



A visco-elastic phase-field model for calving in ice

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Abstract. Iceberg calving accounts for around half of the ice lost annually from Antarctica, but realistic representation of fracture and calving in large-scale ice sheet models remains a major unsolved problem in glaciology. We present a new phase-field visco-elastic model for fracture. The model can capture both the long timescale visco-elastic deformation of ice, as well as the nucleation and evolution of fractures. By incorporating the effect of pore pressure into the degraded tensile energy, the model can also nucleate basal crevasses. We show that this approach is capable of simulating the physical process of calving. Numerical experiments suggest that, for a floating ice shelf with no variation across flow the calving rate for thin-end calving scales with the ice thickness to a power of around 5.5 to 9.5. The equations make no assumptions about the style of calving, and can therefore also simulate numerous more realistic settings in Antarctica for which material parameters and three-dimensional effects can be expected to influence the calving rate.

1 Introduction

Calving of tabular icebergs from ice shelves accounts for approximately half the ice lost from Antarctica each year (Greene et al., 2022). Failure of calving ice cliffs is also hypothesised to make the Antarctic Ice Sheet vulnerable to an instability that could drive rapid sea-level rise (DeConto and Pollard, 2016; Pattyn et al., 2018). Despite its central importance to the problem of predicting global sea level, the fracture process that drives calving, comprising both initiation and evolution of cracks, is complex and not yet well understood. Rather than simulating the physical process of fracture in detail, most models of calving and cliff retreat rates have been based on heuristic arguments or limited observations scaled up to the whole of Antarctica (DeConto and Pollard, 2016). There is no guarantee that current methods will accurately capture sea-level contributions when conditions deviate from present-day observations, so a focus on physically-based modelling of fracture in glaciological settings is needed.

It has been almost thirty years since Meier (1997) emphasised that “iceberg calving is largely a problem in fracture mechanics coupled to ice dynamics”. However, coupling the dynamics of ice introduces a number of challenges for fracture mechanics. Firstly compared to many engineering applications ice shelf fracture can occur on a much larger scale: the Larsen B ice shelf collapse occurred over a characteristic length of ~ 60 km and an average thickness of 250 m (Rack and Rott, 2004). Secondly, the changes in body configuration that lead to fracture are governed by gradual viscous deformation, occurring over timescales of years, whereas fracture-mechanics literature tends to focus on the elastic problem in isolation, because the timescale over which many engineering samples undergo catastrophic failure can be considered short enough to neglect viscous deformation.



Thirdly, hydrofracture, through the presence of water in cracks, plays a key role in calving. Large meltwater ponds on Larsen B are hypothesised to have caused its sudden collapse (Banwell et al., 2013). Furthermore, large tensile crevasses can form at the base of an ice shelf due to the presence of water pressure in microcracks on the bottom surface, despite the material being under highly compressive loading from the weight of the ice above (Jezek, 1984).

Previously, fracture and crevasse propagation in ice have been addressed using a combination of analytical and computational techniques. Approximations for crevasse depths in idealised geometries have been described using a zero-stress model developed by Nye (1955), Meier (1958), and Weertman (1973). Under this paradigm, crevasses are assumed to form under any tensile stress, no matter how small, but to stop propagating at the ‘Nye depth’, where the compressive component of stress generated by the weight of overlying ice exceeds the tensile component generated by longitudinal stretching. Further extensions have been made to include the effect of water pressure acting in defects on the bottom surface, leading to large basal crevasses which can penetrate to around half the ice thickness (Weertman, 1980; Jezek, 1984).

Computational modelling of damage evolution in ice shelves and ice sheets has been investigated by Bassis and Walker (2012), Clerc et al. (2019), Lhermitte et al. (2020). Models based on linear elastic fracture mechanics have been used to estimate crevasse depths on the assumption that ice behaves elastically on short time scales (van der Veen, 1998; Lipovsky, 2020; Zarrinderakht et al., 2022). These models can make useful predictions, but are most applicable to idealised geometries with simple boundary conditions and explicitly specified initial flaws. The heuristic criteria needed to predict the onset of crack nucleation can make such models difficult to parameterise. Implementing numerical methods that capture bifurcation and coalescence of cracks also remains a challenge within the framework of linear elastic fracture mechanics. Together, these factors have discouraged widespread use of linear elastic fracture mechanics for large-scale ice sheet modelling.

Recently, phase-field models for fracture have gained a large following due to their ability to predict complex cracking phenomena such as crack branching and coalescence, or crack nucleation. These models use variational approaches that are based on energy minimisation (Francfort and Marigo, 1998). Introducing a phase field for fracture, one effectively has a variable that takes a value of zero within the solid material and one within a fracture, while interpolating between those values in a sufficiently smooth manner. Crucially, the regularised phase-field problem does not require an explicit treatment of the crack configuration. This alleviates issues related to the complex crack topology. All computations can be executed on a fixed mesh using standard finite-element techniques. Further, phase-field methods converge to the sharp-crack behaviour as the regularisation lengthscale is decreased. The details of this convergence are described by Ambrosio and Tortorelli (1990). In practice, this means that phase-field methods can be designed so the transfer of energy between stored elastic energy and surface energy is computed to a good approximation within the numerical model.

Since it was initially conceived (Bourdin et al., 2000) several extensions to the phase-field model have been proposed. A considerable amount of research has been focused on accounting for different failure mechanisms, particularly in brittle elastic solids, such as preventing interpenetration of crack surfaces (Amor et al., 2009), ensuring the appropriate tensile/compressive asymmetry (Miehe et al., 2010a; Freddi and Royer-Carfagni, 2010) or including particular failure surfaces (Navidtehrani et al., 2022; De Lorenzis and Maurini, 2022). Some recent work has also focused on modifications to apply the phase-field model to large structures (Lo et al., 2023). Recent work by Ulloa et al. (2022) developed a variationally consistent way to account for



fractures driven by fluid pressure in microcracks, representing both mode-I (brittle) and mode-II (ductile) fracture. The above phase-field models aim to retain variational consistency in the formulation. There has also been research which sacrifices this in return for reduced complexity and computational cost (Ambati et al., 2015).

65 In ice, the phase-field method has been applied in poro-elastic (Sun et al., 2021) and in a poro-visco-elastic (Clayton et al., 2022, 2025) models over timescale of minutes for simulating both ice cliffs (where the bottom boundary is constrained by the bedrock) and floating ice shelves. More recent work has combined the phase-field method for mode-I fractures with a Cosserat continuum for mode-II shear bands, applied to laboratory experiments and ice cliffs (Hageman, 2026).

70 In this work, we present a phase-field formulation of fracture for Maxwell poro-visco-elastic materials. We aim to maintain variational consistency as much as possible and ensure that the mass of the system remains conserved. This model is capable of capturing viscous deformation over timescales of years as well as instantaneous elastic deformation, which is crucial for modelling calving from ice shelves. Furthermore, we propose a split of the free energy to incorporate the effect of pore-pressure in fractures, such that the model is able to nucleate both basal and surface crevasses.

85 This article is organised as follows: Section 2 discusses the poro-visco-elastic phase-field fracture model for ice shelf dynamics. For convenience, we divide the presentation into three subsections. Subsection 2.1 presents the poro-visco-elastic model for the fully intact material. In Subsection 2.2 we review relevant concepts from phase-field modelling of fracture, incorporate these into the rheology, and present a non-dimensional form of the full system. In Subsection 2.3 we describe how we include the effect of pore pressure in driving tensile fractures. In Section 3 we discuss implementation details. Numerical experiments are presented in Section 4, accompanied by discussion of the results. We present concluding remarks in Section 5. The article is accompanied by appendices that address technical details.

2 Model description

85 In this section, we describe the relevant equations governing the evolution of ice sheets and fracture phase-field equations. We begin by describing the rheological properties of ice flow and introduce their energy storage and dissipation potential functions. We then briefly review the fracture phase-field equations and include the additional variables into the system of equations that govern ice sheet evolution, including a nondimensionalised system of equations. Finally, we describe the split in the poro-elastic energy necessary to represent both surface and basal crevasses.

2.1 Poro-visco-elastic model

90 We first focus only on the behaviour of the intact material and neglect any fractures in the material, as illustrated in Fig. 1. We do however consider the ice to be potentially porous, containing microcracks within the material with an associated pore pressure. Throughout this article we consider a body of ice Ω with boundary $\partial\Omega$. We indicate the height of the domain as H and the height of the water level measured from the base as H_w .

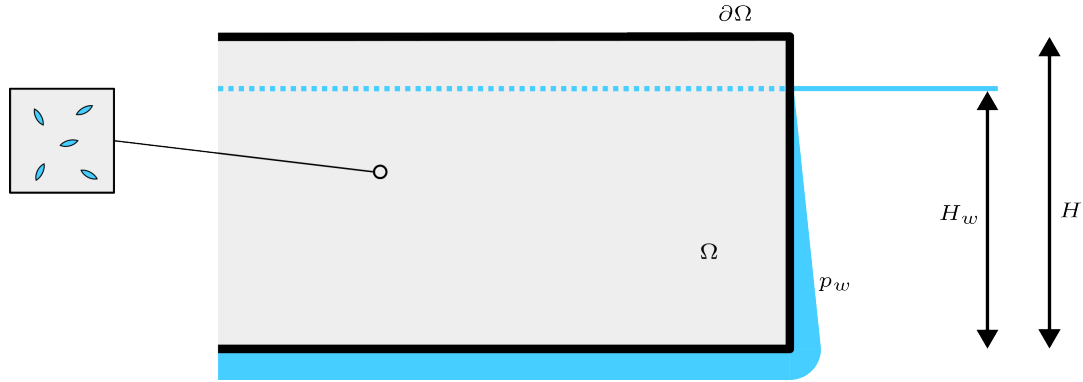


Figure 1. Illustration of the intact geometry and associated notation. We consider the ice as potentially porous, as illustrated by the inset. The domain Ω has height H and boundary $\partial\Omega$ on which ocean water pressure p_w acts. The height underwater is given by H_w

We assume that the macroscopic motions of the body Ω are given by the displacement field $\mathbf{u} \in \mathbb{R}^3$. In the small strain context we assume that the total strain is given by the symmetric part of the displacement gradient

$$\varepsilon = \nabla_s \mathbf{u},$$

defined as $\nabla_s \mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$, with superscript T denoting the transpose. We further assume that the total displacement $\mathbf{u}$ can be additively decomposed into a recoverable elastic part $\mathbf{u}_e$ and an irrecoverable viscous part $\mathbf{u}_v$

$$\mathbf{u} = \mathbf{u}_e + \mathbf{u}_v.$$

Therefore,

$$\varepsilon = \nabla_s \mathbf{u} = \nabla_s \mathbf{u}_e + \nabla_s \mathbf{u}_v = \varepsilon_e + \varepsilon_v,$$

where ε_e and ε_v are the elastic and viscous components of strain. In order to derive the system of equations governing the behaviour of the viscoelastic material, we choose the independent constitutive state variables to be $\mathbf{u}, \mathbf{u}_v$, so that $\varepsilon_e = \nabla_s(\mathbf{u} - \mathbf{u}_v)$ and $\varepsilon_v = \nabla_s \mathbf{u}_v$. We will make a standard assumption that the external forces act only on the total displacements $\mathbf{u}$ (Miehe, 2011), while the viscous part $\mathbf{u}_v$ is considered an internal variable (Maugin, 1990).

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We now consider the rheological relations governing the evolution of polycrystalline ice. The behaviour of ice sheets and ice shelves is typically represented by a Maxwell visco-elastic model with nonlinear stress-dependent viscosity referred to as Glen's flow law (Glen, 1955; Cuffey and Paterson, 2010). In the Maxwell model it is assumed that the total strain in a material body Ω can be decomposed additively into the elastic and viscous strains, whereas the total deviatoric stress acting on Ω is equal to both the elastic and viscous stresses:

110

$$\begin{aligned} \varepsilon &= \varepsilon_e + \varepsilon_v, \\ \boldsymbol{\sigma}^D &= \boldsymbol{\sigma}_e^D = \boldsymbol{\sigma}_v^D. \end{aligned}$$



The elastic stress σ_e is typically related to the elastic strain ε_e through Hooke's law

$$\sigma_e = \mathbf{C} : \varepsilon_e = \lambda \text{tr}(\varepsilon_e) \mathbf{I} + 2\mu \varepsilon_e = K \text{tr}(\varepsilon_e) \mathbf{I} + 2\mu \varepsilon_e^D \quad (1)$$

115 where $\mathbf{C}$ is the stiffness tensor, $\mathbf{I}$ is the second-order identity tensor, $\varepsilon_e^D = \varepsilon_e - \frac{1}{3} \text{tr}(\varepsilon_e) \mathbf{I}$ is the deviatoric part of the elastic strain tensor. Here, $K = \lambda + 2\mu/3$ is the bulk modulus, λ, μ are the first and second Lamé parameters that are related to the Young's modulus E and Poisson ratio ν :

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

We consider the ice to be a porous material. From Biot's theory of poroelasticity, the effective stress is given by:

$$120 \quad \sigma' = \sigma - B p_w \mathbf{I} \quad (2)$$

where σ' is the effective stress, $B = 1 - K/K_s$ is the Biot coefficient, K_s is the bulk modulus of the fully intact solid, and p_w is the pore water pressure. This equation could be coupled to a separate equation to solve for the evolution of the pore pressure (e.g. Ulloa et al., 2022). However, in this work we make a simplifying assumption that the pore water pressure is always given by the ocean water pressure:

$$125 \quad p_w = \begin{cases} -\rho_w |g| (z - H_w), & \text{for } z < H_w \\ 0, & \text{for } z \geq H_w \end{cases} \quad (3)$$

where ρ_w is the water density and g is the gravitational acceleration. We now turn to the viscous stress σ_v , this is also decomposed into the volumetric and deviatoric parts. The volumetric part is given by the effective pressure p , such that $p = -1/3 \text{tr}(\sigma_v)$. The deviatoric part σ_v^D is related to the deviatoric part of the viscous strain rate $\dot{\varepsilon}_v^D$ by Glen's flow law (Glen, 1955),

$$130 \quad \sigma_v = -p \mathbf{I} + \sigma_v^D = -p \mathbf{I} + 2\eta(\dot{\varepsilon}_v^D) \dot{\varepsilon}_v^D. \quad (4)$$

In effect, Glen's flow law (Glen, 1955) defines a viscosity that depends upon strain rate as follows.

$$\eta(\dot{\varepsilon}_v^D) = \frac{1}{2} A^{-1/n} \left(\frac{1}{2} \dot{\varepsilon}_v^D : \dot{\varepsilon}_v^D \right)^{\frac{1-n}{2n}}, \quad (5)$$

where n is usually considered to be $n = 3$ and A is a positive constant dependent on temperature via the Arrhenius law (Cuffey and Paterson, 2010).

