}$	C	ka	34.77	3.477	—
$^{231}\text{Pa}/^{235}\text{U}$	B	ka	47.26	4.726	24.11
$^{230}\text{Th}/^{238}\text{U}$ ($A_{21} = 1$)	B	ka	109.0	10.90	55.64
$^{230}\text{Th}/^{238}\text{U}$ ($A_{21} > 1$)	B	ka	109.0–157.5	10.9–15.8	55.6–80.4
$^{234}\text{U}-^{230}\text{Th}$ model age	B	ka	157.5	15.75	80.37
^{81}Kr	A	ka	330.4	33.04	168.6
$^{40}\text{K}-^{40}\text{Ar}$	C	Ga	1.800	0.180	—
$^{235}\text{U}-^{207}\text{Pb}$	C	Ga	1.015	0.102	—
$^{238}\text{U}-^{206}\text{Pb}$	C	Ga	6.446	0.645	—
$^{87}\text{Rb}-^{87}\text{Sr}$	C	Ga	71.57	7.157	—

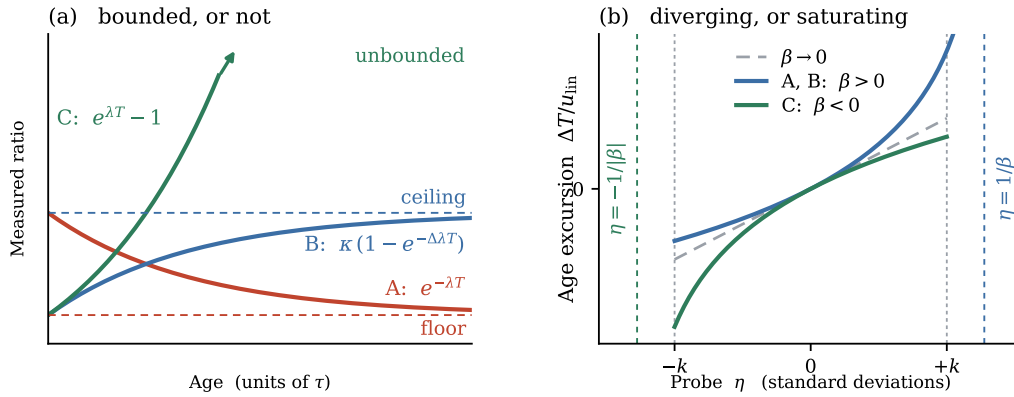


Figure 6. Schematic representation of three families of monotonic chronometers. **(a)** The measured ratio versus age in units of τ . Families A and B are bounded by a floor at $R = 0$ and a ceiling at $R = \kappa$ respectively, and family C accumulates without bound. **(b)** The resulting age excursion for a probe of η standard deviations, at $|\beta| = 0.4$. For the bounded families $\beta > 0$, the excursion would diverge at $\eta = 1/\beta$, the dating limit, which lies outside the coverage range drawn here. For family C ($\beta < 0$) the excursion saturates and no upper coverage limit can fail; its lower limit fails only at $|\beta| > 1/k$, the same threshold as the upper limit of families A and B, which for an accumulating ratio would need a precision worse than 51 %, which does not arise in practice.

7.3 Thresholds

The family C entries of Table 7 include the parent–daughter pairs used in nuclear forensics, where the daughter outlives the parent and the atom ratio therefore accumulates without bound (Pommé, 2015; Pommé, Jerome and Venchiarutti, 2014). Their thresholds are short: 0.1τ is 2.1 a for ^{241}Pu – ^{241}Am and 12.7 a for ^{238}Pu – ^{234}U , so these are chronometers for which $|\beta| \lesssim 0.1$ is not satisfied automatically, unlike the geological cases where an uncertainty of that size does not occur. The same structure dates americium materials through the ^{239}Pu – ^{243}Am atom ratio ($\tau = 15.3$ ka), for which the parent half-life was recently redetermined by absolute and relative activity measurements (Marouli et al., 2020).

One property is shared by all three families. Scaling every decay constant by a common factor rescales the clock, so at fixed measured ratio the age scales inversely. Differentiating this inverse scaling gives a sum rule for the decay-constant sensitivities, $\sum_i \frac{\partial T}{\partial \lambda_i} \frac{\lambda_i}{T} = -1$, independent of age and of family.



1 Families A and C have a single constant, whose relative sensitivity is therefore -1 : the age inherits the half-
2 life uncertainty one-for-one. Family B divides the same total of -1 between parent and daughter, and
3 beyond secular equilibrium the two shares individually leave the interval $[-1, 0]$; for ^{234}U – ^{230}Th at 500 ka,
4 they are -2.76 and $+1.76$. Whether a covariance between the constants reduces or inflates their combined
5 contribution therefore depends on the signs of the two sensitivities and of the correlation.

