



# **A note on solution of ZK-Burger equation by Rossby-Khantadze waves**

**Laila Z Kahlon<sup>1\*</sup>, Hassan A Shah<sup>1</sup>,**

**<sup>1</sup>Physics Department, Forman Christian College (A Chartered University), Ferozepur**

**Road, Lahore 54600, Pakistan**

**\*E-mail: [lailakahlon@fccollege.edu.pk](mailto:lailakahlon@fccollege.edu.pk) and [lailakahlon192@gmail.com](mailto:lailakahlon192@gmail.com)**

## **Abstract**

In the present manuscript the investigation of Rossby-Khantadze equation in a dissipative ionospheric plasma is carried out. By using the tangent hyperbolic (tanh) approach, a damping solitary type solution is found for the ZK-Burger equation. Additionally, the dependency of obtained solitary structures on different ionospheric parameters namely, temporal scales and dissipation is investigated.

## **1. Introduction**

A common phenomenon in planetary atmospheres and laboratory plasmas is that of zonal flows (Gills, 1982; Petviashvili, 1992). In planetary atmospheres, the presence of zonal flows is verified by terrestrial and satellite observations. The velocities of these flows are varying vertically (along the meridians), linked by planetary-scale perturbations of ultra-low-frequency (Gershman, 1973). The nonlinear interactions occur as a result of interaction of zonal flows with planetary Rossby waves. Such sheared flow can affect on the properties of waves and can formed many nonlinear structures solitary waves, vortices and zonal flows etc.

Rossby waves have a special importance due to their significant role in atmospheric circulation with the inclusion of sheared zonal flow. In the ionosphere, Rossby waves may couple to the Khantadze waves (Khintadze, 2008) and the dynamics of such coupling of a neutral fluid and the magnetized ionized component are investigated in the work of Kaladze et al. 2004; Kaladze & Horton, 2006. In a recent work of Futatani et al. 2013, it is demonstrated that in the Hall conductive Earth ionospheric E-region, the self-organization of Rossby-Khintadze electromagnetic (EM) waves leads to localized dipole structures of solitary type waves, which propagate longitudinally and are driven by background mean flows. In Kaladze et al. (2014), the analytic solutions for the vortex streets are obtained; for partially ionized plasmas. In the work of Fagundes et al. (2005), the dependency of such waves on charged particles's number density in F-region was established.

The equatorial sheared Rossby wave have soliton solutions which have been confirmed by observations obtained from the Freja and Viking satellites, (Qiang et al. 2001; Bostrom, 1992; Dovner et al. 1994; Lindqvist, 1994). A non-linear modified Korteweg-de Vries (mKdV)



35 by sheared Rossby waves propagating in stratified neutral fluids was found by Jian et al. <sup>[14]</sup>.  
36 In the following papers Kaladze, 2012; Kahlon & Kaladze (2015), the generation of intense  
37 magnetic field ( $\sim 10^3$  nT) accompanied by zonal flow generation by Rossby–Khantadze EM  
38 was made in the ionospheric E-region plasma. Kaladze et al. (2019) examined the nonlinear  
39 interaction of magnetized Rossby waves with sheared zonal flows in the Earth’s ionospheric E  
40 region and derived a modified Korteweg–de Vries (mKdV) equation incorporating cubic non-  
41 linearity to describe the propagation of solitons. Kahlon et al. (2024) investigated a modified  
42 KdV (MKdV) for the Rossby-Khantadze (RK) waves in ionospheric E region describing  
43 solitary solitons. Forbes (1996) discussed the presence of planetary Rossby waves in the  
44 ionospheric dynamo E region and provided theoretical interpretations for associated  
45 experiments. Additionally, Vukcevic and Popovic (2020) gave the idea of formation of  
46 multiple solitons in diverse ionospheric layers with different latitudes using direct ground-  
47 based and satellites observations.

48 Since the ionosphere is partially ionized, therefore dissipation effects are important. We  
49 formulate the Zakharov-Kadomtsev-Burger (ZKB) equation for the Rossby-Khantadze (RK)  
50 waves in a dissipative ionospheric plasma. The nonlinear analysis of the dissipative Rossby-  
51 Khantadze (DRK) equation is carried out both analytically and numerically. We note that such  
52 an investigation has not been carried out thus far. In Sec. 2, the basic mathematical formulation  
53 of the problem is presented and linear dispersion is found. In the Sec. 3, by using reductive  
54 perturbation approach, the Zakharov-Kuznetsov-Burger (ZKB) equation for Rossby-  
55 Khantadze is derived which in the absence of inhomogeneity in magnetic field gives the  
56 Zakharov-Kuznetsov-Burger (ZKB) equation for Rossby Waves (Yin et al. 2019). In the Sec.  
57 4, its analytical solutions are obtained. Sec. 5 contains results and in Sec. 6 discussion and  
58 concluding remarks are made.

59

## 60 **2. Mathematical Formulation**

61 We consider the propagation of Rossby-Khantadze waves in a weakly ionized  
62 ionospheric E region plasma. The terrestrial ionosphere consists of electrons, ions, and neutrals  
63 and coupling among them can be described by single fluid model made in Song *et al.* (2001)  
64 and related studies (Song et al. 2009; 2005; Luo et al. 2023). The nonlinear Rossby-Khantadze  
65 waves are supposed to propagate in the  $xy$  plane i.e. as  $\mathbf{k} = k_x \mathbf{x} + k_y \mathbf{y}$ . The equilibrium  
66 magnetic field has components in  $y$  and  $z$  directions  $B_0 = (0, B_{0y}, B_{0z})$  and the perturbed field  
67 is assumed to be polarized only in the  $z$  direction,  $B_{1z} = h \mathbf{z}$ .