135 The deviatoric viscous strain rate $\dot{\varepsilon}_v^D$ is typically given by $\nabla_s^D \dot{\mathbf{u}}_v$, the deviatoric part of the symmetric gradient of $\dot{\mathbf{u}}_v$, where $\dot{\mathbf{u}}_v$ is the rate of change of $\mathbf{u}_v$, and $\mathbf{u}_v$ is the irrecoverable part of the total displacement $\mathbf{u}$. Note that the viscosity η can be equivalently represented as a function $\hat{\eta}$ with dependency on the viscous shear stress (Greve and Blatter, 2009),

$$\eta(\dot{\varepsilon}_v^D) = \hat{\eta}(\sigma_v^D) = \frac{1}{2} A^{-1} \left(\frac{1}{2} \sigma_v^D : \sigma_v^D \right)^{\frac{1-n}{2}}. \quad (6)$$



Within the material, the fracture process is intimately related to the stored elastic energy, which may be converted into fracture energy or dissipated during viscous deformation. For this reason it proves useful to define the free energy and dissipation potential functions of the poro-visco-elastic system. The free energy function of the recoverable elastic energy is:

$$\begin{aligned}\psi(\boldsymbol{\varepsilon}_e) &= \frac{1}{2} \lambda \text{tr}(\boldsymbol{\varepsilon}_e)^2 + \mu \boldsymbol{\varepsilon}_e : \boldsymbol{\varepsilon}_e \\ &= \frac{1}{2} K \text{tr}(\boldsymbol{\varepsilon}_e)^2 + \mu \boldsymbol{\varepsilon}_e^D : \boldsymbol{\varepsilon}_e^D\end{aligned}\quad (7)$$

The total free energy of the system is given by a combination of the elastic free energy and the contribution from pore pressure (Coussy, 2004; Anand, 2015; Ulloa et al., 2022):

$$\Psi(\boldsymbol{\varepsilon}_e, p_w) = \psi(\boldsymbol{\varepsilon}_e) - \frac{p_w^2}{2M} - B p_w \text{tr} \boldsymbol{\varepsilon}_e \quad (8)$$

where M is the Biot modulus, $1/M = (B - v_0^f)/K_s$ where v_0^f is the porosity. We note that the Cauchy stress is given by $\boldsymbol{\sigma} = \partial_{\boldsymbol{\varepsilon}_e} \psi(\boldsymbol{\varepsilon}_e)$ and Biot's effective stress is given by $\boldsymbol{\sigma}' = \partial_{\boldsymbol{\varepsilon}_e} \Psi(\boldsymbol{\varepsilon}_e, p_w)$.

A viscous dissipation potential function $\phi(\dot{\boldsymbol{\varepsilon}}_v^D)$ can be defined so that $\boldsymbol{\sigma}_v^D = \partial_{\dot{\boldsymbol{\varepsilon}}_v^D} \phi$, which is satisfied by:

$$\phi(\dot{\boldsymbol{\varepsilon}}_v^D) = \frac{2n}{n+1} \eta (\dot{\boldsymbol{\varepsilon}}_v^D) \dot{\boldsymbol{\varepsilon}}_v^D : \dot{\boldsymbol{\varepsilon}}_v^D. \quad (9)$$

There are multiple reasons for defining the free energy and dissipation potentials of the poro-visco-elastic system. Primarily, this allows us to incorporate the fracture variable into the system and easily alter the system for different rheological assumptions. Secondly, it allows us to understand the thermodynamic consistency of the system (Miehe et al., 2010b). Finally, it serves as a basis for the numerical implementation.

The stored energy and dissipation potential functionals of the system are the integrals of the free energy and dissipation potential functions over the domain Ω (Maugin, 1990):

$$\mathbf{E} = \int_{\Omega} \Psi \, dV, \quad (10)$$

$$\mathbf{D} = \int_{\Omega} \phi \, dV. \quad (11)$$

The constitutive functions $\mathbf{E}$ and $\mathbf{D}$, defined above, are related respectively to energy storage and dissipation due to the deformation of the material (Miehe, 2011). The quantity $\mathbf{E}$ represents the free energy stored in Ω . The rate of energy storage at the state $\mathbf{u}, \mathbf{u}_v$ is the time derivative of the energy functional.

$$\frac{d}{dt} \mathbf{E} = \int_{\Omega} \delta_{\mathbf{u}} \Psi(\mathbf{u}, \mathbf{u}_v) \cdot \dot{\mathbf{u}} + \delta_{\mathbf{u}_v} \Psi(\mathbf{u}, \mathbf{u}_v) \cdot \dot{\mathbf{u}}_v \, dV + \int_{\partial\Omega} [\partial_{\nabla_s \mathbf{u}} \Psi \cdot \mathbf{n}] \cdot \dot{\mathbf{u}} + [\partial_{\nabla_s \mathbf{u}_v} \Psi \cdot \mathbf{n}] \cdot \dot{\mathbf{u}}_v \, dS,$$

where we introduce functional derivatives of the free-energy and dissipation potential functions as follows

$$\delta_{\mathbf{u}} \Psi := -\nabla \cdot [\partial_{\nabla_s \mathbf{u}} \Psi], \quad \delta_{\mathbf{u}_v} \Psi := -\nabla \cdot [\partial_{\nabla_s \mathbf{u}_v} \Psi], \quad \delta_{\dot{\mathbf{u}}_v} \phi := -\nabla \cdot [\partial_{\nabla_s^D \dot{\mathbf{u}}_v} \phi].$$



165 Note that these functional derivatives do not contain terms with $\partial_{\mathbf{u}}\Psi$, $\partial_{\mathbf{u}_v}\Psi$ or $\partial_{\dot{\mathbf{u}}_v}\phi$, since these are zero, which is a requirement of frame invariance (Maugin, 1990; Miehe, 2011).

Following Maugin (1990) we emphasise that $\mathbf{D}$ is a dissipation potential, and does not equate to the power dissipated within the volume. In our case, though, it can be related to the internal viscous power dissipation Π_v within the material as

$$170 \quad \mathbf{D} = \int_{\Omega} \phi \, dV = \frac{n}{n+1} \int_{\Omega} 2\eta(\dot{\epsilon}_v^D) \dot{\epsilon}_v^D : \dot{\epsilon}_v^D \, dV = \frac{n}{n+1} \int_{\Omega} \boldsymbol{\sigma}_v^D : \dot{\epsilon}_v^D \, dV = \frac{n}{n+1} \Pi_v. \quad (12)$$

A dual potential also exists that is a functional of the deviatoric viscous stress $\boldsymbol{\sigma}_v^D$.

$$\mathbf{D}^* = \int_{\Omega} \phi^* \, dV = \frac{1}{n+1} \int_{\Omega} \frac{\boldsymbol{\sigma}_v^D : \boldsymbol{\sigma}_v^D}{2\hat{\eta}(\boldsymbol{\sigma}_v^D)} \, dV = \frac{1}{n+1} \int_{\Omega} \boldsymbol{\sigma}_v^D : \dot{\epsilon}_v^D \, dV = \frac{1}{n+1} \Pi_v, \quad (13)$$

where $\phi^*(\boldsymbol{\sigma}_v^D)$ is provided by the Legendre-Fenchel transform of the dissipation potential function:

$$\phi^*(\boldsymbol{\sigma}_v^D) = \sup_{\dot{\epsilon}_v^D} [\boldsymbol{\sigma}_v^D : \dot{\epsilon}_v^D - \phi(\dot{\epsilon}_v^D)]. \quad (14)$$

175 One consequence is that Glen's flow law can be expressed in two forms: either as $\boldsymbol{\sigma}_v^D = \partial_{\dot{\epsilon}_v^D} \phi$, or as $\dot{\epsilon}_v^D = \partial_{\boldsymbol{\sigma}_v^D} \phi^*$. A further consequence is that any combination of deviatoric viscous stress $\boldsymbol{\sigma}_v^D$ and viscous strain rate $\dot{\epsilon}_v^D$ that satisfies the Glen relationship will also satisfy

$$\Pi_v = \mathbf{D} + \mathbf{D}^*. \quad (15)$$

This allows the viscous power dissipation to be split into a part $\mathbf{D}$ that depends only on the rate variable $\dot{\epsilon}_v^D$, and a part $\mathbf{D}^*$ that will ultimately depend only upon state variables, albeit via the viscous deviatoric stress $\boldsymbol{\sigma}_v^D$. Since both components are positive, the viscous power dissipation Π_v is positive, which is desirable on thermodynamic grounds.

We define the internal power as the sum of reversible and dissipated components,

$$\Pi_{\text{int}} = \frac{d}{dt} \mathbf{E} + \Pi_v = \frac{d}{dt} \mathbf{E} + \mathbf{D} + \mathbf{D}^* \quad (16)$$

The external power is given by body and surface forces $\mathbf{f}$, $\mathbf{t}$ that act only on the external variable $\mathbf{u}$

$$185 \quad \Pi_{\text{ext}} = \int_{\Omega} \mathbf{f} \cdot \dot{\mathbf{u}} \, dV + \int_{\partial\Omega} \mathbf{t} \cdot \dot{\mathbf{u}} \, dS. \quad (17)$$

Energy conservation can be expressed as

$$\Pi = \Pi_{\text{int}} - \Pi_{\text{ext}} = 0, \quad (18)$$



Here, we consider a variational approach, demanding that the first variation of Π with respect to rate variables $\dot{\mathbf{u}}$ and $\dot{\mathbf{u}}_v$ vanishes, so that

$$\begin{aligned}
 190 \quad 0 = \delta\Pi = \delta\Pi_{\text{int}} - \delta\Pi_{\text{ext}} = & \int_{\Omega} [\delta\mathbf{u}\Psi - \mathbf{f}] \cdot \delta\dot{\mathbf{u}} + [\delta\mathbf{u}_v\Psi + \delta\dot{\mathbf{u}}_v\phi] \cdot \delta\dot{\mathbf{u}}_v \, dV \\
 & + \int_{\partial\Omega} [\partial_{\nabla_s\mathbf{u}}\Psi \cdot \mathbf{n} - \mathbf{t}] \cdot \delta\dot{\mathbf{u}} \, dS \\
 & + \int_{\partial\Omega} [\partial_{\nabla_s\mathbf{u}_v}\Psi \cdot \mathbf{n} + \partial_{\nabla_s^D\dot{\mathbf{u}}_v}\phi \cdot \mathbf{n}] \cdot \delta\dot{\mathbf{u}}_v \, dS.
 \end{aligned} \tag{19}$$

This statement represents the principle of virtual power (Maugin, 1990). In applying this principle, the rates $\dot{\mathbf{u}}$ and $\dot{\mathbf{u}}_v$ are perturbed by small amounts $\delta\dot{\mathbf{u}}$ and $\delta\dot{\mathbf{u}}_v$, considered virtual quantities, while the state variables $\mathbf{u}$ and $\mathbf{u}_v$ (and the force-related quantities $\mathbf{f}$, $\mathbf{t}$, σ_e , σ_v^D) remain fixed. The rates and their virtual perturbations satisfy homogeneous Dirichlet boundary conditions on $\partial_D\Omega$:

$$\dot{\mathbf{u}}, \delta\dot{\mathbf{u}} \in \{\mathbf{v} | \mathbf{v} = \mathbf{0} \text{ on } \partial_D\Omega\}, \quad \dot{\mathbf{u}}_v, \delta\dot{\mathbf{u}}_v \in \{\mathbf{v} | \mathbf{v} = \mathbf{0} \text{ on } \partial_D\Omega; \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega\}.$$

The argument behind Eq. (19) is that we seek a state such that arbitrary perturbations $\delta\dot{\mathbf{u}}$ and $\delta\dot{\mathbf{u}}_v$ applied to the rate variables $\dot{\mathbf{u}}$ and $\dot{\mathbf{u}}_v$ have no impact upon Eq. (18). Because the contribution from $\mathbf{D}^*$ has no explicit dependence on the rates $\dot{\mathbf{u}}$ and $\dot{\mathbf{u}}_v$, an alternative definition of the internal potential,

$$200 \quad \hat{\Pi}_{\text{int}} = \frac{d}{dt}\mathbf{E} + \Pi_v - \mathbf{D}^* = \frac{d}{dt}\mathbf{E} + \mathbf{D}, \tag{20}$$

can serve equally well in place of Π_{int} defined by Eq. (16). Setting $0 = \delta\hat{\Pi} = \delta\hat{\Pi}_{\text{int}} - \delta\Pi_{\text{ext}}$ would also provide Eq. (19).