6 The families differ not only in shape but also in how their nonlinearity can be expressed. For a single-
7 exponential chronometer, β has a simple form that does not require derivatives. The ingrowth function
8 relaxes exponentially toward its bound, $G'(T) = -\mu(R - R_\infty)$, so $\mu = -G''/G'$ is constant. The pair (μ, R_∞) is
9 read from the last two columns of Table 6: $(\lambda_p, 0)$ for family A, $(\Delta\lambda, \kappa)$ for family B, and $(\Delta\lambda, \kappa')$ for family
10 C, which reduces to approximately $(-\lambda_p, -1)$ when the daughter is much longer-lived than the parent. This
11 form of G' gives three useful results. First, the age sensitivity is $\partial T/\partial R = 1/G' = -1/[\mu(R - R_\infty)]$. Second,
12 integration gives $T = -\ln[(R - R_\infty)/(R_0 - R_\infty)]/\mu$, with R_0 the ratio at zero age. This is the logarithm used in
13 Section 9. Finally, because $\sigma_{\text{eq}} = u(R)$ and $\beta = \mu u_{\text{lin}}$, the rate μ cancels, giving the derivative-free form

$$14 \quad |\beta| = \frac{\sigma_{\text{eq}}}{|R - R_\infty|} \quad (23)$$

15 Thus, for a single-exponential chronometer, $|\beta|$ is the equivalent ratio perturbation relative to the distance
16 between the measured ratio and its physical bound.

17 In family A, the bound is zero, so β is simply the relative uncertainty of the measured activity. The
18 criterion $|\beta| < 1/z_p$ is equivalent to $R > z_p \sigma_{\text{eq}}$, the classical criterion for the activity to be significantly above
19 zero. The upper age limit exists if and only if the activity is detected at the same confidence level. Thus,
20 the dating limit of family A is a detection limit (Currie, 1968).

21 In family B the bound is κ , and β is measured against the deficit from secular equilibrium, $\kappa - R$,
22 rather than against the ratio itself. The upper limit exists if and only if that deficit is significant. A sample
23 measured to a fraction of a percent can still be strongly nonlinear if it lies close to equilibrium, as in the
24 case of the $^{230}\text{Th}/^{238}\text{U}$ chronometer in Sections 3–6.

25 In family C the bound is -1 , so the relevant distance is $1 + R$, which exceeds unity at every age. Since
26 $|\beta| < \sigma_{\text{eq}}/R$, β can never exceed the relative uncertainty of the measured daughter/parent ratio. A ratio
27 measured to 1 % gives $|\beta| < 0.01$ regardless of age; reaching $|\beta| = 0.1$ would require a ratio uncertainty
28 exceeding 10 %. This is why accumulation chronometers are intrinsically protected against strong
29 nonlinearity.



1 The table contains three ^{230}Th -based entries derived from two chronometers. The ^{234}U – ^{230}Th model
2 age is one used in nuclear forensics, where the chemical purification date of uranium material is dated from
3 the ingrowth of ^{230}Th through the decay of ^{234}U in the sample (Macsik et al., 2025; Macsik and Pommé,
4 2026). The measured quantity is the activity ratio $[^{230}\text{Th}/^{234}\text{U}]$. Because ^{234}U itself decays while ^{230}Th grows,
5 the ingrowth is governed by the difference of the two rates, and the chronometer is a single exponential in
6 $\lambda_3 - \lambda_2$ ($\tau = 157.5$ ka at every age). Since the ratio is normalised to ^{234}U , the presence of ^{238}U in the sample
7 plays no part in the chronometer.

8 In the $^{230}\text{Th}/^{238}\text{U}$ chronometer, the activity ratio $[^{230}\text{Th}/^{238}\text{U}]$ is obtained from the measured isotope
9 amount ratio by mass spectrometry, converted using the decay-constant factor λ_3/λ_1 . Unlike the purified
10 uranium materials used in some radiochronometric applications, speleothems precipitate from natural drip
11 water and retain ^{238}U , whose activity is effectively constant on Quaternary timescales. The age equation
12 additionally depends on the measured present-day activity ratio $A_{21} = [^{234}\text{U}/^{238}\text{U}]$. When $A_{21} = 1$, ^{234}U is in
13 secular equilibrium with its parent. Eq. (1) then loses its second term and reduces to $1 - e^{-\lambda_3 T}$, a single
14 exponential in λ_3 , with $\tau = 109.0$ ka. Many speleothems, however, have $A_{21} > 1$. This excess is inherited
15 from the cave drip water rather than generated after deposition: α -recoil preferentially mobilises ^{234}U from
16 damaged lattice sites in the host rock and soil, enriching the water from which the calcite precipitates.

