We thank both reviewers for their helpful suggestions on our manuscript: "**A simple dynamical system for representing climate tipping points with hysteresis**".

The original requests are in black font. Our proposed responses in the indented blue font.

From Reviewer #1

1. *General feedback.*

The manuscript presents a valuable and computationally efficient framework for representing complex climate phenomena through low-dimensional models. Its primary strength lies in the detailed algebraic derivation, which serves as a "manual" for researchers to map Earth System Model (ESM) outputs onto a parameter-sparse dynamical system. This approach is particularly relevant for analyzing "overshoot" scenarios and inter-model differences. However, the current version contains several technical inconsistencies and typographical errors that should be addressed before final publication:

> Thank you, Reviewer #1, for taking the time to assess our paper. We are pleased that the reviewer acknowledges the manuscript's strength, which is indeed to help ESM developers and users to map their diagnostics onto a simplified parameter-sparse dynamical system. We recognise the technical inconsistencies and typographical errors identified, and below we set out how we will resolve them.

2. *Comments about the structure.*

The manuscript provides a robust and highly valuable mathematical framework. However, considering the target journal's broader climate science readership (rather than a specialized mathematics journal), the current structure is excessively dense. The extensive step-by-step algebraic derivations, while excellent for reproducibility, significantly hinder the narrative flow and may alienate readers interested in the physical implications rather than the calculus. We strongly suggest a major restructuring: - Main Text: Should focus on the conceptual model, the physical meaning of the parameters, and the discussion of the results. - Supplementary Material: Move the detailed algebraic proofs to a dedicated Appendix or Supplementary Information file.

> Although this will require quite a substantial restructuring, we are prepared to undertake this. In the main text, we will retain the two main equations that are currently boxed, namely (1) the dynamical system mapped onto the specifics of the tipping point, and (2) the global warming forcing within the same framework. We will also keep the equations that link the dynamical system parameters to the system of interest. All other equations that form the derivations of (1) and (2), we will move to a fully cross-cited Appendix.

3. *Comments about mathematics.*

The mathematical framework is robust, but its presentation is inappropriate for a general climate science audience. The manuscript includes exhaustive, step-by-step algebraic derivations that overwhelm the narrative. We strongly recommend moving these derivations to a Supplementary Information document. –

> Yes, this is a major change, but we are certainly prepared to do this. As mentioned in the response above, our preference is for a new Appendix, rather than an SI.

Page 8 please indicate the implicit solver used.

Page 9 please indicate the smooth algorithm.

Figure 2 please use a colorbar indicating the velocity in the arrows.

> We will amend the manuscript to describe the solver and smoothing algorithm, and amend Figure 2 as suggested to provide clearer visual guidance of the pseudo-equilibrium solution velocities.

4. *Grammatical errors*

"Hystersis": On Page 6 (line 6), the word is misspelled as "hystersis"

"Pre-industrial" Inconsistencies

> We will correct these “typos”.

5. *Summary*

The manuscript provides a valuable and elegant mathematical framework for simplifying climate tipping points and hysteresis. However, in its current state, it is too mathematically dense for a non-mathematical journal. The paper reads more like a derivation manual than a research article. I recommend accepting only after a Major Revision that focuses on restructuring the content and fixing significant typographical and rendering errors

> We are pleased that the manuscript is regarded as valuable. We agree that parts of the paper may initially appear more like a book or manual. However, when searching for this particular mathematical construct to support the analysis of tipping and hysteresis, we were unable to find it online, in other research papers, or in textbooks. Yet there remains a continuing need to use dynamical systems to characterise tipping behaviour in complex ESMs, or in data such as paleo measurements. Hence, the algebra required to map onto such dynamical systems needs to be made available. Arguably, determining the risk of tipping remains the most important requirement for climate change research.
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> To place the algebra in the literature and in journal format, we are happy to adopt the Major Revision suggestion to move the algebraic derivations into an Appendix. The equations retained in the main body of the paper will be strictly those that other researchers can deploy directly to link climate-related time series to our underlying dynamical system.

From Reviewer #2

The authors present a study of a simplified dynamical system displaying tipping points, with the claim that it can be calibrated to represent some aspects of the behaviour of more complex Earth Systems Models under global warming. The explicit aim is to include effects of 'inertia' and 'hysteresis'. Only one of the hysteresis branches is explored (the return path to the base state is not considered).

> We thank Reviewer #2 for their assessment of our manuscript. Regarding the specific point about the return path, we will clarify in the text that, with sufficient time (and once the forcing falls below the level at which two pseudo-equilibrium solutions are possible), the solution will return to its base state. This applies to the orange and red curves.