Applying the fundamental lemma of the calculus of variations to Eq. (19) results in a coupled system of equations in a domain Ω with Neumann type boundary conditions:

$$\begin{aligned}
 -\nabla \cdot (\lambda \nabla \cdot (\mathbf{u} - \mathbf{u}_v) \mathbf{I} + 2\mu \nabla_s (\mathbf{u} - \mathbf{u}_v) - Bp_w \mathbf{I}) &= \mathbf{f} \quad \text{in } \Omega \\
 2\eta(\nabla_s^D \dot{\mathbf{u}}_v \nabla_s^D \dot{\mathbf{u}}_v - 2\mu \nabla_s^D (\mathbf{u} - \mathbf{u}_v)) &= 0 \quad \text{in } \Omega \\
 \nabla \cdot \dot{\mathbf{u}}_v &= 0 \quad \text{in } \Omega \\
 (\lambda \nabla \cdot (\mathbf{u} - \mathbf{u}_v) \mathbf{I} + 2\mu \nabla_s (\mathbf{u} - \mathbf{u}_v) - Bp_w \mathbf{I}) \cdot \mathbf{n} &= \mathbf{t} \quad \text{on } \partial\Omega.
 \end{aligned} \tag{21}$$

205 The first equation corresponds to the momentum balance equation of the effective elastic stress. The second equation corresponds to the balance of elastic and viscous deviatoric stresses, with ∇_s^D representing the deviatoric part of the symmetric gradient. The third equation is the standard incompressibility condition and the fourth equation is a traction boundary condition.

We assume that the external loading functions $\mathbf{f}$, $\mathbf{t}$ are the gravitational force and depth varying hydrostatic water pressure, respectively, given by

$$\mathbf{f} = \rho_i \mathbf{g}, \text{ in } \Omega \tag{22}$$

$$\mathbf{t} = -p_w \mathbf{n}, \text{ on } \partial\Omega \tag{23}$$



where ρ_i is ice density, $\mathbf{n}$ is the unit outward pointing normal vector, and p_w is the water pressure.

2.2 Phase field fracture formulation

Before describing the details of the implementation we first highlight the principal differences from the situation without fracture considered in the previous section. We will briefly introduce the energetic approach to fracture, then extend the equations from Subsection 2.1 to incorporate fracture into the viscoelastic constitutive equations via a phase-field variable d that takes values of $d = 1$ at cracked locations and $d = 0$ away from the cracks. In doing so, we eliminate treatment of evolving geometry due to fracture evolution.

Conceptually, we consider regions where $d > 0$ as still part of the ice mass, unlike Clayton et al. (2022) who conceptually consider regions of $d \approx 1$ as voids in the ice, filled by air or water. Below the water line, we consider the effect of water pressure inside microcracks (at a much smaller scale than the grid) in the material.

In the standard theory of brittle fracture, the drop in stored elastic energy GdS in the surrounding material when a specific crack $\mathcal{C}$ is incremented in area by an infinitesimal amount dS , is compared to the critical energy $G_c dS$ that must be supplied to extend the crack by the same amount. For pure brittle failure G_c is envisioned to be the surface energy that is required to separate the crack faces per unit area, but this can also be generalised for more ductile materials in which additional energy must be expended to form the crack (Irwin, 1957). Propagation of the crack $\mathcal{C}$ occurs when $G \geq G_c$ so that sufficient energy is recovered from the elastic field to advance the crack. This is known as the Griffith criterion (Griffith, 1921). The material parameter G_c is commonly known as the critical energy release rate and can be estimated from laboratory experiments.

In the phase-field approach, outlined by Francfort and Marigo (1998), Bourdin et al. (2012) and Miehe (2011), the energy $\Gamma(\mathcal{C})$ needed to create a crack $\mathcal{C}$ is approximated by an elliptic functional that depends upon the critical energy release rate G_c , and the phase-field variable d ,

$$\Gamma_\ell(d) = \int_{\Omega} G_c \gamma(d) dV. \quad (24)$$

The crack density function γ plays a role analogous to specific surface area (i.e. the surface area of cracks per unit volume). In this work, we used the higher-order phase field model of Borden et al. (2014), such that γ is defined by

$$\gamma(d) = \frac{1}{2\ell} \left(d^2 + \frac{1}{2}\ell^2 |\nabla d|^2 + \frac{1}{16}\ell^4 |\nabla^2 d|^2 \right). \quad (25)$$

The regularisation parameter ℓ , which has dimensions of length, acts to control the lengthscale over which the phase-field variable d varies in the neighbourhood of cracks. Larger ℓ corresponds to a smoother regularised transition between fully-fractured material ($d = 1$) and fully-intact material ($d = 0$). It is this regularisation that allows sharp cracks to be represented on a finite computational mesh. Although this approach approximates the specific surface area of a sharp crack network with γ , there are theoretical reasons to expect Γ_ℓ to become ever closer to the true energy Γ as ℓ is decreased (Ambrosio and Tortorelli, 1990).



To capture the release of elastic energy upon crack formation, alternative degraded forms for the free-energy function
245 $\tilde{\Psi}(\epsilon_e, p_w, d)$ and the dissipative potential $\tilde{\phi}(\dot{\epsilon}_v^D, d)$ are used. These functions play the same role as $\Psi(\epsilon_e, p_w)$ and $\phi(\dot{\epsilon}_v^D)$
in the previous section, but now account for the reduced ability to maintain stresses within the cracked material.

The net result is that we can define the internal potential of the system as modified by the presence of the phase field d as follows:

$$\begin{aligned} 250 \quad \hat{\Pi}_{\text{int}} &= \frac{d}{dt} \tilde{\mathbf{E}} + \tilde{\mathbf{D}} + \frac{d}{dt} \Gamma_\ell \\ &= \frac{d}{dt} \int_{\Omega} \tilde{\Psi}(\epsilon_e, p_w, d) dV + \int_{\Omega} \tilde{\phi}(\dot{\epsilon}_v^D, d) dV + \frac{d}{dt} \int_{\Omega} G_c \gamma(d) dV. \end{aligned} \quad (26)$$

We split the original free energy into a tensile and compressive part:

$$\Psi(\epsilon_e, p_w) = \Psi_+(\epsilon_e, p_w) + \Psi_-(\epsilon_e, p_w) \quad (27)$$

and apply the effect of the phase-field only to the tensile part (Miehe et al., 2010a):

$$255 \quad \tilde{\Psi}(\epsilon_e, p_w, d) = g(d) \Psi_+(\epsilon_e, p_w) + \Psi_-(\epsilon_e, p_w) \quad (28)$$

For now, we leave the closed form of Ψ_+ to section 2.3. We define $g(d)$ as the degradation function. In line with the literature we choose

$$g(d) = (1 - \kappa)(1 - d)^2 + \kappa, \quad (29)$$

where κ is a small regularisation parameter. This function depends on the phase-field variable d and smoothly interpolates
260 between the undamaged state $d = 0$ and a fully damaged state $d = 1$. The motivation for this choice of function is that it
satisfies the following criteria,

$$g(0) = 1, \quad g(1) \approx 0, \quad g'(1) = 0.$$

These conditions impose limits on the amount of degradation for the undamaged and fully damaged states. The last condition ensures that the fracture force vanishes when $d \rightarrow 1$ (Miehe et al., 2010a).

265 Next, we define the dissipative potential $\tilde{\phi}(\dot{\epsilon}_v^D, d)$. Contrary to the free energy, we do not decompose the dissipative potential
function into tensile and compressive parts, but simply degrade the dissipative potential $\phi(\dot{\epsilon}_v^D)$ as follows,

$$\begin{aligned} \tilde{\phi}(\dot{\epsilon}_v^D, d) &= \frac{2n}{n+1} \tilde{\eta}(\dot{\epsilon}_v^D, d) \dot{\epsilon}_v^D : \dot{\epsilon}_v^D. \\ \tilde{\eta}(\dot{\epsilon}_v^D, d) &= (1 - d)^2 \eta(\dot{\epsilon}_v^D) + k_v \end{aligned} \quad (30)$$

This is appropriate if the failed material cannot support a deviatoric viscous stress. Here, we have slightly modified the degradation function $g(d)$ such that the regularisation parameter, in this instance k_v , acts separately from the non-linear viscosity in
270 $\tilde{\eta}(\dot{\epsilon}_v^D, d)$ for numerical stability reasons.



Finally, we do not modify the external load functional, which we assume to be given as in Eq. (17):

$$\Pi_{\text{ext}} = \int_{\Omega} \mathbf{f} \cdot \dot{\mathbf{u}} \, dV + \int_{\partial\Omega} \mathbf{t} \cdot \dot{\mathbf{u}} \, dS \quad (31)$$

In this expression, we have not multiplied the body and external forces by the degradation function $g(d)$, unlike Duddu et al. (2020); Clayton et al. (2022); Hageman (2026). We examine the effect of this choice in Section 4.2. This ensures that the total mass of the system remains conserved, which is critical for the deformation of an ice mass which is driven by the body force rather than an externally applied load. However, this means that hybrid approximations of the elastic stress (Ambati et al., 2015) which degrade the whole of the elastic stress tensor cannot be applied, as there is then no residual stress to balance the body force inside regions where $g(d) \approx 0$.

Equipped with the modified expressions for $\hat{\Pi}_{\text{int}}$ and Π_{ext} , we proceed as in Subsection 2.1, defining $\Pi = \hat{\Pi}_{\text{int}} - \Pi_{\text{ext}}$ and using the same variational principle (Maugin, 1980, 1990) to derive the strong form of the equations for the viscoelastic phase-field fracture system. Now, as well as all admissible virtual rates $\delta\dot{\mathbf{u}}$, $\delta\dot{\mathbf{u}}_v$ we must also consider $\delta\hat{\Pi}_{\text{int}}$ and $\delta\Pi_{\text{ext}}$ to balance for all admissible variations $\delta\dot{d}$, that perturb the rate of change of the phase-field variable. Taking the first variation of Π for the revised system with respect to $\delta\dot{\mathbf{u}}$, $\delta\dot{\mathbf{u}}_v$ and $\delta\dot{d}$, then using the fundamental lemma of the calculus of variations to set $\delta\Pi = 0$, provides an additional equation that must be satisfied by the phase-field whenever $\dot{d} > 0$,

$$\begin{aligned} \frac{G_c}{2\ell} \left(2d - \ell^2 \nabla^2 d - \frac{1}{8} \ell^4 \nabla^4 d \right) &= 2(1-d)\Psi_+ \quad \text{in } \Omega, \\ \nabla \left(\frac{1}{8} \ell^4 \nabla^2 d - \ell^2 d \right) \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (32)$$

As in most models, fracture is designed to be irreversible, with a requirement on thermodynamic consistency (Miehe et al., 2010b). The rate of change of $\Gamma_\ell(d)$ defines the dissipation of power used in creating the crack field

$$\dot{\Gamma}_\ell(d) = \int_{\Omega} G_c \delta_d \gamma(d) \dot{d} \, dV,$$

where $\delta_d \gamma(d)$ is the variational derivative of the crack density function. The reference domain Ω in this expression is considered fixed since we adopt a Lagrangian approach.

To ensure thermodynamically-consistent, irreversible fracture, we require that the dissipated power is positive

$$\dot{\Gamma}_\ell(d) \geq 0.$$

This condition can be satisfied by locally imposing that

$$\delta_d \gamma(d) \geq 0, \quad (33)$$

$$\dot{d} \geq 0. \quad (34)$$



The second inequality is a natural constraint that assumes the phase-field evolution is locally irreversible. This does not account for any healing of the material.

300 In order to satisfy the conditions Eq. (33) and Eq. (34), we follow Miehe et al. (2010b) and introduce a local history field of maximum tensile energy over time or loading steps

$$\mathcal{H}(\mathbf{x}, t) := \max_{0 \leq s \leq t} \langle \Psi_+(\mathbf{x}, s) - \Psi_{crit} \rangle_+, \quad (35)$$

where $\langle \cdot \rangle_\pm$ is the Macaulay bracket given by $\langle \cdot \rangle_\pm = 1/2(\cdot \pm |\cdot|)$ and s ranges over earlier times. We also include the threshold value Ψ_{crit} following Miehe et al. (2016) as it allows the model to have the correct tensile strength, even for large values of ℓ . Figure 2 shows the stress/strain response of a single element under tension such that $\nabla^2 d$ and $\nabla^4 d = 0$ (Kristensen et al., 2021) and neglecting pore pressure. The free energy then reduces to $\Psi = \sigma^2/2E$. With $\Psi_{crit} = 0$ (Fig. 2a) the maximum tensile strength σ_t is dependent on ℓ and exhibits the correct tensile strength only for $\ell = \ell_{crit} = \frac{3EG_c}{(16\sigma_t/3)^2} \sim 2$ cm for ice, requiring mesh sizes ~ 1 cm or less. For smaller scale problems this mesh size is typically not an issue. However considering icebergs and ice shelves which exist on the kilometre scale, we will necessarily be using a mesh size and ℓ values that are orders of
310 magnitudes greater than ℓ_{crit} . By introducing the threshold value Ψ_{crit} , and setting it equal to

$$\Psi_{crit} = \frac{\sigma_t^2}{2E} \quad (36)$$

we recover the correct tensile strength and stress/strain curve when $\ell \gg \ell_{crit}$ (Fig. 2b). Compared to other approaches for implementing the phase-field at large scales such as modifying the degradation function (Lo et al., 2023) or increasing the value of G_c (Hageman, 2026) we believe this approach provides a simple way to specify the tensile strength while retaining
315 experimentally measured values of G_c .