17 Speleothem calcite can therefore have a wide range of initial $^{234}\text{U}/^{238}\text{U}$ activity ratios, reflecting the
18 U-isotope composition of the drip water and its interaction with the host rock and soil. This contrasts with
19 marine carbonates such as corals, which acquire uranium from seawater and consequently start close to the
20 marine value (Cheng, Adkins, Edwards and Boyle, 2000). There is no universal initial $^{234}\text{U}/^{238}\text{U}$ value for
21 speleothems that could serve, as it does for corals, as a screening criterion for open-system behaviour. The
22 excess ^{234}U decays away with its own time constant while feeding additional ^{230}Th , so Eq. (1) carries two
23 exponentials, μ becomes the weighted mean of Eq. (8). For example, τ is approximately 111 ka for $A_{21}^0 =$
24 1.05, about 129 ka at 2.0, approaching 158 ka in the limit of large A_{21}^0 . Values $A_{21} < 1$ occur in uranium-
25 depleted material, for which τ falls below 109.0 ka and the bracketing of μ by Eq. (9) no longer holds.
26 Section 3 bounds the excess of μ over λ_3 at 15 % for a mild depletion at $A_{21} \geq 0.9$.

27 A three-nuclide chain such as Eq. (1) is not governed by a single exponential, and therefore belongs
28 to family B only as a superposition of two rates. Here, μ is the weighted mean of two rates given by Eq. (8),
29 and τ takes a range rather than a single value, which explains why the general $^{230}\text{Th}/^{238}\text{U}$ entry in Table 7
30 spans 109 to 158 ka. Any other two-exponential ingrowth is handled similarly using the appropriate pair of
31 rates. The identity in Eq. (23) then holds approximately with the bound taken as A_{31}^{max} of Eq. (2); by Eq.



(21) the error is no more than a few percent. The ^{95}Zr – ^{95}Nb chronometer is of this type, the branch through $^{95\text{m}}\text{Nb}$ contributing a second exponential (Pommé and Collins, 2014).

8 The criterion applied to real measurements

This section illustrates the scale of β in representative published measurement campaigns. The geochronological cases are treated first, followed more briefly by short-lived nuclear chronometers, for which the same criterion applies with much shorter time constants. These examples demonstrate the practical use of the criterion; they are not an independent validation dataset and do not reproduce all components of the original analytical uncertainty budgets.

For ^{14}C , the framework reproduces a rule the radiocarbon community arrived at independently (Stuiver and Polach, 1977): when the measured activity is not significantly above blank, the upper age limit becomes unbounded and the result is reported as a minimum age. Stuiver and Polach (1977) recommend that when the sample activity falls below twice its standard deviation, the age should be reported as a minimum computed at $A + 2\sigma$. Since family A has $\beta = \sigma_{\text{eq}}/R$, their condition is $\beta > 0.5$ (at $k=2$), which is close to Eq. (22): $\beta > 1/z_p = 0.51$ (at $k=1.96$). Their minimum age is therefore compatible with the lower coverage limit at $\mathcal{G}^{-1}(R + z_p\sigma_{\text{eq}})$. The framework thus independently recovers the established radiocarbon practice. The radiocarbon comparison illustrates the same mathematical structure: as activity approaches background, the transformed age uncertainty becomes one-sided and reporting convention shifts toward a minimum age. Practical radiocarbon calculations include additional blank, normalisation, and reporting conventions that are outside the scope of the present treatment.

Within geochronology, for last-interglacial speleothems based on $^{230}\text{Th}/^{238}\text{U}$, the criterion is comfortably satisfied across the ages compiled by Dumitru et al. (Dumitru, Polyak, Asmerom and Onac, 2021). The same holds for the individual ages that enter age-depth models such as StalAge (Scholz and Hoffmann, 2011). The compiled ages are quoted at $k = 2$, so u_{lin} is half the stated uncertainty. Since β remains below 0.03 even at the shortest possible τ of 109 ka, the asymmetry never exceeds 1.5 % at $k=1$. Since the geochronology community routinely quotes expanded uncertainties rather than standard uncertainties, the asymmetry doubles for the same β , to about 3 % on the quoted 95 % intervals. Family C extends the range much further. The U–Pb ages of phreatic overgrowths on speleothems used to constrain Pliocene sea level (Dumitru et al., 2019) are some thirty times older, between 3.27 and 4.39 Ma, yet the atom ratio accumulates without bound, so β is negative and negligibly small, and symmetric propagation of the age uncertainty remains appropriate.