68 Following Song et.al (2001) we obtain the governing one fluid equation for a weakly  
69 ionized single fluid plasma,

$$70 \quad n_n m_n \frac{D\mathbf{v}_n}{Dt} = -\nabla P + \mathbf{j} \times \mathbf{B} - 2 \boldsymbol{\Omega} \times \mathbf{v} + \mathbf{g} - \mu_0 \nabla^2 \mathbf{v}, \quad (1)$$

71 where  $\frac{D}{Dt}$  is the total derivative  $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)$ , with  $\mathbf{v}$  denotes the fluid velocity of the neutral  
72 gas. The neutral fluid mass density is  $\rho_n = n_n m_n$ .  $P$  is the gas pressure,  $\boldsymbol{\Omega}$  is the Earth's angular  
73 velocity,  $\mu_0$  is the permeability and the symbol  $\mathbf{g}$  represents the gravitational acceleration.

74 The curl of Eq. (1) allows one to express the equation as

$$75 \quad \frac{\partial \zeta}{\partial t} + \beta \frac{\partial \psi}{\partial x} + J(\psi, \zeta) - \frac{\beta_B}{\mu_0 \rho} \frac{\partial h}{\partial x} = -\mu_0 \zeta, \quad (2)$$

76  
77 where  $\psi(x, y, t)$  denotes the stream function. For 2D incompressible motion we assume i.e.  
78  $\mathbf{v} = (u, v, 0)$  where  $u = -\frac{\partial \psi}{\partial y}$ ,  $v = \frac{\partial \psi}{\partial x}$ ,  $\zeta = \nabla^2 \psi$ ,  $\beta_B = \frac{\partial B_{0z}}{\partial y}$ ,  $0 \leq \mu_0 \leq 1$  is the dissipative  
79 coefficient, and  $J(a, b) = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial a}{\partial y} \frac{\partial b}{\partial x}$  is the vector nonlinearity (Jacobian), which is known  
80 as the vector nonlinearity.

81 The generalized Ohm's law is

$$82 \quad \mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{1}{en} (\mathbf{j} \times \mathbf{B}) = \frac{\rho}{en} \mathbf{F}_A \quad (3)$$

83 where  $\mathbf{F}_A = \frac{1}{\rho} \mathbf{j} \times \mathbf{B} = \frac{1}{\rho \mu_0} \nabla \times \mathbf{B} \times \mathbf{B}$ .

84 The Maxwell's equations are

$$85 \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \quad \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \quad (4)$$

86 From Eqs. (1)-(4) we obtain

$$87 \quad \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{en\mu_0} \nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] = 0. \quad (5)$$

88 By considering the z-component of magnetic field induction and further simplification  
89 gives

$$90 \quad \frac{\partial h}{\partial t} + J(\psi, h) + \beta_B \frac{\partial \psi}{\partial x} + c_B \frac{\partial h}{\partial x} = 0, \quad (6)$$

91 where  $h$  is the z-component of the magnetic field perturbation,  $c_B = \frac{\beta_B}{en\mu_0}$  is the speed at which  
92 fast Khantadze waves propagate, and the shear in the magnetic field is expressed  $\beta_B = \frac{\partial B_{0z}}{\partial y}$ .

93 The equations (2) and (6) describe the complete set of the system of nonlinear equations  
94 for sheared Rossby-Khantadze waves in the dissipative partially ionized ionospheric plasma.

95 We use rigid wall boundary conditions, which are described as



$$\psi(0) = \psi(1) = 0 \quad (7)$$

This explains the northern and southern edges of the zonal flow, which can be considered latitude boundaries.

### 3. ZKB equation

To derive the nonlinear ZKB equation, we employ the reductive perturbation approach (Khomenko, 2020; Infeld & Rowlands, 2000). We consider a steady parallel shear flow  $U(y)$  in a Cartesian coordinate  $(x, y)$  system where  $x$  and  $y$  are along the east and the north, respectively. The total stream function  $\Psi(y)$  is taken as

$$\begin{cases} \psi = \Psi(y) + \varepsilon^2 \psi', \\ h = \varepsilon^2 h'. \end{cases} \quad (8)$$

where  $\psi$  represents the sum of the background and disturbed stream functions,  $\Psi(y) = \int [U(y) - c_0] dy$ ,  $\psi'$ . Additionally, the basic flow is  $U(y)$ , and the phase speed of the Rossby waves is denoted by  $c_0$ , which is constant, and the small parameter  $\varepsilon \ll 1$  indicates the strength of perturbation.

Using Eqs. (8)-into above system of equations (2) and (6), we get

$$\begin{cases} \frac{\partial \zeta'}{\partial t} + [U(y) - c_0] \frac{\partial \zeta'}{\partial x} + p(y) \frac{\partial \psi'}{\partial x} - \frac{\beta_B}{\mu_0 \rho} \frac{\partial h'}{\partial x} + \varepsilon^2 J(\psi', \zeta') = -\varepsilon^3 \lambda \zeta', \\ \frac{\partial h'}{\partial x} + \varepsilon^2 J(\psi', h') + [U(y) - c_0] \frac{\partial h'}{\partial x} + \beta_B \frac{\partial \psi'}{\partial x} + c_B \frac{\partial h'}{\partial x} = 0 \end{cases} \quad (9)$$

Where  $\zeta' = \nabla^2 \psi'$ .