The fracture phase-field evolution is then modified to be

$$\begin{aligned} \frac{G_c}{2\ell} \left(2d - \ell^2 \nabla^2 d - \frac{1}{8} \ell^4 \nabla^4 d \right) &= 2(1-d)\mathcal{H} \quad \text{in } \Omega, \\ \nabla \left(\frac{1}{8} \ell^4 \nabla^2 d - \ell^2 d \right) \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (37)$$

We will use this formulation for the implementation. If the positive part of the elastic free energy Ψ_+ nowhere exceeds the critical value Ψ_{crit} , then $\Psi_+ - \Psi_{crit}$ is negative everywhere. In that case, $\mathcal{H} = 0$ and the original state $d = 0$ is a solution.
320 During loading, $\Psi_+ - \Psi_{crit}$ is more positive than any previous value, so $\mathcal{H} = \Psi_+ - \Psi_{crit}$ and Eq. (37) is equivalent to Eq. (32). During unloading, $\Psi_+ - \Psi_{crit}$ is more negative than some previous value, so $\mathcal{H}$ is unchanged, hence d is unchanged and its rate of change is zero. Because $\mathcal{H} \geq 0$, inequality Eq. (33) is satisfied whenever $d \leq 1$.

For completeness, we present the full system of equations derived from the modified expressions for $\hat{\Pi}_{int}$ and Π_{ext} below.
325 To simplify the dependence of these equations upon material parameters, we present them in non-dimensional form. This is

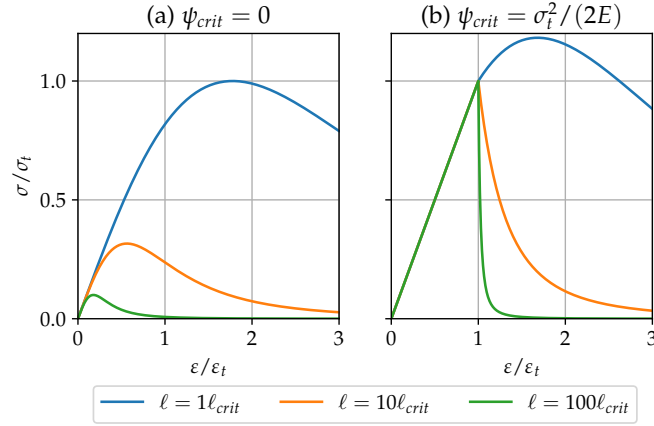


Figure 2. Stress-strain response for a single element under tension, both with and without Ψ_{crit} , for values of $\ell = \ell_{crit}$, $10\ell_{crit}$ and $100\ell_{crit}$, where $\ell_{crit} = \frac{3EG_c}{(16\sigma_t/3)^2} \sim 2$ cm for ice. σ_t is the tensile strength and ε_t is the strain at which the undamaged material would reach this stress.

obtained by introducing non-dimensional scaled quantities (labelled by asterisks) defined as follows:

$$\begin{aligned}
 \mathbf{x} &= H\mathbf{x}^*, \quad t = \tau t^*, \quad \mathbf{u} = u_c \mathbf{u}^*, \quad \mathbf{u}_v = u_c \mathbf{u}_v^*, \quad \ell = H\ell^*, \\
 \lambda &= \mu_c \lambda^*, \quad \mu = \mu_c \mu^*, \quad p_w = p_{wc} p_w^*, \quad \mathbf{f} = f_c \mathbf{f}^*, \quad \mathbf{t} = p_c \mathbf{t}^*, \\
 \rho_i &= \rho_c \rho_i^*, \quad \rho_w = \rho_c \rho_w^*, \quad \eta = \eta_c \eta^*, \quad \boldsymbol{\sigma} = p_c \boldsymbol{\sigma}^*, \quad p = p_c p^*, \\
 \mathcal{H} &= \mathcal{H}_c \mathcal{H}^*, \quad \Psi = \mathcal{H}_c \Psi^*, \quad \Psi_{\pm} = \mathcal{H}_c \Psi_{\pm}^*, \quad \sigma_t = p_c \sigma_t^*.
 \end{aligned} \tag{38}$$

The scaling of x , y , z is to be understood as the scaling of all physical dimensions by a characteristic scale for the ice thickness H . Similarly, time is scaled by τ and displacements by u_c . The other scales are chosen in terms of H , τ , u_c and material constants as follows:

$$\begin{aligned}
 p_c &= \mu u_c H^{-1}, \quad \eta_c = A^{-\frac{1}{n}} \left(\frac{u_c}{H\tau} \right)^{\frac{1-n}{n}}, \quad \mathcal{H}_c = \mu \left(\frac{u_c}{H} \right)^2, \\
 \mu_c &= \mu, \quad \rho_c = \rho_i, \quad f_c = \rho_c g, \quad p_{wc} = \rho_c g H.
 \end{aligned}$$

Having derived the full system in exactly the same way as described previously for the situation without fracture, we substitute these relationships, then drop the asterisks on non-dimensional quantities to provide the following non-dimensional system



of equations.

$$\begin{aligned}
 -\nabla \cdot \tilde{\boldsymbol{\sigma}} &= \mathbf{f} && \text{in } \Omega \\
 2\tilde{\eta}(\nabla_s^D \dot{\mathbf{u}}_v, d) \nabla_s^D \dot{\mathbf{u}}_v - g(d)p\mathbf{I} &= \tilde{\boldsymbol{\sigma}} && \text{in } \Omega \\
 g(d) \nabla \cdot \dot{\mathbf{u}}_v &= 0 && \text{in } \Omega \\
 \tilde{\boldsymbol{\sigma}} \cdot \mathbf{n} &= \mathbf{t} && \text{on } \partial\Omega \\
 d - \frac{1}{2} \ell^2 \nabla^2 d - \frac{1}{16} \ell^4 \nabla^4 d &= C \ell^2 (1-d) \mathcal{H} && \text{in } \Omega \\
 \nabla \left(\frac{1}{8} \ell^4 \nabla^2 d - \ell^2 d \right) \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega.
 \end{aligned} \tag{39}$$

where $\tilde{\boldsymbol{\sigma}} = \partial \tilde{\Psi} / \partial \boldsymbol{\varepsilon}_e = g(d) \boldsymbol{\sigma}_+ + \boldsymbol{\sigma}_-$. The constant C is defined below in Eq. (43). The history function $\mathcal{H}$ is defined as in Eq. (35).

340 We then make one approximation from the above variationally consistent form, and replace the second equation in Eq. (39) with:

$$2\tilde{\eta}(\nabla_s^D \dot{\mathbf{u}}_v, d) \nabla_s^D \dot{\mathbf{u}}_v - g(d)p\mathbf{I} = g(d) \boldsymbol{\sigma}' \tag{40}$$

where $\boldsymbol{\sigma}' = K/\mu \text{tr}(\boldsymbol{\varepsilon}_e) \mathbf{I} + 2\boldsymbol{\varepsilon}_e^D - Bp_w \mathbf{I}$ is the non-dimensionalised undamaged effective stress tensor. This is similar to the hybrid approximation of Ambati et al. (2015) but for the viscous part only. This is necessary as we have made a simplistic
 345 assumption that the whole viscous dissipation term is degraded whereas this may not be the case. We still keep the full variational form in the first equation, to ensure there is some residual elastic stress $\boldsymbol{\sigma}_-$ to balance the body force inside regions of $g(d) \approx 0$.

Choosing the scale of the stress as $\delta \rho_i g H$ based on the analytic solution in Appendix B, where $\delta = 1 - \rho_i / \rho_w$, dimensional analysis of the first equation in Eq. (39) gives:

$$350 \quad u_c = \frac{\delta \rho_i g H^2}{\mu} \approx 2.5 \text{ cm} \tag{41}$$

Similarly, considering that the elastic and viscous stresses must balance in the second equation of Eq. (39), and given that we have assumed that the elastic and viscous displacements have the same scale u_c , means the timescale τ corresponds to the characteristic time:

$$\tau = \frac{\eta_c}{\mu} = \frac{(\delta \rho_i g H)^{1-n}}{A\mu} \approx 3 \text{ hour.} \tag{42}$$

355 In agreement with Hageman et al. (2024) for our temperature of -10°C . Non-dimensionalisation of the phase field equation in Eq. (39) reveals a constant C which controls the growth of fracture:

$$C = \frac{\delta^2 \rho_i^2 g^2 H^3}{G_c \mu} \approx 10^4 \tag{43}$$

There are four non-dimensional parameters which enter into the system Eq. (39): the aforementioned constant C controlling the growth of fractures, the Poisson ratio ν , the regularisation length ℓ^* , and the non-dimensional tensile strength σ_t^* that enters



Parameters	Values
ρ_i	900 [kg/m ³]
ρ_w	1000 [kg/m ³]
$ g $	9.81 [m/s ²]
B	0 [-]
E	9.33×10^9 [N/m ²]
ν	0.325 [-]
A	3.56×10^{-25} [Pa ⁻³ s ⁻¹]
n	3 [-]
G_c	0.96 [N/m]
σ_t	2×10^5 [N/m ²]

Table 1. Table of material parameters and their characteristic values. The parameters for Young's modulus E , and Poisson ratio ν are from Greve and Blatter (2009), flow constant A is for ice at -10°C from Cuffey and Paterson (2010), and critical energy release rate G_c is from measured values of $K_{IC} = 100$ KPa from the Brunt ice shelf (Pearce et al., 2026), where $G_c = K_{IC}^2(1 - \nu^2)/E$.

360 via Eq. (35). The Poisson ratio for ice is known approximately (Greve and Blatter, 2009). The same is also true for the tensile strength of ice (Wells-Moran et al., 2025), and hence Ψ_{crit}^* through Eq. (36). Finally, the regularisation length ℓ^* is effectively set by computational cost limits, as it must be at least twice the size of the mesh size.

2.3 Incorporating pore pressure into the free energy split

We now move on to our expression for the free energy split in Eq. (27), Eq. (28). A conceptual picture for the physics we aim to represent is shown in Fig. 3. As mentioned in the introduction, ice can be considered a brittle material where fracture is also strongly influenced by the presence of water pressure, through the presence of microcracks (e.g. Fig. 3b and d). As the material becomes damaged $d > 0$ we consider these microcracks to open and the Biot coefficient to increase (Fig. 3a and c). In particular, the bulk material can be under significant compressive stress from its self weight, but microcracks, particularly on the bottom surface (Fig. 3d), can have very high pore pressure which balances this, causing basal crevassing (Jezek, 1984).

370 The idea of Jezek (1984) is that crevasses do not form just when the stress is tensile, as in Nye (1955), but, due to the presence of microcracks on the bottom surface filled with water (Fig. 3), crevasses can form in solid, non-porous ice when the water pressure plus the tensile component of stress is greater than zero, i.e.

$$\sigma_3 + p_w > 0 \quad (44)$$

where σ_3 is the largest eigenvalue of the stress. We can consider this a “modified stress”:

$$\hat{\sigma} = \sigma + p_w \mathbf{I} \quad (45)$$

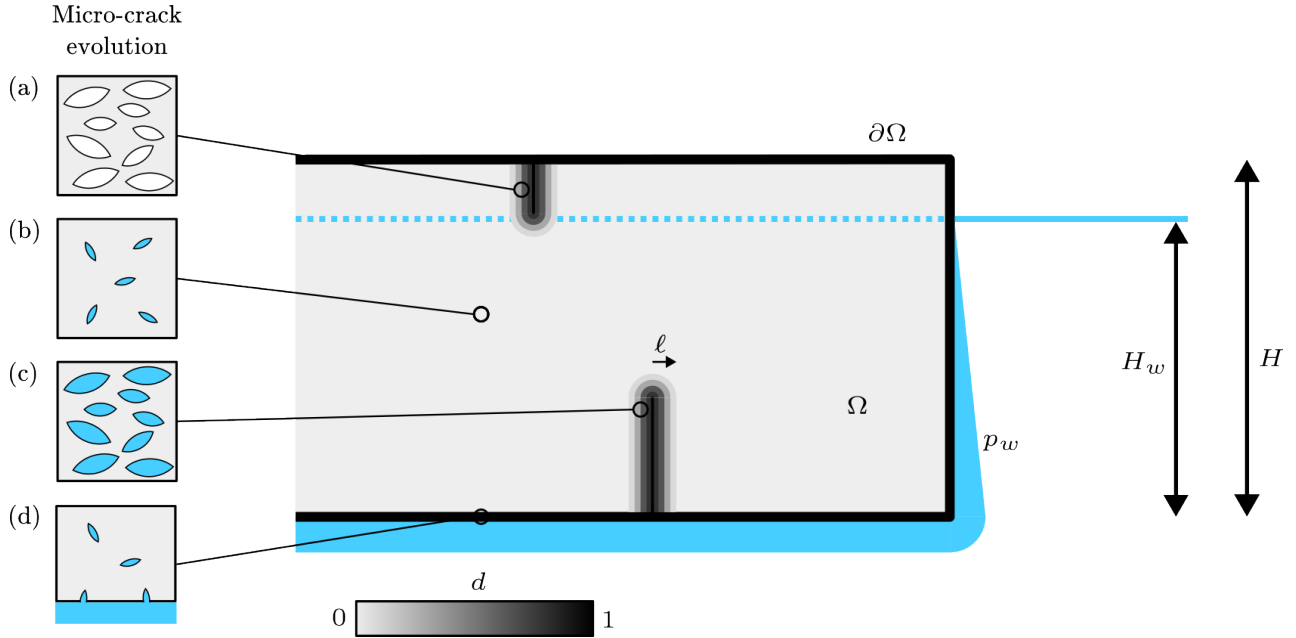


Figure 3. Schematic of the model representation of fracture and the associated (sub grid) micro-crack physics we intend to represent. Fractures are represented by a smooth phase-field d which takes values 0 in intact material and 1 in fractures, with fractures ‘smeared’ over the regularisation length ℓ . Above the waterline, microcracks have zero pore pressure (a). Below the waterline, microcracks are assumed to have pore pressure equal to the hydrostatic ocean water pressure p_w (b-d). Increasing d causes opening of micro-cracks ((b) to (c)). In particular we are interested in representing the effect of micro-cracks on the bottom surface (d), which lead to basal crevasses.

where failure occurs when $\hat{\sigma}_3 > 0$. This can be written in terms of the elastic strain:

$$\hat{\sigma} = \mathbf{C} : \varepsilon_e + p_w \mathbf{I} = \mathbf{C} : \left(\varepsilon_e + \frac{p_w \mathbf{I}}{3K} \right) \quad (46)$$

where $\mathbf{C}$ is the stiffness tensor defined in Eq. (1). In this way, the extra tensile stress caused by pore pressure can also be considered as an extra volumetric strain.