1 The same criterion applies, with much shorter time constants, to short-lived nuclear-metrology
2 chronometers used by the authors. The ^{140}Ba – ^{140}La pair has $\tau = 2.8$ d, so linear propagation remains adequate
3 while the age uncertainty stays below about 6.7 hours. Asymmetry was not considered in the original ^{140}Ba –
4 ^{140}La work (Pommé, Collins, Harms and Jerome, 2016), which quoted standard uncertainties only,
5 consistent with the conventional first-order treatment. Eq. (15) predicts that the two excursions depart from
6 u_{lin} by about $\beta/2$ on either side. The median $\beta = 0.086$ over the 25 spectra of that campaign therefore
7 corresponds to a modest ~ 4 % increase in the upper uncertainty and a reduction of the same size in the
8 lower one. Nonlinearity becomes substantial only at the end of the campaign, where β reaches 0.71: the
9 upper excursion is then underestimated by 73 % of u_{lin} , 7.8 % of the Gaussian distribution assigned to the
10 measured ratio lies above the ceiling, and no finite upper coverage limit exists, so the result should be
11 reported as a lower age limit.

12 The remaining chronometers stay comfortably linear over practical campaign durations. β remains
13 below 0.03 for ^{95}Zr – ^{95}Nb over 122 days (Pommé and Collins, 2014) and below 0.1 for ^{227}Th – ^{223}Ra over
14 about twelve weeks, rising to 0.17 only after 105 days (Pommé, Collins, Harms and Jerome, 2016); the
15 same ^{227}Th – ^{223}Ra dating equation also underlies activity calibration for a therapeutic $^{227}\text{Th}/^{223}\text{Ra}$ mixture in
16 nuclear medicine (Pommé, 2024). For the ^{234}U – ^{230}Th chronometer applied to nuclear forensics, $\beta \approx 7 \times 10^{-7}$
17 and nonlinearity is negligible (Macsik et al., 2025; Macsik and Pommé, 2026).

18 **9 Recommendations for laboratory reporting**

19 The proposed workflow retains the conventional uncertainty budget and central age calculation, but
20 adds a coverage interval. The procedure is appropriate when the uncertainty budget can be represented
21 adequately by an equivalent ratio perturbation; otherwise, full distributional propagation should be used.
22 When additive corrections, such as the detrital correction, dominate the uncertainty budget, the quantile
23 interval remains robust (Section 6) but β no longer predicts the bias, and a Monte Carlo check is advisable.

- 24 1. Calculate the central age T . Solve the dating equation using the normal data-reduction procedure
25 and report the central solution without an empirical bias correction. For a single-exponential
26 chronometer (and for Eq. (1) when $A_{21} = 1$) the inverse is the logarithm of Section 7, $T = -\ln[(R -$
27 $R_{\infty})/(R_0 - R_{\infty})]/\mu$. The general two-exponential case of Eq. (1) has no elementary inverse and
28 needs Newton iterations, for which six to eight steps from the logarithmic starting estimate
29 (Section 4.1) suffice to reach machine precision (the final step and the refinement of Appendix A
30 reuse the same column). In particular, do not shift the central age by the approximate bias term:
31 that term is a diagnostic of nonlinear transformation, not a correction toward a preferred age.



- 1 2. Uncertainty bookkeeping. Construct the uncertainty budget in terms of statistically independent
 2 underlying input quantities. When a shared source of uncertainty affects multiple calculated
 3 terms, propagate that source once through the complete calculation rather than treating the
 4 affected terms as independent contributions. Splitting a systematic source over several budget
 5 lines and summing them in quadrature would artificially reduce its contribution. Propagating it
 6 once preserves its full weight and permits quadrature summation over the genuinely independent
 7 inputs, with no covariance matrix required. Obtain u_{lin} from Eq. (5) with analytic sensitivity
 8 coefficients $\partial T / \partial R = 1 / G'$ evaluated in closed form via Eq. (4) rather than by finite differences.
 9 Retain a per-input two-sided Kragten block, probing each input at $\pm u_i$, as an independent
 10 numerical check on those coefficients and on the curvature. These individual probes serve a
 11 different purpose from the combined probe used later to construct the final coverage interval.
- 12 3. Compute the linearity criterion $\beta = u_{\text{lin}} / \tau$, check whether it is below 0.1, and report its value
 13 alongside the age. For a single-exponential chronometer $\tau = 1 / \mu$ is a constant read from Table 7;
 14 for Eq. (1) with $A_{21} \neq 1$ it is $1 / \mu$ with μ the weighted mean of Eq. (8). Equivalently, Eq. (23)
 15 gives β without μ , from the equivalent perturbation (incorporating the full uncertainty budget)
 16 and the distance remaining to the physical ceiling. The value of β tells any user whether the age
 17 can be treated as a symmetric Gaussian.
- 18 4. Report the coverage interval from Kragten at $z_p \sigma_{\text{eq}}$, obtained by solving the age equation at
 19 $A_{31} \pm z_p \sigma_{\text{eq}}$ as in Eq. (13), and flag the case where no root exists, which is the loss of the upper
 20 coverage limit in families A and B and of the lower one in family C. If a combined two-sided
 21 Kragten block is already in place, this is the same block with the probe enlarged from $1 \sigma_{\text{eq}}$ to
 22 $z_p \sigma_{\text{eq}}$ and the scaling by z_p dropped.