We now apply the reductive perturbation technique and define stretched variables as

$$\begin{cases} X = \varepsilon x, \\ Y = \varepsilon y, \\ T = \varepsilon^3 t \end{cases} \quad (10)$$

Taking the perturbation expansion as

$$\begin{cases} \psi' = \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \dots \\ \text{and} \\ h' = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \dots \end{cases} \quad (11)$$



121 Using (10) and (11) into (9) and collecting the lowest order  $\varepsilon^0$ :

122

$$123 \quad \begin{cases} (U - c_0) \frac{\partial}{\partial x} \frac{\partial^2 \psi_0}{\partial y^2} + (\beta - U'') \frac{\partial \psi_0}{\partial x} - \frac{\beta_B}{\mu_0 \rho} \frac{\partial h_0}{\partial x} = 0; \\ (U - c_0) \frac{\partial h_0}{\partial x} + \beta_B \frac{\partial \psi_0}{\partial x} + c_B \frac{\partial h_0}{\partial x} = 0 \end{cases} \quad (12)$$

124

125 which gives

$$126 \quad (U - c_0) \frac{\partial}{\partial x} \frac{\partial^2 \psi_0}{\partial y^2} + p(y) \frac{\partial \psi_0}{\partial x} + \frac{\alpha(y)}{(U - c_0 + c_B)} \frac{\partial \psi_0}{\partial x} = 0. \quad (13)$$

127 where  $\beta - U'' = p(y)$ , and  $\frac{\beta_B^2}{\mu_0 \rho} = \alpha(y)$ .

128 Here  $U''$  is second order derivative with respect to  $y$ .

129 To obtain a solution for Eq.13 we employ a separation of variables as following

$$130 \quad \psi_0 = A(X, Y, T) \phi_0(y) \quad (14)$$

131 Using Eq. (14) into (13) gives

$$132 \quad \phi_0'' + \frac{p(y)}{(U - c_0)} \phi_0 - \frac{\alpha(y)}{(U - c_0)(U - c_0 + c_B)} \phi_0 = 0 \quad (15)$$

133 with

$$134 \quad \phi(0) = \phi(1) = 0$$

135 Eq.(15) constitutes an eigenvalue problem governing the meridional structure of the

136 Rossby-Khantadze waves. By substituting the plane wave solution  $\psi_0 \sim e^{i(k_x x + k_y y - \omega t)}$  we

137 obtain the corresponding dispersion relation for the Rossby-Khantadze waves

$$138 \quad [(\frac{\omega}{k_x} - U(y))k_{\perp}^2 + p(y)][\frac{\omega}{k_x} - U(y) - c_B] - \alpha(y) = 0 \quad (16)$$

139 where  $k_{\perp}^2 = k_x^2 + k_y^2$ .

140 Eq. (16) gives the linear dispersion relation for Rossby-Khantadze waves, due to the inclusion

141 of zonal flow  $U$ . Here the coupling factor  $\alpha = \frac{\beta_B^2}{\mu_0 \rho}$  and if  $\alpha = 0$  we obtain two independent

142 modes i.e. the Rossby wave and Khantadze wave.

143 To obtain the evolution equation for the wave amplitude, we collect next order  $\varepsilon^1$  terms  
144 from Eq. (9)

145 **First order  $\varepsilon^1$ :**

$$146 \quad \begin{cases} (U - c_0) \frac{\partial}{\partial x} \frac{\partial^2 \psi_1}{\partial y^2} + (\beta - U'') \frac{\partial \psi_1}{\partial x} - \frac{\beta_B}{\mu_0 \rho} \frac{\partial h_1}{\partial x} = -2(U - c_0) \frac{\partial}{\partial x} \frac{\partial^2 \psi_0}{\partial y \partial y}, \\ \frac{\partial h_1}{\partial x} = -\frac{\beta_B}{(U - c_0 + c_B)} \frac{\partial \psi_1}{\partial x} \end{cases} \quad (17)$$



147

148 On solving further

$$149 \quad (U - c_0) \frac{\partial}{\partial X} \frac{\partial^2 \psi_1}{\partial y^2} + (\beta - U'') \frac{\partial \psi_1}{\partial x} = -2(U - c_0) \frac{\partial}{\partial X} \frac{\partial^2 \psi_0}{\partial y \partial Y} - \frac{\alpha(y)}{(U - c_0 + c_B)} \frac{\partial \psi_1}{\partial X}. \quad (18)$$

150 Now employing a separation of variables of the form,

$$151 \quad \psi_1 = \tilde{B}(X, Y, T) \phi_1(y)$$

152 The analysis gives

$$153 \quad \psi_1 = A_Y(X, Y, T) \phi_1(y) \quad (19)$$

154 Using this in Eq.(15), we get

$$155 \quad \phi_1'' = -2\phi_0' - \frac{(\beta - U'')}{(U - c_0)} \phi_1 - \frac{\alpha(y)}{(U - c_0)(U - c_0 + c_B)} \phi_1, \quad (20)$$

156 where the boundary conditions  $\phi_1(0) = \phi_1(1) = 0$  must be satisfied.

