We thank both reviewers for their helpful suggestions on our manuscript: "**A simple dynamical system for representing climate tipping points with hysteresis**".

The original requests are in black font. Our responses are described in indented blue font, and where revised text is quoted, this is in italics.

From Reviewer #1

1. *General feedback.*

The manuscript presents a valuable and computationally efficient framework for representing complex climate phenomena through low-dimensional models. Its primary strength lies in the detailed algebraic derivation, which serves as a "manual" for researchers to map Earth System Model (ESM) outputs onto a parameter-sparse dynamical system. This approach is particularly relevant for analyzing "overshoot" scenarios and inter-model differences. However, the current version contains several technical inconsistencies and typographical errors that should be addressed before final publication:

> Thank you, Reviewer #1, for taking the time to assess our paper. We are pleased that the reviewer acknowledges the manuscript's strength, namely its ability to help ESM developers and users map their diagnostics onto a simplified, parameter-sparse dynamical system. We recognise the technical inconsistencies and typographical errors identified, and below we set out how we have resolved them.
>
> During this revision, we also took the opportunity to review all points of clarity and have consequently made a small number of minor wording changes to improve clarity.

2. *Comments about the structure.*

The manuscript provides a robust and highly valuable mathematical framework. However, considering the target journal's broader climate science readership (rather than a specialized mathematics journal), the current structure is excessively dense. The extensive step-by-step algebraic derivations, while excellent for reproducibility, significantly hinder the narrative flow and may alienate readers interested in the physical implications rather than the calculus. We strongly suggest a major restructuring: - Main Text: Should focus on the conceptual model, the physical meaning of the parameters, and the discussion of the results. - Supplementary Material: Move the detailed algebraic proofs to a dedicated Appendix or Supplementary Information file.

> We have implemented the suggested restructuring. The paper now includes an extensive Appendix with three sections. These cover (1) the background algebraic derivations that enable a user to map their system of interest onto the dynamical system, (2) the equations to be solved to ensure the future warming "overshoot" trajectory extends smoothly from the historical warming record and (3) conditions that allow a short overshoot that avoids a change of state and hysteresis.

> The main paper now retains only the key equations the user needs for their own application to an Earth system component of interest. The extensive underlying mathematical derivations are now all in the Appendix. Please see the "track changes" (i.e., difftex) document, which illustrates how the equations are now split between the two parts of the paper.
>
> We really appreciate this suggestion, and it has made the main paper far smoother and more focused for the reader.

3. *Comments about mathematics*.

The mathematical framework is robust, but its presentation is inappropriate for a general climate science audience. The manuscript includes exhaustive, step-by-step algebraic derivations that overwhelm the narrative. We strongly recommend moving these derivations to a Supplementary Information document. –

> Please see the response above. Most background algebra is now in the Appendix. Only the key equations a user will need are retained in the main body of the manuscript. Three key equations are now shown more clearly (in boxes), namely (1) the mapping from parameters that define a system to the main dynamical system equation, (2) features of a future overshoot forcing scenario and (3) an inequality that determines whether a state change will occur, along with hysteresis.

Page 8 please indicate the implicit solver used.

> The text now states: "*A simple explicit solver is used…with a timestep of 0.1 years*".

Page 9 please indicate the smooth algorithm.

> We now write in the paper: "*Also included in that timeseries data is a version that is smoothed in time (5-year lowess smoothing method; for details, see NASA-GISS (2025)), which forms the basis for the historical values we employ*"
>
> We also aim to have a continuous, smooth, and identical gradient in the rate of global warming in the last years of the historical record, as at the start of the future quadratic form. The text states: "*We take the gradient as….. the change between the year 2014 and 2024, which we divide by ten*".

Figure 2 please use a colorbar indicating the velocity in the arrows.

> We have considered this suggestion and the alternative representation in which the arrow lengths are proportional to the pseudo-equilibrium solution velocities. We prefer the latter, as it keeps the diagram in a similar format to Figure 1. We therefore respectfully request that the velocities be presented in this way.
>
> To make the presentation absolutely clear, however, we now add to the paper: "*In Fig. 2, each arrow corresponds to a solution, with the beginning of the arrow representing the starting conditions and the end of the arrow indicating the final value derived*

> *after the simulated 15-year period. Hence, arrow length is representative of solution "velocity" "*

4. *Grammatical errors*

"Hystersis": On Page 6 (line 6), the word is misspelled as "hystersis"

> Thanks. Typo corrected.

"Pre-industrial" Inconsistencies

> We now use that phrase "*pre-industrial*" at all mentions.

5. *Summary*

The manuscript provides a valuable and elegant mathematical framework for simplifying climate tipping points and hysteresis. However, in its current state, it is too mathematically dense for a non-mathematical journal. The paper reads more like a derivation manual than a research article. I recommend accepting only after a Major Revision that focuses on restructuring the content and fixing significant typographical and rendering errors

> We are pleased that the manuscript is regarded as valuable and elegant.
>
> The paper has been completely restructured, with the majority of the algebraic derivations now placed in the three-part Appendix. This version has been carefully scanned to eliminate any final "typos", and there is now a consistent (rendered) format throughout.
>
> The main body of the paper retains only the equations that a user will implement directly. We are grateful for the encouragement to do this and acknowledge that the main part of the paper now has much greater focus. Thank you.