380 We seek to develop a free energy split which accurately captures the formation of fractures in this case, while also retaining the expected tensile/compressive asymmetry without pore pressure. Recent work by Ulloa et al. (2022) has developed a variational formulation for the effect of pore pressure for both open and closed microcracks, giving mode-I and mode-II fractures respectively. You and Yoshioka (2023) also combine the effect of pore pressure with certain kind of free energy decompositions. As basal crevasses are mode-I, we first describe the formulation for this in Ulloa et al. (2022) and then detail where we
385 depart from this.

Following Ulloa et al. (2022) we consider the degradation of the Biot coefficient B and modulus M as:

$$\tilde{B}(d) = 1 - \frac{g(d)K}{K_s} = B + (1 - B)(1 - g) \quad (47)$$



The degraded Biot modulus is then:

$$\frac{1}{\widetilde{M}(d)} = \frac{\widetilde{B}(d) - v_0^f}{K_s} = \frac{1}{M} + \frac{(1-B)^2}{K}(1-g) \quad (48)$$

390 For mode-I crevasses, Ulloa et al. (2022) degrade the whole of the elastic part of the free energy to give:

$$\widetilde{\Psi}(\epsilon_e, p_w, d) = g(d)\psi(\epsilon_e) - \frac{p_w^2}{2\widetilde{M}(d)} - \widetilde{B}(d)p_w \text{tr}(\epsilon_e) \quad (49)$$

Before we propose our modification to Ulloa et al. (2022), we first show how Eq. (49) can be recast in the form of a split like Eq. (27) and Eq. (28). The details of this are in Appendix A. After some algebra Eq. (49) can be rewritten as:

$$\begin{aligned} \widetilde{\Psi}(\epsilon_e, p_w, d) &= g(d)\Psi_+(\epsilon_e, p_w) + \Psi_-(\epsilon_e, p_w) \\ \Psi_-(\epsilon_e, p_w) &\equiv \Psi(\epsilon_e, p_w) - \Psi_+(\epsilon_e, p_w) \\ \Psi_+(\epsilon_e, p_w) &= \psi(\widehat{\epsilon}_e) = \frac{1}{2}K \text{tr}(\widehat{\epsilon}_e)^2 + \mu\widehat{\epsilon}_e^D : \widehat{\epsilon}_e^D \end{aligned} \quad (50)$$

395 where we define a modified strain:

$$\widehat{\epsilon}_e \equiv \epsilon_e + (1-B)\frac{p_w \mathbf{I}}{3K} \quad (51)$$

This modified strain represents the extra volumetric strain caused by the presence of water filled microcracks, and matches the idea of Jezek (1984) in Eq. (46) for solid ice ($B = 0$). We note that in this case, it can be seen that pore pressure causes the largest increase in tensile energy when the Biot coefficient $B = 0$, and has no extra effect for $B = 1$, i.e. for materials like
400 soils. This may seem counter intuitive however we are considering some material with a micro-crack with pore pressure p_w . For $B = 1$, the pore pressure is fully incorporated into the effective stress, so no additional fracture driving force arises from our modified strain. For an almost fully solid material, for example our case of ice with a microcrack at the bottom surface, the pore pressure provides a significant extra driving term for fracture.

Ulloa et al. (2022) couple Eq. (50) with another differential equation for the plastic strain to represent mode-II cracks.
405 This comes with a significant burden of complexity and computational cost. For cracks induced by pore-pressure which are known to be mode-I, such as basal crevasses in ice, it should be possible to represent these without recourse to an equation for plastic strains. Unfortunately, disregarding the plasticity equation from Ulloa et al. (2022) and taking Eq. (50) on its own is not sufficient, as without the plasticity condition it does not capture any tensile/compressive asymmetry in behaviour. Indeed setting the pore pressure $p_w = 0$ shows Eq. (50) gives $\widetilde{\Psi} = g(d)\psi(\epsilon_e)$ i.e. the whole elastic energy is degraded, and even
410 compressive states will cause fracture.

To include the effect of pore pressure, but also to allow material behaviour which exhibits significant tensile/compressive asymmetry for mode-I cracks, we propose a modification of the scheme in Ulloa et al. (2022). Using the idea that the effect of pore pressure can be represented as an extra volumetric elastic strain, we consider a split in the total free energy of the form:

$$\Psi_+(\epsilon_e, p_w) = \psi_+(\widehat{\epsilon}_e) \quad (52)$$



415 where the form of ψ_+ can be given by any of the elastic energy splits available in the literature (e.g. Miehe et al., 2010a; Navidtehrani et al., 2022; Vicentini et al., 2024, and references therein), but acting on the modified elastic strain $\hat{\epsilon}_e$ rather than the original elastic strain ϵ_e . For no pore pressure, this recovers the forms in these references, and for the choice of degrading the whole elastic energy, i.e. $\psi_+ = \psi$, this recovers the variational form of Ulloa et al. (2022) with open microcracks, and for a split that can be expressed in terms of the invariants, it recovers You and Yoshioka (2023).

420 In this work, we combine this idea with the ‘no-tension’ split (Freddi and Royer-Carfagni, 2010). This aims to ensure that only tensile stresses contribute to fracture. In this way it functions in a similar manner to other phase-field applications in ice, such as the stress-spectral split used in (Hageman, 2026) and the stress based driving force used in (Clayton et al., 2022), while ensuring variational consistency such that some residual stress can be kept to balance the body force in the momentum equation.

$$425 \quad \Psi_+(\epsilon_e, p_w) = \begin{cases} \Psi(\epsilon_e, p_w), & \text{if } e_1 > 0, \\ \Psi(\epsilon_e, p_w) - E \frac{e_1^2}{2}, & \text{if } e_1 \leq 0 \text{ and } e_2 + \nu e_1 > 0, \\ (1 + \nu) \frac{((1 - \nu)e_3 + \nu(e_2 + e_1))^2}{(1 - 2\nu)(1 - \nu^2)}, & \text{if } e_2 + \nu e_1 \leq 0 \text{ and } (1 - \nu)e_3 + \nu(e_2 + e_1) > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (53)$$

Where traditionally $e_3 \geq e_2 \geq e_1$ would represent the eigenvalues of the elastic strain tensor ϵ_e but in our formulation they represent the eigenvalues of the modified elastic strain tensor $\hat{\epsilon}_e$.

We highlight the applicability of this split by considering the analytical solution for an idealised infinitely long ice shelf of uniform thickness. The details of this analytic solution are available in Appendix B. In Fig. 4 we plot the original (i.e. without the effect of pore pressure) and modified free energy split for an infinitely long spreading iceberg, with an undamaged Biot coefficient $B = 0$. Shaded regions show where $\sigma_{xx} + p_w > 0$; it is in these regions that surface and basal crevasses are expected to form (Nye, 1955; Jezek, 1984). The original no-tension split $\psi_+(\epsilon_e)$ matches this depth for surface crevasses, but is not positive for basal crevasses as it does not include the effect of water pressure. However, by combining the effect of pore-pressure with the no tension split of Freddi and Royer-Carfagni (2010), the free energy becomes positive when $\sigma_{xx} + p_w > 0$,
435 allowing the modelling of basal crevasses as well as surface crevasses.

3 Implementation Details

In this section, we briefly describe the implementation details of the viscoelastic phase field fracture system Eq. (39). For timescales less than the Maxwell time of approximately 3 hours, we can usually discount differences between the original and current configurations as the differences are negligibly small. However, evolution over long time periods will eventually
440 invalidate the small displacement assumption. In such cases we need to address the differences between the body configurations at different times. We adopt an approach described by Bathe et al. (1975) referred to as material-nonlinearity-only based on

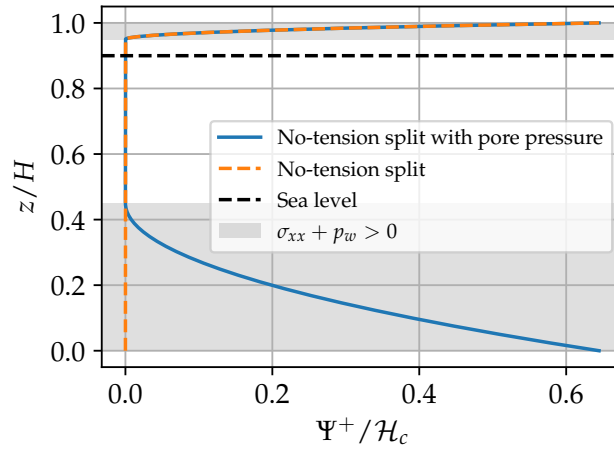


Figure 4. Comparison between the no tension split (Freddi and Royer-Carfagni, 2010) both with and without the effect of pore pressure. Highlighted also is the sea level and regions where the tensile stress plus the water pressure is positive, which is taken by Jezek (1984) as the condition for both basal and surface crevasses to form.

the updated Lagrangian formulation. Essentially, this means that at each timestep we solve for the change in displacements $(\Delta \mathbf{u}, \Delta \mathbf{u}_v)$, and update the mesh in a Lagrangian manner based on the change in displacements at the end of each timestep. We also define the elastic strain tensor as the integral of the increment in the elastic strain tensor over time, rather than simply
445 as $\nabla_s(\mathbf{u} - \mathbf{u}_v)$, to account for changes in the body configuration. At each time it is updated using:

$$\boldsymbol{\varepsilon}_e^n = \boldsymbol{\varepsilon}_e^{n-1} + \nabla_s(\Delta \mathbf{u}^n - \Delta \mathbf{u}_v^n). \quad (54)$$

We note that the system Eq. (39) is nonlinear and nonconvex due to the proposed split into tensile-compressive parts and the degradation function $g(d)$. Consequently, a simultaneous solution of the phase-field and displacement variables in Eq. (39) is not guaranteed to converge. In order to alleviate this problem we propose an extension of the staggered scheme (Miehe et al.,
450 2010a) to an augmented system of equations. In this staggered algorithm, we solve for the change in displacements and pressure variables while keeping the phase field variable fixed, then solve for the phase-field variable, keeping the displacements and pressure fixed. Alternating in this way, we iteratively converge towards a solution for the coupled system at each timestep.

The weak form of the system of equations in Eq. (39) is implemented in FEniCSx (Alnæs et al., 2014; Scroggs et al., 2022a, b; Baratta et al., 2023). The computational domain is decomposed into a triangular mesh. For the momentum equation,



455 the weak form is given by:

$$\begin{aligned}
 0 = & \int_{\Omega} (g(d)\boldsymbol{\sigma}_+(\widehat{\boldsymbol{\varepsilon}}_e) + \boldsymbol{\sigma}_-(\widehat{\boldsymbol{\varepsilon}}_e)) : \nabla_s(\mathbf{v}) - \mathbf{f} \cdot \mathbf{v} \, d\Omega + \int_{\partial\Omega} p_w \mathbf{n} \cdot \mathbf{v} \, d\Gamma \\
 & + \int_{\Omega} 2\tilde{\eta} \nabla_s \left(\frac{\Delta \mathbf{u}_v}{\Delta t} \right) : \nabla_s(\mathbf{w}) - g(d)p \nabla \cdot \mathbf{w} - g(d)\boldsymbol{\sigma}'(\widehat{\boldsymbol{\varepsilon}}_e) : \nabla_s(\mathbf{w}) \, d\Omega \\
 & - \int_{\Omega} g(d) \nabla \cdot (\Delta \mathbf{u}_v) q \, d\Omega.
 \end{aligned} \tag{55}$$

where $\mathbf{v}$, $\mathbf{w}$ and q are test functions. The weak formulation is implemented using a mixed finite element method in space, using piecewise polynomial functions to approximate the solution. In order to ensure numerical stability of the resulting system we use Taylor-Hood elements, where piecewise quadratic functions are used for the change in displacements $\Delta \mathbf{u}$, $\Delta \mathbf{u}_v$ and
 460 piecewise linear functions are used for the pressure p . Another subtlety is that the external water pressure must also include the displacement to give the correct iceberg deformation:

$$p_w = \begin{cases} -\rho_w |\mathbf{g}| (z + \Delta \mathbf{u} \cdot \mathbf{e}_z - H_w), & \text{for } z < H_w \\ 0, & \text{for } z \geq H_w \end{cases} \tag{56}$$

Equation Eq. (55) is non-linear due to the split in the stress and the power law viscosity and we solve it within each fixed-point iteration using a Newton method.

465 For the higher-order phase-field equation, we introduce an auxiliary variable $\chi = \nabla^2 d$ as part of a mixed function space to account for the fourth-order derivative. We use piecewise quadratic elements for the phase-field variable d and auxiliary variable χ

$$\begin{aligned}
 0 = & \int_{\Omega} dr - \frac{1}{2} \ell^2 \chi r - \frac{1}{16} \ell^4 \nabla \chi \cdot \nabla r - 2C\ell(1-d)r\mathcal{H} \, d\Omega \\
 & - \int_{\Omega} \chi s + \nabla d \cdot \nabla s \, d\Omega,
 \end{aligned} \tag{57}$$

where r and s are piecewise quadratic test functions.

470 In order to obtain mesh independent results, the mesh size near the crack path needs to be at least twice as small as the regularisation parameter ℓ (Miehe et al., 2010b). It is important to keep in mind that we are not trying to resolve any timescale attached to the process of brittle failure. Rather, at each timestep, we iteratively seek a simultaneous solution of the change in displacements, pressure and phase field variables that together satisfy the balance equations. This means that cracks can advance suddenly from one timestep to the next if this is energetically favourable.