157 In next  $\varepsilon^2$  order, we obtain the set of equations from Eq. (9) follows as

$$158 \quad \begin{cases} \frac{\partial}{\partial T} \frac{\partial^2 \psi_0}{\partial y^2} + (U - c_0) \frac{\partial}{\partial X} \left\{ \nabla^2 \psi_0 + 2 \frac{\partial^2 \psi_1}{\partial y \partial Y} \right\} + (U - c_0) \frac{\partial}{\partial X} \frac{\partial^2 \psi_2}{\partial y^2} + p(y) \frac{\partial \psi_2}{\partial x} - \\ \frac{\beta_B}{\mu_0 \rho} \frac{\partial h_2}{\partial X} - \lambda \frac{\partial^2 \psi_0}{\partial y^2} + \frac{\partial \psi_0}{\partial X} \frac{\partial}{\partial y} \frac{\partial^2 \psi_0}{\partial y^2} - \frac{\partial \psi_0}{\partial y} \frac{\partial}{\partial X} \frac{\partial^2 \psi_0}{\partial y^2} = 0, \\ \frac{\partial h_0}{\partial T} + \frac{\partial \psi_0}{\partial X} \frac{\partial h_0}{\partial y} - \frac{\partial \psi_0}{\partial y} \frac{\partial h_0}{\partial X} + (U - c_0) \frac{\partial h_2}{\partial X} + \beta_B \frac{\partial \psi_2}{\partial X} + c_B \frac{\partial h_2}{\partial X} = 0 \end{cases} \quad (18)$$

159

160 Using equations 14, 16 and 18b into Eq. (18a), we get

$$161 \quad \frac{\partial}{\partial X} \frac{\partial^2 \psi_2}{\partial y^2} + \frac{p(y)}{(U - c_0)} \frac{\partial \psi_2}{\partial x} - \frac{\alpha(y)}{(U - c_0)(U - c_0 + c_B)} \frac{\partial \psi_2}{\partial X} = \frac{F}{(U - c_0)} = \frac{p(y)}{(U - c_0)^2} \phi_0 \frac{\partial A}{\partial T} + \\ 162 \quad \frac{\alpha(y)}{(U - c_0)^2 (U - c_0 + c_B)} \phi_0 \frac{\partial A}{\partial T} - \phi_0 \frac{\partial^3 A}{\partial X^3} - [\phi_0 + 2\phi_1'] \frac{\partial^3 A}{\partial X \partial Y^2} + \frac{\alpha(y) \phi_0}{(U - c_0)(U - c_0 + c_B)^2} \frac{\partial A}{\partial T} + \lambda A \phi_0'' - \\ 163 \quad \frac{1}{(U - c_0)} A \frac{\partial A}{\partial X} [\phi_0 \phi_0''' - \phi_0' \phi_0''] \quad (19)$$

164

165 Further we let  $F = (U - c_0) \frac{\partial}{\partial X} \frac{\partial^2 \psi_2}{\partial y^2} + p(y) \frac{\partial \psi_2}{\partial x} + \frac{\alpha(y)}{(U - c_0 + c_B)} \frac{\partial \psi_2}{\partial X}$ , and multiply Eq. (19) with

166  $\phi_0$  and integrate with respect to y, the left hand side of Eq. (19) by using orthogonality of the

167 eigen function becomes

$$168 \quad \int_0^1 \phi_0 \frac{F}{(U - c_0)} dy = 0. \quad (20)$$

169 Therefore Eq. (19) takes the form



$$\begin{aligned}
 & \int_0^1 \left[ \frac{p(y)}{(U-c_0)^2} + \frac{\alpha(y)}{(U-c_0)^2(U-c_0+c_B)} + \frac{\alpha(y)}{(U-c_0)(U-c_0+c_B)^2} \right] \phi_0^2 dy \frac{\partial A}{\partial T} - \int_0^1 \frac{\phi_0}{(U-c_0)} [\phi_0 \phi_0''' - \\
 & \phi_0' \phi_0''] dy A \frac{\partial A}{\partial X} - \int_0^1 \phi_0^2 dy \frac{\partial^3 A}{\partial X^3} - \int_0^1 [\phi_0 + 2\phi_1'] \phi_0 dy \frac{\partial^3 A}{\partial X \partial Y^2} - \lambda \int_0^1 \left[ \frac{p(y)}{(U-c_0)^2} + \right. \\
 & \left. \frac{\alpha(y)}{(U-c_0)^2(U-c_0+c_B)} \right] \phi_0^2 dy A = 0. \tag{21}
 \end{aligned}$$

173

174 Further rearranging gives

$$\partial_\tau A + \gamma_1 (A \partial_X A) + \gamma_2 \partial_X^3 (A) + \gamma_3 \partial_X \partial_Y^2 (A) + \gamma_4 A = 0. \tag{22}$$

176 where

$$\left\{ \begin{aligned}
 I &= \int_0^1 \left[ \frac{p(y)}{(U-c_0)^2} + \frac{\alpha(y)}{(U-c_0)^2(U-c_0+c_B)} + \frac{\alpha(y)}{(U-c_0)(U-c_0+c_B)^2} \right] \phi_0^2 dy, \\
 \gamma_1 &= -\frac{1}{I} \int_0^1 \frac{\phi_0}{(U-c_0)} [\phi_0 \phi_0''' - \phi_0' \phi_0''] dy, \\
 \gamma_2 &= -\frac{1}{I} \int_0^1 \phi_0^2 dy, \\
 \gamma_3 &= -\frac{1}{I} \int_0^1 [\phi_0 + 2\phi_1'] \phi_0 dy, \\
 \gamma_4 &= -\frac{\lambda}{I} \int_0^1 \left[ \frac{p(y)}{(U-c_0)^2} + \frac{\alpha(y)}{(U-c_0)^2(U-c_0+c_B)} \right] \phi_0^2 dy.
 \end{aligned} \right. \tag{23}$$