23 A supplementary workbook, archived in the repository given under Code and data availability and
 24 described in Appendix C, implements six methods side by side: linear GUM, the second-order series, two-
 25 sided Kragten, the direct quantile transformation, Gauss–Hermite moments, and a VBA Monte Carlo,
 26 together with the input parameters of the three validation cases and the coverage test of Table 5.

27 10 Limitations

28 The proposed approach contains two conceptually separate elements. The direct transformation of
 29 ratio quantiles through a strictly monotonic inverse age equation is always valid for the assigned one-
 30 dimensional equivalent-ratio model. By contrast, representing a complete uncertainty budget by a single



1 equivalent perturbation is a first-order approximation and may be inadequate when significant contributions
2 act along different parameter directions or have strongly non-Gaussian, correlated, or additive behaviour.
3 This section first distinguishes the curvature diagnostic β from a commonly used Kragten-based
4 comparison, and then discusses four limitations of the present treatment.

5 A tempting shortcut is to read the difference between the analytically propagated uncertainty and the
6 Kragten value as a measure of nonlinearity (Macsik and Pommé, 2026). This difference is a useful check
7 on the first-order approximation, but it is not a proxy for β , for two reasons. First, the Kragten–analytical
8 difference can be dominated by a single input that contributes little to the total variance but has strong
9 individual curvature: for young material, a decay constant carrying a few percent of the variance may bend
10 the age curve a thousand times more sharply than the ratio direction carrying the rest, since the decay-
11 constant curvature is independent of age while that of the ratio is not. Second, the comparison generally
12 underestimates the curvature, since β is the curvature along the combined perturbation σ_{eq} , whereas δ_i
13 probes each u_i separately. Equation (24) is useful for understanding why: if a budget has n comparable
14 contributions of similar curvature, then $\langle \delta_i \rangle \simeq 1/2 \mu u_{lin}/\sqrt{n}$ while $\beta = \mu u_{lin}$, so

$$\beta \simeq 2\sqrt{n} \left(\frac{u_{Kragten}}{u_{analytic}} - 1 \right). \quad (24)$$

16 The Kragten–analytical difference should therefore not be used as a substitute for calculating β .
17 Beyond this, four limitations remain:

18 *The moment expansions.* Eq. (11) is asymptotic rather than convergent. Table 2 shows the skewness
19 prediction failing by a factor of 1.8 at $\beta = 0.31$, well before the bias prediction does. Where $|\beta| \gtrsim 0.1$, the
20 quantile solution of Eq. (13) should be used instead: β is a good diagnostic but a poor correction.

21 *Scalar reduction of the uncertainty budget.* Appendix A bounds the residue of the reduction relative
22 to the half-width and Eq. (A.8) removes most of what remains, but the bound has been derived only for the
23 A_{21} direction. In the decay-constant direction the residue is demonstrably small: for a single-exponential
24 chronometer the curvature parameter reduces to $-2u(\lambda)/\lambda$, twice the relative half-life uncertainties (0.05 %
25 to 0.4 %) and therefore mostly negligible. The detrital parameter is a separate matter: it enters additively
26 and lies outside the reduction altogether, as Section 6 shows.

27 *Gaussian inputs.* The analysis above assigns a Gaussian to the equivalent perturbation. In practice,
28 some budget components rest on few observations and are more properly t -distributed, and the tails are
29 what the quantile transformation is most sensitive to. The transformation itself is untouched: Eq. (13)
30 requires only the quantile of the input distribution, so z_p is simply replaced by $t_p(v_{eff})$, making the quantile



method more robust than a moment-based one. The dating limit in Eq. (22) tightens because $t_p > z_p$ ($t_p(4) = 2.776$, $t_p(10) = 2.228$), thus the bound at τ/t_p is smaller: the Gaussian precision thresholds of 56–80 ka reduce to approximately 49–71 ka for $v_{\text{eff}} = 10$ and 39–57 ka for $v_{\text{eff}} = 4$. The criterion $|\beta| \lesssim 0.1$ is unaffected, being a statement about the linearity of the transformation and not about the input distribution.