475 The system of equations we have described is three-dimensional but we solve them on a two-dimensional grid with two-dimensional fields. We assume that there is no out of plane *viscous* strain-rate. Consequently, if we also impose the assumption of no out-of-plane elastic strain, then, from our Maxwell assumption $2\eta \dot{\boldsymbol{\varepsilon}}_v^D = 2\mu \boldsymbol{\varepsilon}_e^D$, this leads to the trivial case of $\varepsilon_{exx} = -\varepsilon_{ezz}$ (with y the out of plane dimension). Therefore, we allow there to be some out of plane elastic strain, which is always



given as:

$$\varepsilon_{e_{yy}} = \frac{1}{2}(\varepsilon_{e_{xx}} + \varepsilon_{e_{zz}}) \quad (58)$$

We account for this numerically by noting that the trace is modified as

$$\text{tr}_{3D}(\varepsilon_e) = \frac{3}{2} \text{tr}_{2D}(\varepsilon_e) \quad (59)$$

and the eigenvalues of the elastic strain tensor required for the no-tension split are given as:

$$\{\lambda_1, \lambda_2, \lambda_3\}_{3D} = \{\lambda_1, \frac{1}{2}(\lambda_1 + \lambda_2), \lambda_2\}_{2D} \quad (60)$$

4 Numerical Experiments

In this section we present numerical experiments to demonstrate the capabilities of the model. Our primary focus is on ice shelves that float freely in the water. We ignore any tidal effects and neglect the presence of firn.

4.1 Crevasses in a spreading ice shelf

We consider an infinitely long spreading ice shelf, i.e. the same problem as Fig. 4, but in this case coupled to the phase-field equation to examine where crevasses form. Over time, the ice shelf gradually thins and spreads; however, we do not consider the time evolution as a later time corresponds only to the same infinite shelf with a reduced height. For thickness $H \lesssim 550$ m crevasses do not nucleate as the maximum value of $\Psi_+ < \sigma_t^2/2E$. We show a domain with $H = 600$ m and $L = 1200$ m and $\ell^* = 0.01$ and impose the solution for $\mathbf{u}$ and $\mathbf{u}_v$ for infinite length ice shelf as a boundary condition, which can be found in Appendix B.

With an initial condition of an undamaged ice shelf, the modification of the free energy split in Section 2.3 allows both surface and basal crevasses to nucleate and form, resulting in the fracture pattern shown in Fig. 5. We show a contour plot of the phase field variable d . Also plotted are horizontal lines indicating where both $\sigma_{xx} + p_w$ and ψ_+ become positive for an undamaged infinite ice shelf, as shown previously in Fig. 4. The model predicts a number of closely spaced crevasses, at both the base and the surface. The crevasses also propagate to the critical depth where the stress plus water pressure switches from tensile to compressive. This result matches the original work of Nye (1955); Jezek (1984) who hypothesised that basal and surface crevasses would propagate to this depth.

4.2 Degrading or keeping the body force

One choice we have made in Subsection 2.2 is to not degrade (i.e. not multiply by $g(d)$) the body force, unlike Clayton et al. (2022, 2025); Hageman (2026). We examine the effect of this over a range of ℓ values by comparing two models: (a) a model with the body force degraded:

$$-\nabla \cdot \tilde{\sigma} = g\mathbf{f}$$

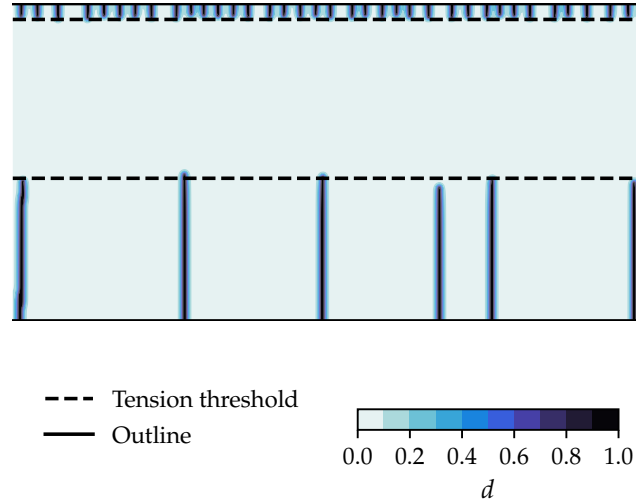


Figure 5. Damage field d for an infinitely long spreading ice shelf, with $H = 600$ m. From an initially uncrevassed state, fractures nucleate as shown by contours of d , alongside dashed lines indicating where $\sigma_{xx} + p_w = 0$, hypothesised by Nye (1955); Weertman (1980); Jezek (1984) as the depth to which closely spaced crevasses will propagate.

and (b) the model we have described above and use in the rest of this paper.

We run both these models over an ice shelf/iceberg domain of $H = 400$ m, $L = 2$ km. The iceberg is floating, with a symmetry boundary condition on the left boundary and with water pressure acting on the bottom and right boundaries, as in
 510 Fig. 3. Figure 6 shows both the exaggerated (by a factor of 300) displacement u and a contour of $d = 0.5$ showing surface crevasses, for a range of values of ℓ^* . The non-dimensional mesh size is set at $\ell^*/2$. For both cases, a number of surface crevasses form towards the end of the ice shelf.

Figure 6a shows the model with the body force degraded. For $\ell^* = 0.01$, four surface crevasses form close to the end of the iceberg. As the body force is degraded, regions where $d > 0$ effectively remove mass from the system. This removal of
 515 mass affects the hydrostatic equilibrium and deformation of the iceberg, causing the iceberg to rise in the water and the end to bend upwards. As ℓ^* is reduced and correspondingly the mesh size decreases, each crevasse removes less mass from the system, affecting the hydrostatic equilibrium less and causing less upward bending. Consequently, as $\ell^* \rightarrow 0$ the displacement and crevasse pattern for Fig. 6a tends more towards Fig. 6b.

We note that the runs in Fig. 6 had to be done with no water pressure acting in the cracks, as the upward deflection seen in
 520 Fig. 6a causes the bottom surface to become more tensile, and, if the effect of water pressure is included, causes unstable basal crevasse formation for only Fig. 6a.

For Fig. 6b, with the undegraded body force as we use in the rest of the paper, a large number of shallow surface crevasses form above the waterline, and the displacement is unaffected by the presence of crevasses. Consequently, there is mesh independence between the results for $\ell^* = 0.01$ and $\ell^* = 0.0025$.

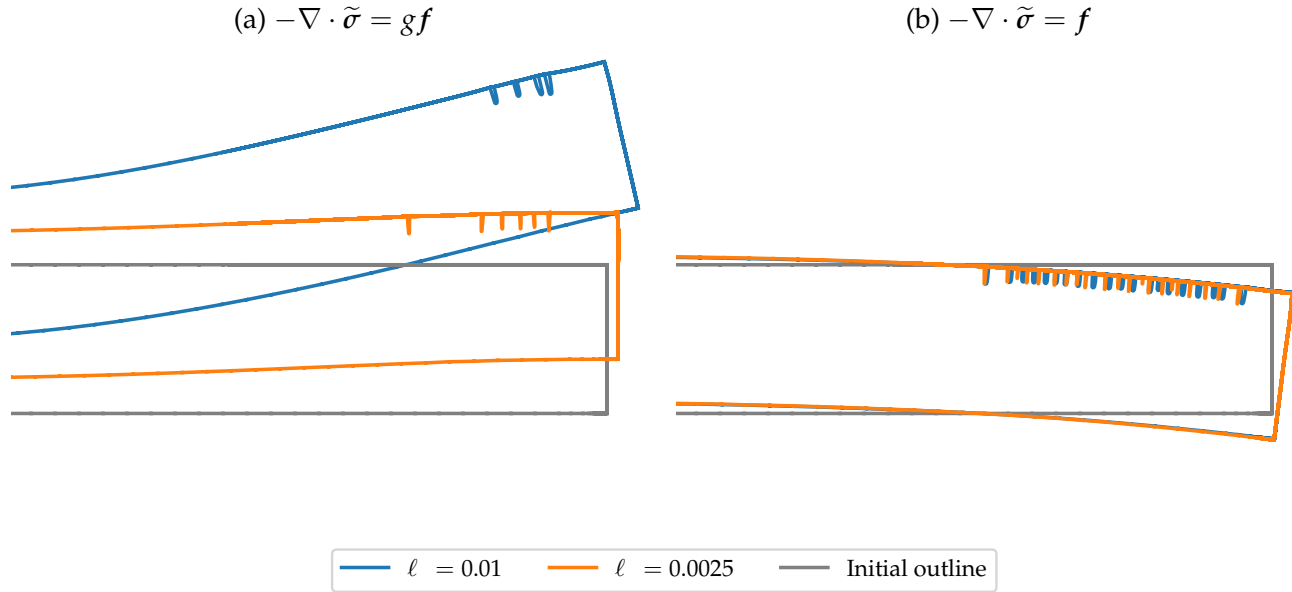


Figure 6. Comparison of the model both (a) with and (b) without the body force in damaged regions. The iceberg geometry is shown in grey. Coloured lines show both the iceberg displacement u , exaggerated by a factor of 300, and contours for $d = 0.5$ highlighting crevasse location on the deformed iceberg, for different values of ℓ^* (and corresponding mesh size $= \ell^*/2$).

525 In our model we have retained the full expression for the stress $\tilde{\sigma} = g(d)\sigma_+ + \sigma_-$ in the first equation of Eq. (39). Consequently, in damaged regions where $g(d) \rightarrow 0$ there remains a residual stress σ_- to balance the body force. We note that both Hageman (2026) and Clayton et al. (2022) degrade the whole of the effective elastic stress tensor following Ambati et al. (2015), such that the stress in the first equation of Eq. (39) is approximated as $\tilde{\sigma} = g\sigma_0$. This approximation is often made as it provides significant computational cost benefits and can be coupled with any form for the fracture driving energy Ψ_+ . If this approximation is made, the body force must be degraded as otherwise there is no residual stress to balance the self-weight inside a crack. Whereas many phase-field applications are driven by external loads, the deformation of ice sheets is driven entirely by their own weight, and ice shelves and icebergs are further affected by hydrostatic equilibrium. Applications limited to ice cliffs, such as Clayton et al. (2025); Hageman (2026), where the base is in contact with the ground and hence hydrostatic equilibrium does not play a role may be less affected. However, generally, we find for floating ice and for other gravity-driven deformations, retaining the full body force and ensuring mass conservation is key.

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535

4.3 Failure of an ice shelf

We now consider the timescales to failure caused by a calving event at the end of an ice shelf. We consider thicknesses from 400 to 550 m, with the length $L = 5H$ set sufficiently large such that the results are independent of length, a mesh size half the regularisation length in the fractured regions and a non-dimensional timestep of 10. As before, a symmetry condition is



540 applied at the left boundary, and water pressure is applied at the bottom and right boundaries below the water line. The initial configuration, and the value of Ψ_+ before the phase-field is switched on, is shown in Fig. 7(a), over a logarithmic scale. Areas where $\Psi_+ > \Psi_{crit}$ are highlighted in red.

At time $t = 0$ a number of surface crevasses form at the end of the ice shelf. The ice shelf with surface crevasses then evolves slowly through viscous creep, whereby the ice shelf spreads, thins, and bends over time. Fig. 7(b) - (e) shows this temporal evolution and also shows a gradual growth of the crevasse closest to the end of the ice shelf, which, due to bending is furthest below the waterline. In our approach any crack beneath the sea level is assumed to be in hydrological connection to the ocean so that it floods with water. The water pressure both acts on the faces of the crack causing increased tensile stresses, and contributes to an increased crack driving force Ψ_+ in the phase field equation. Eventually, these processes cause a second phase of rapid, unstable crack propagation. This ultimate stage of failure occurs within one time-step of the model (Fig. 7(f)) and can be considered a type of hydro-fracture, occurring almost instantaneously.

The simulations demonstrate that even brittle failure with no inherent timescale can have its timing controlled by the slow viscous deformation of ice. In the simulations, crevasses can exist for a significant time in a fairly stable configuration with only gradual growth before very rapidly hydro-fracturing through the full depth due to viscous bending. It is important to distinguish between brittle propagation of a fracture to form a crevasse, which in our model is a rate-independent process and happens instantaneously, and the gradual change of the body configuration and advection of pre-existing crevasses which happens quite slowly on timescales governed by creeping viscous flow. The latter process is absent from zero-stress models of Nye (1955); Meier (1958); Weertman (1973); Benn et al. (2007).

The timing of calving varies strongly with the ice thickness. Through the non-dimensionalisation, changing the thickness H affects the non-dimensional timescale, which is $\propto H^{1-n}$, the fracture driving constant $C \propto H^3$ and the non-dimensional energy threshold Ψ_{crit}^* which is $\propto H^{-2}$. Fig. 8 shows the time to failure, calving distance and calving rate against thickness, for realistic values of fracture toughness values of $K_{IC} = \{50, 100, 150\}$ kPa m^{0.5} (Pearce et al., 2026) and tensile strength values of $\sigma_t = \{100, 150, 200\}$ kPa. We limit our study to these values of tensile strength as it allows the same set of thicknesses to be examined and provides a reasonable range, though there is a possibility for higher values (Wells-Moran et al., 2025). We find a power law relation between thickness and time to calving, but the power law exponent varies with fracture toughness and particularly tensile strength, with the time to failure proportional to $H^{-5.3}$ for $\sigma_t = 100$ kPa and proportional to $H^{-8.6}$ for $\sigma_t = 200$ kPa.