178

179 Equation (22) is a  $(2+1)$ -dimensional ZKB equation with both dispersive and  
 180 dissipation effect along with a generalized beta effect, for Rossby-Khantadze wave equation.  
 181 Wave damping is represented by the dissipation coefficient  $\gamma_4$ , wave spreading is controlled  
 182 by the dispersion coefficients ( $\gamma_2$  and  $\gamma_3$ ), and the steepening effect is measured by the  
 183 nonlinear coefficient  $\gamma_1$ . There are three components to the ZKB equation: Burgers dissipation  
 184 ( $\gamma_4 A$ ), Zakharov-Kuznetsov (ZK) dispersion  $\left( \frac{\partial^3 A}{\partial X^3} + \frac{\partial^3 A}{\partial X \partial Y^2} \right)$  and KdV-type nonlinearity  $\left( \frac{\partial A}{\partial X} \right)$ .  
 185 It describes solitary Rossby-Khantadze waves in a magnetized fluid with dissipation. It reduced  
 186 to the ZK equation, which supports solitary waves, if  $\eta = 0$ . Shock-like structures, or decaying  
 187 solutions, are the result of dissipation if  $\eta \neq 0$ . The stability and speed of waves are modified  
 188 by magnetic fields ( $\beta_B$ ). When  $\gamma_4$  equals zero, the equation becomes the KdV-Burger equation,  
 189 which supports localized solitary-wave solutions

$$A = a \operatorname{sech}^2 \left( \frac{\gamma_1 a}{12\gamma_2} \right)^{1/2} \xi, \tag{24}$$

191 Here  $a$  is constant wave amplitude. Solution is representing that the speed of solitary  
 192 waves is dependent of amplitude.  $\gamma_4$  is the dissipation term. Otherwise, the wave will lose its



solitary like identification. The solution stays valid if the amplitude is variable in temporal time  $\tau$  scale (Demiray, 2002).

195

#### 196 4. Analytic Solution

To investigate the analytical aspects of the problem, we employ the well-known Demiray (2002) approach, which, to the best of our knowledge, has not previously been applied to the study of solution of dissipative nonlinear Rossby–Khantadze wave propagation.

By using separation of variables techniques and comoving frame of reference

$$A(\eta, \tau) = a(\tau) V(\eta) \quad \text{with} \quad \eta = \alpha(\tau) (k_x x + k_y y - \varphi(\tau)) \quad (25)$$

Here  $a(\tau)$ ,  $\alpha(\tau)$  and  $\varphi(\tau)$  denote functions of slow time variable  $\tau$  while  $V(\eta)$  denotes another function of travelling wave  $\eta$ . Using above expression into Eq. (22), we get

$$\left( \frac{a'(\tau)}{a(\tau)} + \gamma_4 \right) V(\eta) + \frac{a'(\tau)}{a(\tau)} \eta V'(\eta) + [-\alpha(\tau) V'(\eta) \varphi(\tau) + \gamma_1 k_x a(\tau) V V' + (\gamma_2 k_x^2 + \gamma_3 k_y^2) \alpha^3(\tau) V'''] = 0, \quad (26)$$

Here  $V'$  indicates the differentiation with respect to  $\xi$ . Here the newly introduced function  $A(\eta, \tau)$  will be described by two unknown functions  $a(\tau)$  and  $V(\eta)$ . Here we use last bracket of Eq. (26) equals to zero (Demiray, 2002). Therefore,

$$\left( \frac{a'(\tau)}{a(\tau)} + \gamma_4 \right) V(\eta) + \frac{a'(\tau)}{a(\tau)} \eta V'(\eta) = 0, \quad (27)$$

$$-\varphi'(\tau) V' + \gamma_1 k_x a(\tau) V V' + (\gamma_2 k_x^2 + \gamma_3 k_y^2) k_x \alpha^2(\tau) V''' = 0. \quad (28)$$

Integrating the above equation (28) with respect to  $\xi$

$$-\varphi'(\tau) V + \gamma_1 k_x a(\tau) \frac{V^2}{2} + (\gamma_2 k_x^2 + \gamma_3 k_y^2) k_x \alpha^2(\tau) V'' = C(\tau), \quad (29)$$

Here  $C(\tau)$  is an unknown function. By using boundary conditions for  $V$  and its derivatives, which vanish as  $\xi \rightarrow \pm\infty$ . In this case  $C(\tau) = 0$  so Eq. (29) will take the form

$$-\varphi'(\tau) V + \gamma_1 k_x a(\tau) \frac{V^2}{2} + (\gamma_2 k_x^2 + \gamma_3 k_y^2) k_x \alpha^2(\tau) V'' = 0. \quad (30)$$

The hyperbolic tangent method (Demiray, 2000) is used to get the solution of (30)

$$V = 1 - z^2, \quad (31)$$

Thus equation (31) is expressed as

$$\frac{d}{d\xi} = (1 - z^2) \frac{d}{dz} \quad (32)$$

Substituting the above it in to Eq. (30) and putting the like powers of  $z$  equals to zero, we obtain

$$\alpha(\tau) = \left[ \frac{\gamma_1 a(\tau)}{12 (\gamma_2 k_x^2 + \gamma_3 k_y^2)} \right]^{\frac{1}{2}}, \quad \varphi(\tau) = \frac{\gamma_1 k_x}{3} a(\tau) + D. \quad (33)$$



223 where D is a constant.