From Reviewer #2

The authors present a study of a simplified dynamical system displaying tipping points, with the claim that it can be calibrated to represent some aspects of the behaviour of more complex Earth Systems Models under global warming. The explicit aim is to include effects of 'inertia' and 'hysteresis'. Only one of the hysteresis branches is explored (the return path to the base state is not considered).

> We thank Reviewer #2 for their assessment of our manuscript.
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> Regarding the return path, and based on this comment, we now add to the paper "*In both these cases, the solution will tip back to the starting branch (given sufficient time) since the global warming drops below the lower fold bifurcation, $\Delta T_{HYS} = 0.4K$*".

The paper is written more as the lecture notes of an elementary course on dynamics and bifurcations than as a research paper. The main result is in Fig. 4: fast crossing of a tipping point and return back of a control parameter does not trigger a jump in the system state, in contrast with the behavior for a slow crossing. This result is very well known. In fact, the authors check that the time scales agree with the result of Ritchie et al 2019, thus confirming that the behaviour has been studied in detail before. Thus, the only part that can be useful for the community is the parametrization of this simplified model in terms of quantities that in principle can be measured from a larger climate model. The manipulations leading to this parametrization are elementary, and do not deserve the many pages devoted to it. But there are more important caveats:

> We agree that parts may read more like a textbook. However, when searching for a mathematical framework to analyse climate tipping and hysteresis, we found no suitable options. Yet there remains a need to employ dynamical systems to characterise tipping behaviour in complex ESMs, as the reviewer notes. The algebra required to map onto dynamical systems proved nontrivial. Given the importance of understanding climate tipping points, we feel the paper will make a valuable contribution to the peer-reviewed literature.
>
> Ritchie *et al*. (2019) do illustrate that, with large inertia, temporary overshoot may occur without triggering key tipping points. What we have is a framework for understanding that further, within an algebraic framework that can entrain directly parameters specific to the system considered.
>
> We have directly addressed the reviewer's concern that the "manipulations…do not deserve the many pages". Also guided by Reviewer 1, the paper now includes a three-part Appendix containing almost all of the algebraic derivations. The main body of the paper is now much more focused and usable, retaining only the main equations.

The manuscript model, Eq. (1) is just the normal form of the codimension-2 cusp bifurcation. It can be argued that any dynamical system with only fold bifurcations can be generically brought to that form. But in general the change in variables needed for that is nonlinear, not the simple linear scaling used here for mu and X. The consequence is that the basins of attraction of the upper and lower stable-solution branches would be quite asymmetric, an important fact neglected in the present approach.

> Thank you for this. We now add to the Discussion the following text: "*Our analysis has some caveats*" and then "*In addition, utilising the normal form and a linear change in variables assumes a certain symmetry to the problem, for example as seen in Figure 1. At the midpoint between the folds of* ΔT_{HYS} *and* ΔT_{TP} *the distances to the basin boundary (the middle unstable branch), from the stable top and lower branches, are the same. However, some problems might require a nonlinear coordinate change, which could lead to asymmetric basins of attraction*."

Only fold bifurcations are taken into account. This excludes possibilities already observed in existing climatic models, such as oscillatory instabilities, homoclinic bifurcations, crisis and other global bifurcations, etc.

> Thank you also for this comment. We now write in the Discussion: "*In our modelling framework, we only consider the passage through fold bifurcations. However, tipping can arise from oscillatory instabilities, which are often attributed to passing through a*

> *Hopf bifurcation. Or, alternatively, tipping can even occur without crossing any bifurcations via rate-induced tipping, such as those attributed to peatland fires (Clarke et al., 2021)*"

Anyway, I would accept that all the above criticisms would become irrelevant if the authors would be able to demonstrate their main claim: that this approach allows to map different complex climate models into a single framework, thus allowing comparison among them. The authors have not demonstrated this, and it seems difficult to do since, as recognized by the authors, most existing studies only consider short simulations not always exploring beyond a bifurcation point. I invite the authors to really follow their proposal and compare outputs from different complex models to show really that their approach is powerful (and in this case the straightforwardness of the calculations will be a welcomed feature). Until they do something like this I do not find the manuscript relevant to the community and I can not recommend the paper for publication.

> Quantifying ESM behaviours and their differences is important, and fortunately, new ESM calculations for "overshoot" forcings are gradually becoming available. Hence, we have added text to the Discussion as: "*We note that some climate research centres are starting to force ESMs with overshoot-type scenarios and any simulated hysteresis effects may be amenable to mapping onto our dynamical system framework*".
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> However, such ESM-based calculations remain comparatively rare, and once available, extracting diagnostics across them and for multiple tipping elements with potential hysteresis will be a substantial research project. Our aim is to provide a traceable structure for their comparison at that point, and to regard our analysis as preparatory work for such a project.
>
> In developing the parameterisation of the dynamical system, we were unable to find the required algebra in any source. Furthermore, when creating the mappings ourselves, we found this task highly non-trivial.
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> We hope that the substantial reworking, by separating the algebraic derivations from the key governing equations, is seen as helping to make the main body of the paper more approachable and usable.