The paper is written more as the lecture notes of an elementary course on dynamics and bifurcations than as a research paper. The main result is in Fig. 4: fast crossing of a tipping point and return back of a control parameter does not trigger a jump in the system state, in contrast with the behavior for a slow crossing. This result is very well known. In fact, the authors check that the time scales agree with the result of Ritchie et al 2019, thus confirming that the behaviour has been studied in detail before. Thus, the only part that can be useful for the community is the parametrization of this simplified model in terms of quantities that in principle can be measured from a larger climate model. The manipulations leading to this parametrization are elementary, and do not deserve the many pages devoted to it. But there are more important caveats:

> As there is a real chance that key global warming thresholds (such as 1.5 or 2.0°C) will be exceeded, interest in "overshoot" scenarios is rapidly growing. It is correct that Ritchie et al. (2019) illustrate that, with large inertia, temporary overshoot may be possible without triggering tipping points that, at a long-term fixed higher temperature, would otherwise jump to new states. For this reason, we have Section 2.6, which, as the reviewer notes, links that idealised analysis to our algebraic framework.
>
> Fortunately, to investigate overshoot, many Earth System Models (ESM) are now being operated for radiative forcings that first increase and then decrease. Such model-based data are slowly becoming available. As preparation for this, we wanted to establish a routine method to map such ESM outputs onto a dynamical system. As the reviewer will know, of course, once realised, a dynamical system can be used to bring more rigour to comparing ESMs, and to facilitate scanning across a much broader range of potential future forcing scenarios.
>
> We carefully checked for the algebra needed to enable this mapping, but we could not find it online, in textbooks, or in other research papers. In our view, the calculations are certainly not trivial. Given the importance of understanding climate tipping points, we believe the mathematics presented will make a valuable standalone paper in the peer-reviewed literature.

> We are, however, very willing to undertake a major rewrite of the paper, presenting the algebraic derivations in an Appendix and keeping the main equations that link attributes of any time series to the underlying dynamical system in the main text. Please also see our response to Reviewer #1.

The manuscript model, Eq. (1) is just the normal form of the codimension-2 cusp bifurcation. It can be argued that any dynamical system with only fold bifurcations can be generically brought to that form. But in general the change in variables needed for that is nonlinear, not the simple linear scaling used here for mu and X. The consequence is that the basins of attraction of the upper and lower stable-solution branches would be quite asymmetric, an important fact neglected in the present approach.

> We will add to the Discussion a sentence noting that, for the "cubic" equation form we use, this implies that the two stable-solution branches will have similar tipping points at both the start and the end of the hysteresis range. We will write in the Discussion that in general, the tipping dynamics need not be the same for both folds.

Only fold bifurcations are taken into account. This excludes possibilities already observed in existing climatic models, such as oscillatory instabilities, homoclinic bifurcations, crisis and other global bifurcations, etc.

> We will carefully note this in the Discussion part of the manuscript. We will state the possibility that in other circumstances, components of the climate system may be such that a Hopf bifurcation could be crossed, which may give rise to oscillatory instabilities.

Anyway, I would accept that all the above criticisms would become irrelevant if the authors would be able to demonstrate their main claim: that this approach allows to map different complex climate models into a single framework, thus allowing comparison among them. The authors have not demonstrated this, and it seems difficult to do since, as recognized by the authors, most existing studies only consider short simulations not always exploring beyond a bifurcation point. I invite the authors to really follow their proposal and compare outputs from different complex models to show really that their approach is powerful (and in this case the straightforwardness of the calculations will be a welcomed feature). Until they do something like this I do not find the manuscript relevant to the community and I can not recommend the paper for publication.

> Quantifying ESM behaviours and their differences is important, and fortunately, new ESM calculations are slowly becoming available for "overshoot" forcings. However, comparing them will be a substantial research project across a broad range of climate characteristics that may exhibit tipping and potentially hysteresis.
>
> In the meantime, we believe there is room to develop dynamical systems to be ready to quantify climate behaviours as ESM-derived data become available. In preparing for this, we were unable to find the necessary algebra in any source. Furthermore, in creating the mappings ourselves, we found this to be non-trivial. We certainly respect the reviewer's view, but we found the algebra sufficiently complex and likely to be

useful to many that we feel it justifies being a contribution to the climate change literature.

As mentioned above, we are prepared to rewrite the manuscript to make it clearer. This Major Revision will retain in the main paper only the equations that a researcher requires to link any timeseries to a parameterised dynamical system. The underpinning algebra leading to these equations will be moved to a new Appendix.