224 To complete the solution, we use

$$225 \quad \frac{a'(\tau)}{a(\tau)} = \frac{a'(\tau)}{2a(\tau)}. \quad (34)$$

226 Further by substituting Eq. (34) into Eq. (29) we get

$$227 \quad \left( \frac{a'(\tau)}{a(\tau)} + \gamma_4 \right) V(\eta) + \frac{a'(\tau)}{2a(\tau)} \eta V' = 0 \quad (35)$$

228 By multiplying Eq. (35) with V and integrate over  $\xi$  our expression becomes

$$229 \quad \left( \frac{a'(\tau)}{a(\tau)} + \gamma_4 \right) \langle V^2 \rangle + \frac{a'(\tau)}{2a(\tau)} \eta \langle V^2 \rangle = 0, \quad (36)$$

230 where

$$231 \quad \langle V^2 \rangle = \int_{-\infty}^{\infty} V^2 d\xi \quad (37)$$

232 Since we are discussing the travelling waves.  $\langle V^2 \rangle$  is bounded. So, equation (36) will  
233 take the form

$$234 \quad a'(\tau) + \frac{4}{3} \gamma_4 a(\tau) = 0 \quad (38)$$

235 The solution of the above gives

$$236 \quad a(\tau) = a_o \exp \left( -\frac{4}{3} \gamma_4 \tau \right), \quad (39)$$

237 Here  $a_o$  is the constant. By using the condition  $\varphi(0) = 0$ , we get

$$238 \quad \varphi(\tau) = \frac{\gamma_1 k_x a_o}{4\gamma_4} \left( 1 - \exp \left( -\frac{4}{3} \gamma_4 \tau \right) \right), \quad (40)$$

239 Now the final solution is given as

$$240 \quad A(\eta, \tau) = a(\tau) \operatorname{sech}^2 \left( \left( \frac{\gamma_1 a(\tau)}{12 (\gamma_2 k_x^2 + \gamma_3 k_y^2)} \right)^{1/2} \eta \right), \quad (41)$$

241 with

$$242 \quad \eta = \alpha(\tau) \left[ k_x x + k_y y - \frac{\gamma_1 k_x a_o}{4\gamma_4} \left( 1 - \exp \left( -\frac{4}{3} \gamma_4 \tau \right) \right) \right]. \quad (42)$$

243 where  $\gamma_1, \gamma_2, \gamma_3$  are defined previously.

244 The solution is exponentially decaying with time parameter.

## 245 5. Results

246 In this section, we investigate the two-dimensional Rossby-Khantadze wave by using the  
247 plasma parameters for the E-layer of the ionosphere, the corresponding magnetic field is  
248  $B_0 \sim 0.5 \times 10^{-4} T$ , corresponding density of charged particles is  $\rho = 10^{-7} - 10^{-8} \text{ kg m}^{-3}$ ,  
249 corresponding temperature range is 800–2600 K, and  $\beta \approx 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$  (Alexandrov, 1984;  
250 Kaladze et al. 2012). The investigation of our analytical solution (41) by varying different  
251 parameters for considered waves is made.