Correlated decay constants. λ_3 and λ_2 have been treated as independent, although the experiments that determine them do not make them so (Cheng et al., 2013; Collins et al., 2015). Both enter σ_{eq} only through first-order sensitivity coefficients, so a covariance between them changes u_{lin} and hence β , but none of the structural relations derived in this paper. The direction of the effect depends on the signs of the two sensitivities and of the correlation (Section 7.3).

Taken together, these limitations indicate that the method should be used as a fast, transparent diagnostic and interval construction for the part of the uncertainty budget entering through the measured ratio, with Monte Carlo retained for strongly non-Gaussian inputs, additive corrections, and validation.

11 Conclusions

Nonlinear uncertainty propagation is a general feature of radiometric dating: whenever a measured ratio is transformed through a curved age equation, a symmetric measurement uncertainty produces an asymmetric age distribution, and for bounded chronometers the upper coverage limit may cease to exist. This paper provides a criterion to decide when nonlinearity becomes significant and a practical solution for calculating coverage intervals without resorting to simulation.

The uncertainty budget is reduced to a single equivalent perturbation of the measured ratio, with a small residue available in closed form, and the degree of nonlinearity is then captured by one dimensionless parameter, β , the age uncertainty expressed in units of the chronometer's own time constant. For the $^{230}\text{Th}/^{238}\text{U}$ cases examined, symmetric propagation remains adequate while $|\beta|$ stays below 0.1, whatever the age. At larger $|\beta|$, the asymmetric coverage interval follows from solving the dating equation at the two ratio quantiles, implemented in a spreadsheet as a two-sided Kragten probe of the combined perturbation at $\pm k\sigma_{\text{eq}}$. Monte Carlo simulation is not required, although it remains valuable for validating the reduction and for inputs outside its scope, including dominant additive corrections, strong correlations, or non-Gaussian inputs. In a direct coverage test using a two-input model (Table 5), the quantile-based interval maintained its nominal coverage across the examined range, whereas the symmetric GUM interval and a one-standard-deviation Kragten probe fell increasingly short as nonlinearity grew.



1 The framework extends to three families of monotonic chronometers: decaying parent activities,
 2 daughter/parent activity ratios approaching secular equilibrium, and accumulating daughter/parent atom
 3 ratios. For bounded chronometers, part of the uncertainty distribution may fall beyond the physical bound.
 4 The finite-age part of the distribution can be used to define conditional moments (mean, variance,
 5 skewness), obtained by conditioning on a finite age, but these depend on that conditioning convention.
 6 Discarding draws outside the bounds is also what Monte Carlo simulations implicitly do. These are useful
 7 mainly as diagnostics, but the reported uncertainty should remain the coverage interval derived from the
 8 measurement model. Once the expanded uncertainty $z_p u_{\text{lin}}$ exceeds τ , the upper coverage limit ceases to
 9 exist. This is a joint property of the chronometer and the precision achieved, and it provides a probabilistic
 10 definition of the dating limit at $|\beta| = 1/z_p$, which is compatible with established practice in radiocarbon
 11 dating. Unbounded accumulating systems, including the classical geochronometers, retain a finite upper
 12 coverage limit at any age in the idealised accumulating-ratio model considered here.

13 These results show that nonlinear behaviour can be diagnosed and handled using quantities already
 14 contained in a conventional geochronological uncertainty budget. Reporting β together with the central age
 15 and, where required, the asymmetric coverage interval or a lower age limit, would improve the consistency,
 16 transparency, and physical interpretation of uncertainty reporting across radiometric dating systems.

17 **Appendix A The residue of the scalar reduction**

18 Section 3 folds the whole uncertainty budget onto a single equivalent perturbation of the measured
 19 daughter ratio. It replaces the measured ratio A_{21} by an equivalent shift in A_{31} and evaluates it at the central
 20 age rather than at the perturbed one. The equivalence itself is valid, but the evaluation point is approximate.
 21 This appendix examines the resulting residue. The bound derived here concerns the residue of the A_{21}
 22 contribution; the decay-constant direction is discussed in Section 10 and the detrital parameter, which lies
 23 outside the scalar reduction, in Section 6. The reduction slightly underestimates the bias and shifts both
 24 coverage limits toward older ages rather than widening or narrowing the interval. Both effects can be written
 25 in closed form and bounded using quantities already on the report sheet. For a well-behaved analysis, the
 26 shift is only a fraction of a percent of the half-width. Readers willing to accept the reduction can use the
 27 bound of Eq. (A.7); those who would rather remove the residue can use the fixed-point refinement in Eq.
 28 (A.8), which requires one further pass through the Newton iteration already implemented in the spreadsheet.