We also find the non-dimensional calving distance from the front is approximately proportional to $H^{-0.5}$, with some variance across parameters but significantly less than the time to failure. This is caused by increasing the thickness causing increased driving energy Ψ_+ , causing the final crevasse as seen in Fig. 7 to nucleate closer to the end of the ice shelf, and it is this final crevasse the eventually calves in most cases. The calving rate is proportional to the dimensional calving distance from the front divided by the time to failure, and is shown in Fig. 8c. We find this is proportional to approximately $H^{5.5}$ to $H^{9.5}$ over the range of values examined, with again more sensitivity to thickness for higher tensile strengths. We perform a sensitivity in Fig. C1 in Appendix C, showing that these results are broadly robust to changes in timestep, domain length, regularisation length and mesh size.

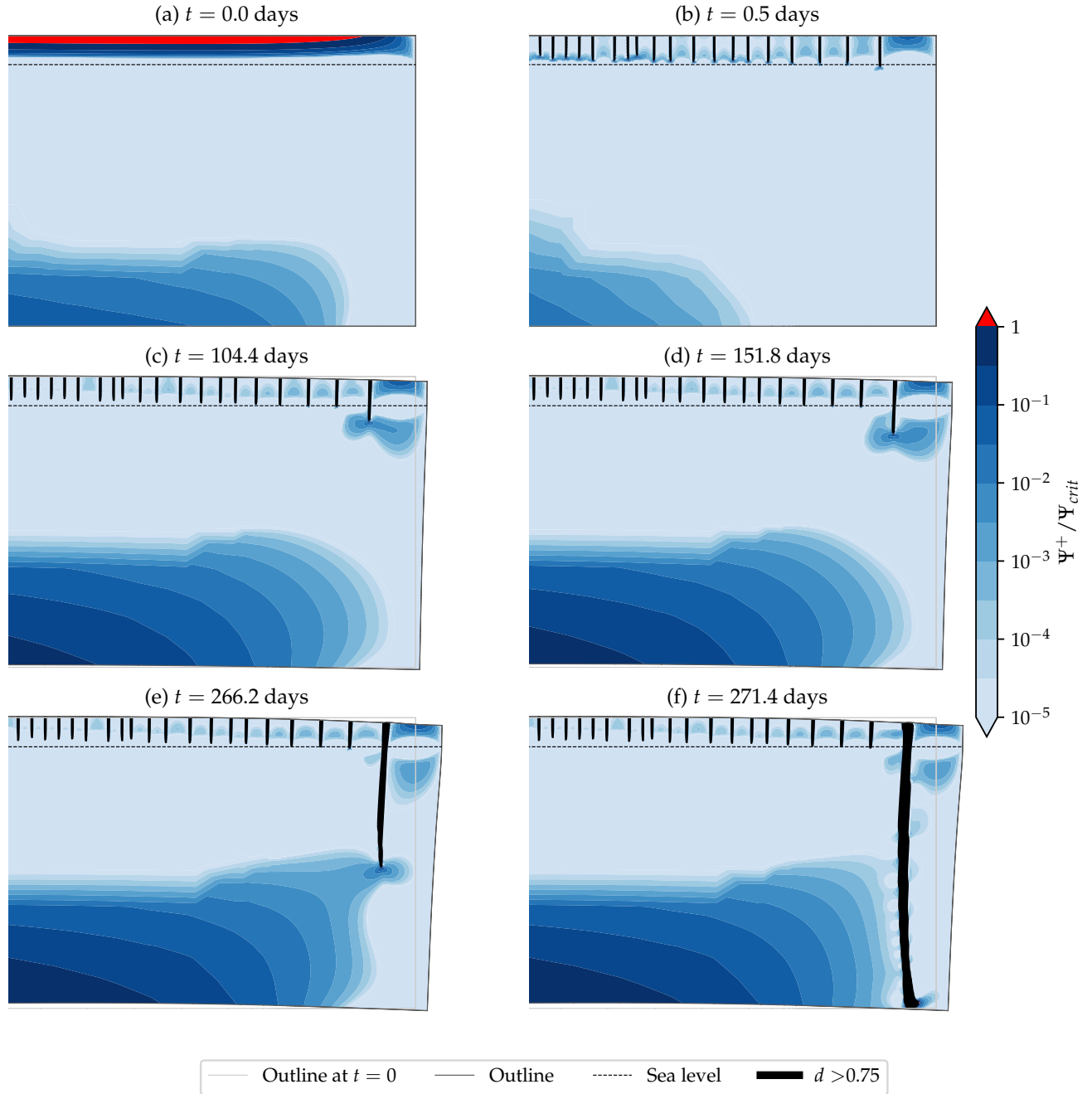


Figure 7. Evolution of an ice shelf of $H = 500$ m and $\ell^* = 0.01$ over time, showing displacement vs the initial outline, the crack driving force Ψ_+/Ψ_{crit} , and the phase field as a filled in contour for $d > 0.75$. Panel (a) shows the initial solution at $t = 0$ days. Panels (b) through (e) show the time evolution of the ice shelf, until the timestep immediately preceding fracture, showing the viscous creep causing a gradual increase in crack driving force and growth of the final crevasse. Figure (f) displays the ice shelf at moment of calving.



575 Because our more complete numerical model contains a detailed representation of the fracture process, and because we
simulate calving in such a simplified geometrical situation (an ice shelf at flotation, with no lateral variations across flow) our
simulations can serve as a simple reference case for comparison with other calving laws: parameterisations of calving which act
as boundary conditions for large-scale simulations of ice sheets. Four different laws are compared by Choi et al. (2018). These
are 1) height above buoyancy, 2) eigencalving, 3) crevasse-depth, and 4) von Mises calving. It is worth briefly considering the
580 qualitative similarities and differences between these four calving laws and the results of our simulations. For reference, Choi
et al. (2018) provide the mathematical expressions that define each of these laws.

The height above buoyancy criterion qualitatively captures the idea that calving rate increases with ice thickness. However,
because the height above buoyancy is zero in our simulations, this law would not predict any calving to occur, which is in
contradiction to our simulations. Similarly, the eigencalving law considers the calving rate to be proportional to the product of
585 two principal horizontal strain rates. When applied to the situation that we simulate, with no lateral strain rate across flow, this
would also predict zero calving rate.

Turning to the crevasse-depth criterion developed by Benn et al. (2007) and used in some large-scale ice sheet models
(DeConto and Pollard, 2016), calving is hypothesised to occur when a crevasse reaches the waterline or surface and basal
crevasses join. However, we find that close to the ice front, with significant bending, surface crevasses can exist below the
590 waterline for a significant time. This calving law also only scales linearly with stress, which is proportional to H^1 , so this law
may significantly underestimate calving rates for large ice thicknesses.

The von Mises criterion as implemented by Choi et al. (2018) assumes the calving rate is proportional to ice speed mul-
tiplied by the tensile stress. While a dependence on speed introduces a problematic frame dependence into the calving rate,
dimensional analysis of how ice speed scales in our model using the scales for u_c in Eq. (41) and τ in Eq. (42), along with the
595 scale for stress shows that the von Mises criterion in this context would give a calving rate proportional to H^5 for $n = 3$, which
is close to our result for low tensile strength of $H^{5.7}$. In this law it seems likely that the ice speed is playing a role as a proxy
for other variables such as ice thickness.

Other calving models have been proposed in addition to those considered by Choi et al. (2018). One example is the calving
parameterisation advanced by Crawford et al. (2021). This exhibits a power-law dependence of calving rate on ice thickness,
600 with an exponent in the range of 6.0 to 7.3 for $n = 3$. This is within the range of our result of 5.7 to 9.4, despite the fact that
Crawford et al. (2021) consider grounded ice fronts, while we consider floating ice fronts.

We have chosen a specific range of thicknesses to exhibit a particular kind of calving. For this case of an unbuttressed ice
shelf with infinite out-of-plane dimension, and over the tensile strengths examined, when the thickness is greater than 550 m
at $t = 0$ there is instant failure of the ice shelf through the nucleation of basal crevasses. Nevertheless, the calving shown here,
605 of thin icebergs calving from the front of an ice shelf, is observed in nature (Scambos et al., 2008).

Unlike Hageman (2026), our model is not currently capable of representing mode-II (shear) fractures, which have been
hypothesised as a mechanism for ice cliff failure (Bassis and Walker, 2012; Schlemm and Levermann, 2019). However Hage-
man (2026) find that mode-II fracture only occurs under very limited conditions, for an ice cliff with a frozen bed, which is a
different problem setup from the freely floating ice shelf. Furthermore, Hageman (2026) do not include the presence of water

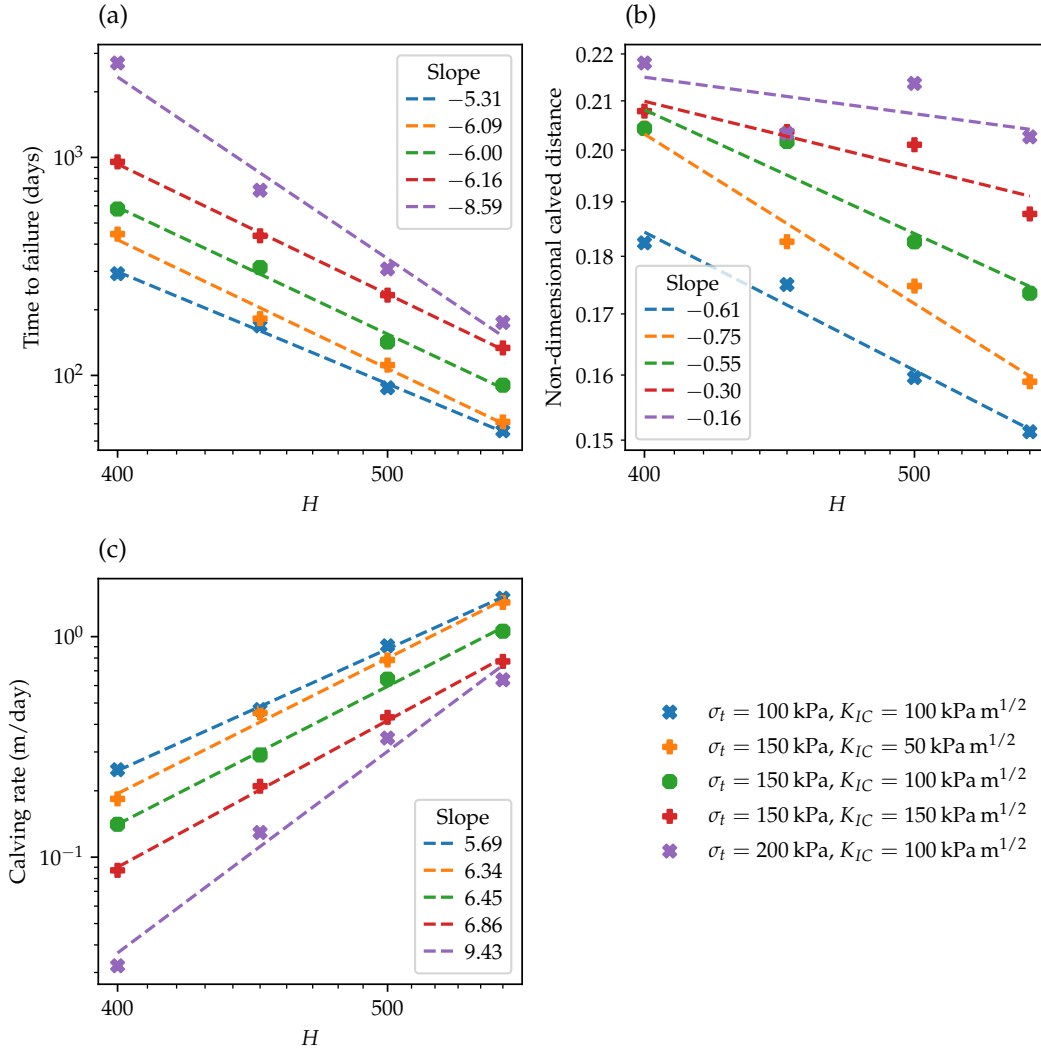


Figure 8. Results for (a) time to failure, (b) non-dimensional calving distance from the front of the ice shelf and (c) calving rate for the thin edge calving event shown in Fig. 7, across varying thickness H , and also for varying tensile strength σ_t and fracture toughness K_{IC} . The model was run with $n = 3$, $L/H = 5$, a regularisation length $\ell^* = 0.016$, non-dimensional timestep $\Delta t^* = 10$, and mesh size $h = \ell^*/2$.

610 pressure in cracks, which provides a leading-order increase in tension, especially as the crevasse propagates deeper into the ice shelf. Consequently, we believe mode-II fracture can be safely neglected for this case.



5 Conclusions

The model presented here simulates the material behaviour of ice from first principles of energy conservation, mass conservation and momentum conservation. By representing the lower-dimensional crack network with a regularised phase-field that can be evaluated on the computational mesh, a standard finite-element approach can be used, even for complicated crack networks. We have demonstrated that the system of equations presented here can be used to model an ice shelf at flotation, capturing both the long timescale viscoelastic deformation as well as the short timescale nucleation and propagation of fractures leading to calving. In particular, we find it is important to ensure that the total mass of the system which drives the deformation remains conserved. By including the effect of pore pressure in cracks inside the free-energy split, we are able to represent mode-I fractures driven by pore pressure even in highly compressive regimes, exemplified by basal crevasses in ice shelves.

The model reproduces the commonly observed phenomenon of thin-edge calving: separation of icebergs with a horizontal dimension smaller than the thickness, leaving a newly-exposed vertical ice front. We find that, for this style of calving, the calving rate is approximately proportional to $H^{5.5}$ to $H^{9.5}$ over the parameter range examined, with higher tensile strength leading to higher power law exponents.