252 By using the ionospheric region parameters, we have plotted the  $A$  versus  $x$ .

253 The amplitude is decreasing with increasing temporal effects.

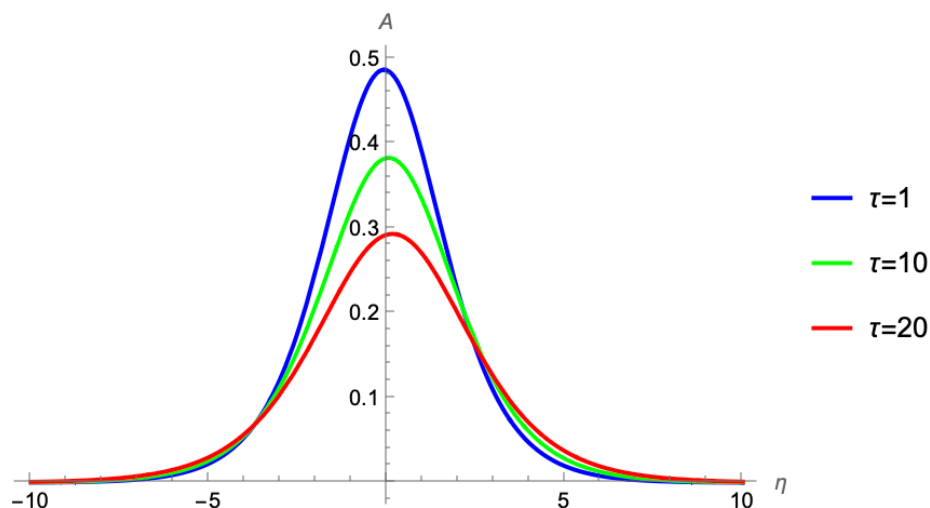


Fig. 1 Dependence of  $A$  on temporal parameter  $\tau$  is shown

258 While it is observed that amplitude of structures is also decreasing by increasing dissipation  
259 effect.

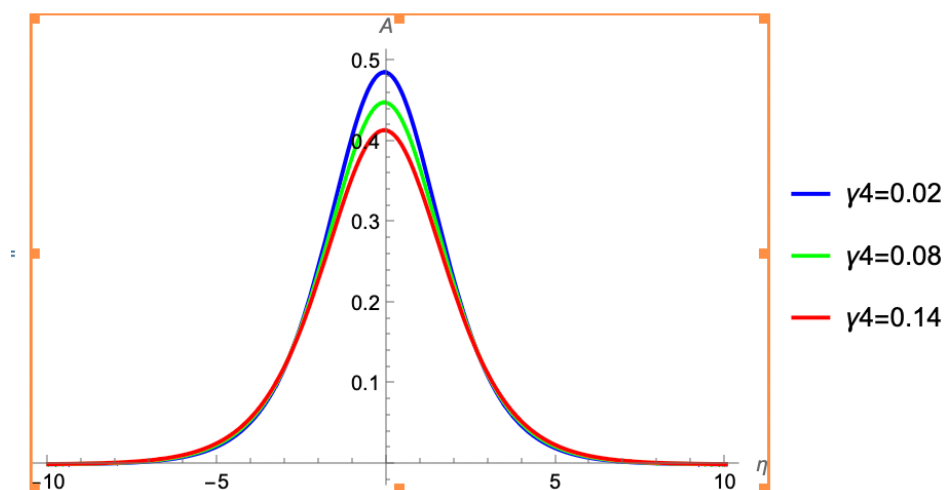


Fig. 2 Dependence of  $A$  on dissipation parameter  $\gamma_4$  is shown

## 6. Discussion and Conclusion



Here we have examined the propagation of Rossby-Khantadze waves in a dissipative ionospheric plasma and apply our results into E region of ionosphere. The combined influence of both inhomogeneities namely, angular velocity and magnetic field are taken. The basic system of equations governing the dynamics of nonlinear Rossby-Khantadze is obtained. In previous work, the authors consider only Rossby waves analysis whereas in the given work we investigate the coupled Rossby - Khantadze waves.

To examine the coupled nonlinear RKWs we employ the multiple scale analysis and asymptotic expansion to lead a new nonlinear ZKB equation for Rossby-Khantadze waves with spatial dependent coefficients. The smallest order,  $O(\varepsilon)$  leads to an eigen-value boundary problem, which for the plane wave solution yields meridional Rossby-Khantadze waves dispersion. It didn't give any idea about the amplitude of considered solitary Rossby-Khantadze waves. Therefore, we proceed to the next order  $O(\varepsilon^1)$ , but we do not get any information about amplitude of solitary Rossby-Khantadze waves. To do so, from the next subsequent order  $O(\varepsilon^2)$ , we obtain the nonlinear ZKB equation for dissipative RK waves. This resulting equation describes a longitudinal dispersion, that competes with a weakly nonlinear and dissipation effect. To make ensure the effective solution then the secular term must be eliminated (Eq. (20)), otherwise the amplitude of solitary structure will be grown indefinitely. From the obtained equation, the coefficients are related to inhomogeneities in rotation and the magnetic field. The effects of dissipations and the inhomogeneities are reflected in the coefficients of the obtained ZKB equation. Furthermore, the analytic and numerical dependence of the amplitude of obtained solitary structures is shown.

To study the analytic part of obtained nonlinear dissipative Rossby-Khantadze waves, we use the well-known Demiray' approach. Furthermore, by employing the hyperbolic tangent method we get damped solitary RKWs solutions, which are subsequently explored numerically.

Overall, our study contributes to our understanding of nonlinear wave propagation in the ionosphere and its applications to other plasma environments, such as interstellar medium, the solar wind plasmas as well as laboratory plasmas systems. The results show that nonlinearity, dispersion, inhomogeneity, and dissipation play important roles in wave dynamics. These findings provide useful insights for future research in plasma physics.

**Data availability.** No data sets were used in this article.



296 **Author contributions.** LZK: conceptualization (equal); formal analysis (equal); investigation  
297 (equal); methodology (equal); writing; original draft (equal); writing – review and editing  
298 (equal). HAS: methodology (equal); writing - review and editing (equal).

299

300 **Competing interests.** The contact author has declared that none of the authors has any  
301 competing interests.

302

### 303 **References**

304 Alexandrov, A. F., Bogdankevich, L. S., and Rukhadze, A. A.: Principles of plasma  
305 electrodynamics, Springer, 1984.

306 Demiray, H.: A note on the solution of perturbed Korteweg–de Vries equation, Appl. Math.  
307 Comput., 132, 643–647, [https://doi.org/10.1016/S0096-3003\(01\)00222-3](https://doi.org/10.1016/S0096-3003(01)00222-3) , 2002.

308 Dovner, P. O., Eriksson, A. I., Boström, R., and Holback, B.: Freja multiprobe observations of  
309 electrostatic solitary structures, Geophys. Res. Lett., 21, 1827–1830,  
310 <https://doi.org/10.1029/94GL00886> , 1994.

311 Fagundes, P. R., Pillat, V. G., Bolzan, M. J., Sahai, Y., Becker-Guedes, F., Abalde, J. R.,  
312 Aranha, S. L., and Bittencourt, J. A.: Observations of F-region ionospheric disturbances  
313 associated with gravity waves, J. Geophys. Res., 110, A12302,  
314 <https://doi.org/10.1016/10.1029/2009JA0144822005> .

315 Forbes, J. M.: Planetary waves in the thermosphere–ionosphere system, J. Geomagn.  
316 Geoelectr., 48, 91–98, <https://doi.org/10.5636/jgg.48.91> , 1996.

317 Futatani, S., Horton, W., and Kaladze, T. D.: Nonlinear propagation of Rossby–Khantadze  
318 electromagnetic planetary waves in the ionospheric E-layer, Phys. Plasmas, 20, 102903,  
319 <https://doi.org/10.1063/1.4826592> , 2013.

320 Gershman, B. N.: Dynamics of the ionospheric plasma, Nauka, Moscow, 1974.

321 Gill, A. E.: Atmosphere–Ocean Dynamics, Academic Press, New York, 1982.

322 Jian, S., Lian-Gui, Y., Chao-Jiu, D. A., and Hui-Qin, Z.: mKdV equation for the amplitude of  
323 solitary Rossby waves in stratified shear flows with a zonal shear flow, Atmos. Ocean. Sci.  
324 Lett., 2, 18–23, <https://doi.org/10.1080/16742834.2009.11446771> , 2009.