29 *The scalar equation.* The reduction of Section 3 is first-order accurate in each input. Its second-order
 30 residue can be obtained in closed form because Eq. (1) is linear in A_{21} :



$$\mathcal{G}(T; A_{21}) = f(T) + (A_{21} - 1) q(T), \quad f = 1 - e^{-\lambda_3 T}, \quad q = \kappa(1 - e^{-(\lambda_3 - \lambda_2)T}), \quad (\text{A.1})$$

with $q'(T) = \lambda_3 e^{-(\lambda_3 - \lambda_2)T}$ and $\partial^2 \mathcal{G} / \partial A_{21}^2 = 0$. Perturbing both measured ratios by δ_D and δ_P and rearranging gives the scalar equation for the perturbed age $\hat{T}$,

$$\mathcal{G}(\hat{T}; A_{21}) = A_{31} + \delta_D - \delta_P q(\hat{T}). \quad (\text{A.2})$$

Equation (A.2) shows where the reduction is accurate and where it leaves a residue. The effect of A_{21} is an equivalent perturbation of A_{31} of size $\delta_P q(\hat{T})$ evaluated at the perturbed solution, whereas Eq. (5) evaluates it at the central age T . The entire residue arises from the difference between $q(\hat{T})$ and $q(T)$.

The residue. Second derivatives with respect to the two measured ratios are written $T_{PP} = \partial^2 T / \partial A_{21}^2$ for the parent ratio and $T_{DD} = \partial^2 T / \partial A_{31}^2$ for the daughter ratio — the latter being $T^{(2)}$ of Eq. (4) — and T_{DP} for the mixed derivative. Writing $p = \mathcal{G}'$ and differentiating $\mathcal{G}(T; A_{21}) = A_{31}$ implicitly,

$$T_{PP} = -\frac{\mathcal{G}'' q^2}{p^3} + \frac{2q'q}{p^2}, \quad T_{DP} = \frac{\mathcal{G}'' q - p q'}{p^3}, \quad T_{DD} = -\frac{\mathcal{G}''}{p^3} = T^{(2)}. \quad (\text{A.3})$$

The reduction replaces T_{PP} by $q^2 T_{DD}$, and

$$T_{PP} - q^2 T_{DD} = \frac{2q'q}{p^2}. \quad (\text{A.4})$$

The second-order bias of the reduced model therefore falls short of the true one by $1/2 (T_{PP} - q^2 T_{DD}) u^2(A_{21})$, that is

$$\Delta E[T] = \nu u_{\text{lin},P}^2, \quad \nu = \frac{q'}{q} = \frac{\lambda_3 - \lambda_2}{e^{(\lambda_3 - \lambda_2)T} - 1} \quad (\text{A.5})$$

where $u_{\text{lin},P} = (q/p) u(A_{21})$ is the contribution of A_{21} to the linear age uncertainty. The residue is positive, thus the reduction always under-states the bias.

The coverage limits are obtained from the second-order expansion of (A.2), at $\delta_P = \mp z_p u(A_{21})$, in place of the expectation that gave (A.5):

$$\Delta t_p = z_p^2 \nu u_{\text{lin},P}^2 + O(u^3) \quad (\text{A.6})$$

for both limits, with the same sign. The interval is displaced toward older ages, not widened. Since $e^x - 1 \geq x$, we have $\nu \leq 1/T$, and dividing (A.6) by the half-width $z_p u_{\text{lin}}$ gives a bound requiring nothing but quantities already on the report sheet:



$$\frac{\Delta t_p}{z_p u_{\text{lin}}} \leq z_p f_p^2 \frac{u_{\text{lin}}}{T}, \quad f_p = \frac{u_{\text{lin},P}}{u_{\text{lin}}}. \quad (\text{A.7})$$

The relative error of the reduction on the coverage interval never exceeds z_p times the relative age uncertainty, weighted by the fraction of the variance carried by A_{21} . For a typical well-behaved analysis ($u_{\text{lin}}/T \approx 1\%$, $f_p^2 \approx 0.3$) this is 0.6 % of the half-width, negligible compared with the interval width.

For the validation cases of Section 6, the bound of Eq. (A.7) evaluates to about 6 a for LIN-1 ($f_p^2 = 0.10$), 5 a for DIRTY-1 ($f_p^2 = 0.036$) and 2.5 ka for OLD-1 ($f_p^2 = 0.23$) — respectively 0.2 %, 0.13 % and 3.4 % of the half-width. For DIRTY-1, Eq. (A.6) predicts a displacement of 4 a; the 0.13 ka observed against the full-input simulation is therefore not this residue but the second-order effect of the detrital parameter a_0 , which lies outside the scalar reduction and is quantified in Section 6. For OLD-1 the bound is loosest, yet the lower coverage limit of the reduced model agrees with the full simulation to better than 0.1 ka, showing the actual residue to lie far below its bound.