Although we have focussed on calving from ice shelves, there is no reason that the same system of equations cannot be applied to other configurations. The influence of firn through increased porosity in the upper layers could be explored, alongside the impact of increased meltwater on calving. A particularly valuable use of the equations that we have derived will be to understand the maximum rate at which calving ice cliffs can retreat in Antarctica (DeConto and Pollard, 2016). This will require two changes to the basal boundary condition: first, to account for the support provided at the lower boundary, which is likely to suppress the influence of bending; and, second, to include the influence of frictional drag at the base.

Having analysed the results from the numerical model, we can conclude that Meier's statement that "iceberg calving is largely a problem in fracture mechanics coupled to ice dynamics" is indeed a useful way to address the problem of calving.

Code availability. The model used to solve the coupled phase-field/poro-viscoelastic equations is available at <https://github.com/dhrichards/kraken> Scripts to reproduce the figures in the paper are available at <https://github.com/dhrichards/Icebergpaperfigures> (Richards, 2026).

Appendix A: Expressing Ulloa as a split

In this appendix we provide the derivation for how Eq. (49) can be rewritten as Eq. (50) Firstly, for clarity we restate Eq. (49), the free energy given by Ulloa et al. (2022):

$$\tilde{\Psi}(\epsilon_e, p_w, d) = g(d)\psi(\epsilon_e) - \frac{p_w^2}{2\tilde{M}(d)} - \tilde{B}(d)p_w \text{tr}(\epsilon_e)$$

Using the expressions for $\tilde{B}(d)$ and $\tilde{M}(d)$ in Eq. (47) and Eq. (48) respectively gives:

$$\tilde{\Psi}(\epsilon_e, p_w, d) = g(d)\psi(\epsilon_e) - \frac{p_w^2}{2M} - Bp_w \text{tr}(\epsilon_e) - (1-g) \left(\frac{(1-B)^2}{2K} p_w^2 + (1-B)p_w \text{tr}(\epsilon_e) \right) \quad (\text{A1})$$



Then $g(d)\psi$ can be rewritten as $\psi - (1 - g)\psi$:

$$\tilde{\Psi}(\epsilon_e, p_w, d) = \psi(\epsilon_e) - \frac{p_w^2}{2M} - Bp_w \text{tr}(\epsilon_e) - (1 - g) \left(\psi(\epsilon_e) + \frac{(1 - B)^2}{2K} p_w^2 + (1 - B)p_w \text{tr}(\epsilon_e) \right) \quad (\text{A2})$$

The first three terms combined are then the original undamaged free energy $\Psi(\epsilon_e, p_w)$ in Eq. (8). The final elastic free energy ψ can be expanded to:

$$645 \quad \tilde{\Psi}(\epsilon_e, p_w, d) = \Psi(\epsilon_e, p_w) - (1 - g) \left(\frac{1}{2} K \text{tr}(\epsilon_e)^2 + \mu \epsilon_e^D : \epsilon_e^D + \frac{(1 - B)^2}{2K} p_w^2 + (1 - B)p_w \text{tr}(\epsilon_e) \right) \quad (\text{A3})$$

and rearranged to give:

$$\tilde{\Psi}(\epsilon_e, p_w, d) = \Psi(\epsilon_e, p_w) - (1 - g) \left(\frac{1}{2} K \left(\text{tr}(\epsilon_e) + (1 - B) \frac{p_w}{K} \right)^2 + \mu \epsilon_e^D : \epsilon_e^D \right) \quad (\text{A4})$$

and

$$\tilde{\Psi}(\epsilon_e, p_w, d) = \Psi(\epsilon_e, p_w) - (1 - g) \left(\frac{1}{2} K \text{tr} \left(\epsilon_e + (1 - B) \frac{p_w \mathbf{I}}{3K} \right)^2 + \mu \epsilon_e^D : \epsilon_e^D \right) \quad (\text{A5})$$

650 where the term inside the trace is the modified strain of Eq. (51). Noting that $\hat{\epsilon}_e^D = \epsilon_e^D$ we then have at:

$$\tilde{\Psi}(\epsilon_e, p_w, d) = \Psi(\epsilon_e, p_w) - (1 - g)\psi(\hat{\epsilon}_e) \quad (\text{A6})$$

and finally arrive at Eq. (50):

$$\begin{aligned} \tilde{\Psi}(\epsilon_e, p_w, d) &= g(d)\Psi_+(\epsilon_e, p_w) + \Psi_-(\epsilon_e, p_w) \\ \Psi_-(\epsilon_e, p_w) &\equiv \Psi(\epsilon_e, p_w) - \Psi_+(\epsilon_e, p_w) \\ \Psi_+(\epsilon_e, p_w) &= \psi(\hat{\epsilon}_e) = \frac{1}{2} K \text{tr}(\hat{\epsilon}_e)^2 + \mu \hat{\epsilon}_e^D : \hat{\epsilon}_e^D \end{aligned}$$

Appendix B: Analytical solution for infinite ice shelf

655 Here we detail the analytical solution for an infinitely long ice shelf of uniform thickness under the Maxwell assumption for viscoelasticity, i.e. $\sigma_e^D = \sigma_v^D$. We aim to derive analytical expressions for the total and viscous displacements starting from the general momentum equation in terms of the deviatoric stress and pressure:

$$\begin{aligned} \frac{\partial \sigma_{xx}^D}{\partial x} + \frac{\partial \sigma_{xy}^D}{\partial y} + \frac{\partial \sigma_{xz}^D}{\partial z} - \frac{\partial p}{\partial x} &= 0 \\ \frac{\partial \sigma_{xy}^D}{\partial x} + \frac{\partial \sigma_{yy}^D}{\partial y} + \frac{\partial \sigma_{yz}^D}{\partial z} - \frac{\partial p}{\partial y} &= 0 \\ \frac{\partial \sigma_{xz}^D}{\partial x} + \frac{\partial \sigma_{yz}^D}{\partial y} + \frac{\partial \sigma_{zz}^D}{\partial z} - \frac{\partial p}{\partial z} &= -\rho_i g \end{aligned} \quad (\text{B1})$$



Using Glens flow law $\sigma^D = 2\eta(\dot{\epsilon}_v)\dot{\epsilon}_v$ and assuming that there is no viscous flow in the y direction, and that there is no vertical
660 variation in the horizontal flow so $\dot{\epsilon}_{vzz} = 0$ simplifies Eq. (B1) to:

$$\begin{aligned}\frac{\partial \sigma_{xx}^D}{\partial x} - \frac{\partial p}{\partial x} &= 0 \\ -\frac{\partial p}{\partial y} &= 0 \\ \frac{\partial \sigma_{zz}^D}{\partial z} - \frac{\partial p}{\partial z} &= -\rho_i g\end{aligned}\tag{B2}$$

To find an expression for the stress, we integrate the third equation above with respect to z to give:

$$\sigma_{zz}^D - p = \rho_i g(z - H).\tag{B3}$$

We then note that $\sigma_{zz}^D = -\sigma_{xx}^D$ due to the deviatoric definition and also note that, as $\partial u_{vx}/\partial x$ is invariant with z , $\sigma_{zz}^D = -\sigma_{xx}^D$
665 is also. Therefore we can integrate the above equation with respect to z again to give:

$$-\sigma_{xx}^D H - \int_0^H p \, dz = \int_0^H \rho_i g(z - H) \, dz = \frac{\rho_i g H^2}{2}\tag{B4}$$

To eliminate the pressure integral from the equation above, we consider a balance of forces between the water pressure at the
end of the ice shelf and the horizontal stress:

$$\int_0^H \sigma_{xx} \, dz = F_w = \int_0^H p_w \, dz\tag{B5}$$

670 Substituting $\sigma_{xx} = \sigma_{xx}^D - p$ and using Eq. (3) for the water pressure:

$$\sigma_{xx}^D H - \int_0^H p \, dz = \frac{\rho_i g H^2}{2}(1 - \delta)\tag{B6}$$

Subtracting Eq. (B6) from Eq. (B4) gives:

$$\sigma_{xx}^D = \delta \rho_i g H / 4 = -\sigma_{zz}^D\tag{B7}$$

From this and Eq. (B3) the pressure can be found:

$$675 \quad p = -\delta \rho_i g H / 4 - \rho_i g(z - H)\tag{B8}$$

The stress tensor is then:

$$\sigma_v = \rho_i g \begin{bmatrix} \delta H / 2 + (z - H) & & \\ & \delta H / 4 + (z - H) & \\ & & z - H \end{bmatrix}\tag{B9}$$



Using the Maxwell assumption $\sigma_v = \sigma_e$ gives a set of simultaneous equations:

$$\sigma_v = \sigma_e = \lambda \text{tr}(\epsilon_e) + 2\mu\epsilon_e \quad (\text{B10})$$

680 which can be solved to give the diagonal elastic strains $\epsilon_{exx}, \epsilon_{eyy}, \epsilon_{ezz}$:

$$\epsilon_{exx} = \rho_i g (-8z\nu + 4z - \delta H\nu + 2\delta H + 8H\nu - 4H) / (8\mu(\nu + 1)) \quad (\text{B11})$$

$$\epsilon_{ezz} = \rho_i g (-8z\nu + 4z - 3\delta H\nu + 8H\nu - 4H) / (8\mu(\nu + 1)) \quad (\text{B12})$$

$$\epsilon_{eyy} = (\epsilon_{exx} + \epsilon_{ezz}) / 2 \quad (\text{B13})$$

These strains can then be used to calculate analytical expressions for different forms of the free energy in Fig. 4.

685 We now seek to calculate the displacements in the x -direction u_x, u_{vx} used as a boundary condition in section 4.1. Assuming that at $x = 0$ the displacements are 0, integrating Eq. (B11) in the x direction gives:

$$u_{ex} = x \rho_i g (-8z\nu + 4z - \delta H\nu + 2\delta H + 8H\nu - 4H) / (8\mu(\nu + 1)) \quad (\text{B14})$$

Where u_e is the elastic displacement. Equation (B7) can also be rewritten using Glens law:

$$\sigma_{xx}^D = 2\eta \frac{\partial \dot{u}_{vx}}{\partial x} = A^{-1} \left(\frac{1}{2} \sigma^D : \sigma^D \right)^{\frac{1-n}{2}} \frac{\partial \dot{u}_{vx}}{\partial x} = \delta \rho_i g H / 4 \quad (\text{B15})$$

690 which simplifies to:

$$\frac{\partial \dot{u}_{vx}}{\partial x} = A \left(\frac{\delta \rho_i g H}{4} \right)^n \quad (\text{B16})$$

and this can be integrated with respect to x to give the viscous velocity in x :

$$\dot{u}_{vx} = x A \left(\frac{\delta \rho_i g H}{4} \right)^n \quad (\text{B17})$$

integrating this equation in time gives:

$$695 \quad u_{vx} = x A \left(\frac{\delta \rho_i g H}{4} \right)^n t + C(x, z) \quad (\text{B18})$$

the total displacement is then:

$$u_x = u_{vx} + u_{ex} = x A \left(\frac{\delta \rho_i g H}{4} \right)^n t + C(x, z) - u_{ex} \quad (\text{B19})$$

we assume that at $t = 0$ the total displacement is 0, hence:

$$C(x, z) = -u_{ex} = -x \rho_i g (-8z\nu + 4z - \delta H\nu + 2\delta H + 8H\nu - 4H) / (8\mu(\nu + 1)) \quad (\text{B20})$$

700 and hence the total displacement in x is:

$$u_x = x A \left(\frac{\delta \rho_i g H}{4} \right)^n t \quad (\text{B21})$$

This is then implemented as a boundary condition at $x = L$ for the results in section 4.1.



Appendix C: Sensitivity to numerical parameters

Here we examine the sensitivity of the observed calving rate to changes in numerical parameters in the model, as well as changes in non-dimensional length. Fig C1 shows this variance. There is some small reduction in calving rate and slight increase in power law exponent for smaller regularisation lengths, but the overall dependence and power law exponent is broadly unchanged.

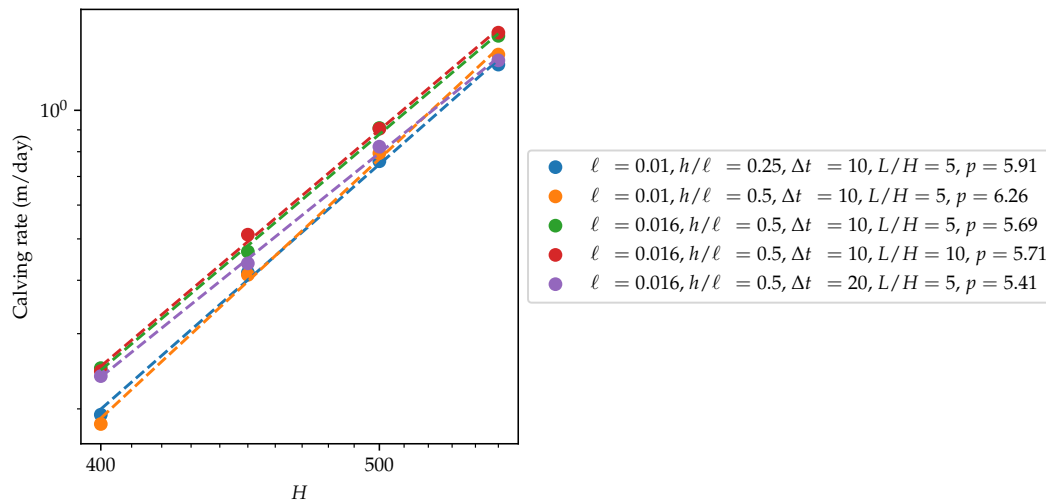


Figure C1. Sensitivity of calving rate parametrization for different regularisation lengths ℓ^* , mesh size h , timestep Δt^* and domain length L/H . The simulations are all performed with $\sigma_t = 100$ KPa and $K_{IC} = 100$ KPa m^{0.5}. The fitted power law exponent is also shown as p .

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Oliver J. Marsh: conceptualisation, methodology, writing (review & editing), and supervision.

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