325 Kahlon, L. Z. and Kaladze, T. D.: Generation of zonal flow and magnetic field in the  
326 ionospheric E-layer, J. Plasma Phys., 81, 905810512,  
327 <https://doi.org/10.1017/S002237781500080X> , 2015.



- 328 Kaladze, T. D., Aburjania, G. D., Kharshiladze, O. A., Horton, W., and Kim, Y.-H.: Theory of  
329 magnetized Rossby waves in the ionospheric E layer, *J. Geophys. Res.*, 109, A05302,  
330 <https://doi.org/10.1029/2003JA010049> , 2004.
- 331 Kaladze, T. D. and Horton, W.: Synoptic-scale nonlinear stationary magnetized Rossby waves  
332 in the ionospheric E-layer, *Plasma Phys. Rep.*, 32, 996–1006,  
333 <https://doi.org/10.1134/S1063780X06120038> , 2006.
- 334 Kaladze, T. D., Kahlon, L. Z., Tsamalashvili, L. V.: Excitation of zonal flow and magnetic  
335 field by Rossby–Khantadze electromagnetic planetary waves in the ionospheric E-layer, *Phys.*  
336 *Plasmas*, 19, 022902, <https://doi.org/10.1063/1.3681370> , 2012.
- 337 Kaladze, T., Kahlon, L., Horton, W., Pokhotelov, O., and Onishchenko, O.: Shear flow driven  
338 Rossby–Khantadze electromagnetic planetary vortices in the ionospheric E-layer, *Europhys.*  
339 *Lett.*, 106, 29001, <https://doi.org/10.1209/0295-5075/106/29001> , 2014.
- 340 Kaladze, T., Tsamalashvili, L., Kaladze, D., Özcan, O., Yeşil, A., and İnç, M.: Modified KdV  
341 equation for magnetized Rossby waves in a zonal flow of the ionospheric E-layer, *Phys. Lett.*  
342 *A*, 383, 125888, <https://doi.org/10.1016/j.physleta.2019.125888> , 2019.
- 343 Khantadze, A. G., Jandieri, V. G., and Jandieri, G. V.: Generation of Rossby-type vortical  
344 perturbations in the ionosphere, *Fiz. Plazmy*, 34, 527–532, 2008.
- 345 Khomenko, E.: Multi-fluid effects in magnetohydrodynamics, in: *Oxford Research*  
346 *Encyclopedia of Physics*, Oxford University Press,  
347 <https://doi.org/10.1093/acrefore/9780190871994.013.4> , 2020.
- 348 Lindqvist, P. A., Marklund, G. T., and Blomberg, L. G.: Plasma characteristics determined by  
349 the Freja electric field instrument, *Space Sci. Rev.*, 70, 593–602,  
350 <https://doi.org/10.1007/BF00756888> , 1994.
- 351 Luo, L., Xu, X., Zhang, Y., Zhou, Z., Chang, Q., Xu, Q., Wang, X., He, P., Yi, S., and Li, H.:  
352 The three-fluid generalized Ohm's law: A theoretical study, *Phys. Fluids*, 35, 017134,  
353 <https://doi.org/10.1063/5.0135612> , 2023.
- 354 Petviashvili, V. I. and Pokhotelov, O. A.: *Ionospheric and Magnetospheric Plasma Physics*,  
355 Gordon and Breach, Reading, 1992.
- 356 Qiang, Z., Zuntao, F., and Shikuo, L.: Equatorial envelope Rossby solitons in a shear flow,  
357 *Adv. Atmos. Sci.*, 18, 418–428, <https://doi.org/10.1007/BF02919321> , 2001.
- 358 Song, P., Gombosi, T. I., and Ridley, A. J.: Three-fluid Ohm's law, *J. Geophys. Res.*, 106,  
359 8149–8156, <https://doi.org/10.1029/2000JA000423> , 2001.
- 360 Song, P., Vasyliūnas, V. M., and Ma, L.: A three-fluid model of solar wind–magnetosphere–  
361 ionosphere–thermosphere coupling, in: *Multiscale Coupling of Sun–Earth Processes*, edited



- 362 by: Lui, A. T. Y., Kamide, Y., and Consolini, G., Elsevier, Amsterdam, 447–456,  
363 <https://doi.org/10.1016/B978-044451881-1/50033-2>, 2005.
- 364 Song, P., Vasyliūnas, V. M., and Zhou, X.-Z.: Magnetosphere–ionosphere/thermosphere  
365 coupling: self-consistent solutions for a one-dimensional stratified ionosphere in three-fluid  
366 theory, J. Geophys. Res., 114, A08213, <https://doi.org/10.1029/2008JA013629>, 2009.
- 367 Vukčević, M. and Popović, L. C.: Solitons in the ionosphere – advantages and perspectives,  
368 in: Proceedings of the XII Serbian-Bulgarian Astronomical Conference (XII SBAC), edited  
369 by: Popović, L. C., Srećković, V. A., Dimitrijević, M. S., and Kovačević, A., Publications of  
370 the Astronomical Society “Rudjer Bošković”, No. 20, 85–91, 2020.
- 371 Yin, X., Yang, L., Liu, G., and Wu, G.: (2+1)-dimensional ZK–Burgers equation with the  
372 generalized beta effect and its exact solitary solution, Comput. Math. Appl., 77, 302–310, 2019.