The repair. Equation (A.2) also supplies a simple refinement. Instead of evaluating the equivalent perturbation at the central age, evaluate it at the coverage limit t_p itself. The resulting fixed-point problem is solved by iteration:

$$\mathcal{G}(t_p; A_{21}) = A_{31} + z_p \sigma_{\text{eq}}(t_p), \quad \sigma_{\text{eq}}(t)^2 = u^2(A_{31}) + q(t)^2 u^2(A_{21}). \quad (\text{A.8})$$

This removes the residue completely when either ratio dominates, and most of it in mixed cases. In a spreadsheet it costs one further pass of the same Newton column, since the function $q(t)$ is already implemented as part of the dating equation. Laboratories may implement Eq. (A.8) to remove the residue or report Eq. (A.6) to show that it was negligible.

Appendix B Series coefficients

The bias, variance and skewness quoted in Eq. (11) come from expanding the inverse of the ingrowth function about the central age. This appendix records that expansion so that the coefficients can be checked. It plays no part in the recommended procedure of Section 9: the series is a diagnostic rather than a correction, and Section 10 explains why it should not be used as one.

Write X for the deviation of the age from its solved value, to second order in the perturbation: $X = T^{(1)}\delta + 1/2 T^{(2)}(\delta^2 - \sigma_{\text{eq}}^2)$. Subtracting the variance centres it, so that its expectation vanishes. Here $\delta \sim N(0, \sigma_{\text{eq}}^2)$, and the Gaussian moments are $E[\delta^4] = 3\sigma_{\text{eq}}^4$ and $E[\delta^6] = 15\sigma_{\text{eq}}^6$,



$$E[X^2] = T^{(1)2}\sigma_{eq}^2 + 1/2 T^{(2)2}\sigma_{eq}^4, \quad E[X^3] = 3T^{(1)2}T^{(2)}\sigma_{eq}^4 + T^{(2)3}\sigma_{eq}^6, \quad (B.1)$$

The variance in Eq. (11) then follows from the second moment, and the skewness on dividing the third by $(T^{(1)2}\sigma_{eq}^2)^{3/2}$, once the $T^{(1)}T^{(3)}\sigma_{eq}^4$ term of the same order is included in the variance.

Appendix C Supplementary workbook

The methods compared for the calculation of coverage intervals are collected in a workbook, so that a reader can reproduce the numbers and try the procedure on their own data. Five of the six methods require no macros; the Monte Carlo simulation and the coverage test are implemented in the supplied VBA code.

To use it, replace the input ratios and uncertainties on the Model sheet with your own, and the other sheets follow. Alternatively, a selector on the Model sheet loads the input parameters of the three synthetic cases of Section 6 from the Cases sheet. Quantile tabulates t_p for nineteen probabilities; GaussHermite gives the conditional moments on forty nodes and an approximate interval through Cornish–Fisher expansion; Kragten gives the two-sided block; Scan sweeps the age at fixed precision, with an advisory colour band at $|\beta| = 0.04$ inside the action threshold of 0.1 used in the text; MC collects the six methods side by side, with the diagnostics and consistency checks, and receives the output of the VBA module NonLinMC.bas (2×10^6 trials by default, a few seconds on an ordinary computer); Coverage reproduces the coverage test of Table 5. The workbook is provided in two editions: nonlinearity_lab.xlsx, which contains no macros and implements the first five methods with formulas alone, and nonlinearity_lab.xlsm, which adds the Monte Carlo and coverage modules. Every result was cross-checked against an independent Python implementation to a relative 10^{-13} or better.

Code and data availability:

The supplementary workbook described in Appendix C (nonlinearity_lab.xlsx and nonlinearity_lab.xlsm, with the VBA code), including the input specification of the three synthetic cases of Section 6, is archived at Zenodo, <https://doi.org/10.5281/zenodo.23021594>. No new measurement data were generated for this work; the published measurements discussed in Section 8 are available from the sources cited there.

Author contributions:



1 SP developed the mathematical framework, derived the curvature parameter and the coverage-
2 interval and dating-limit relations, and implemented the supplementary workbook and the Monte Carlo
3 cross-checks. ZM provided the nuclear-forensic radiochronometry context and uncertainty budgets and
4 tested the procedure against spreadsheet and GUM-Workbench propagation. OAD provided the U-series
5 geochronology context and the speleothem applications. SP prepared the first draft; all authors contributed
6 to the interpretation, the figures and tables, and the revision of the manuscript, and approved the submitted
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14 were manually reviewed and verified for accuracy against Monte Carlo simulation. The authors retain full
15 responsibility for the scientific content and conclusions of the manuscript.

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