



A comparison of the accuracy and stability of numerical schemes used to solve the equations derived from the shallow-shelf approximation

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Abstract. The shallow-shelf approximation (SSA) is widely used to model the evolution of fast-flowing ice that is either floating or sliding rapidly over the bed, resulting in plug flow. However, so far there has been no rigorous, comparative study of the different numerical approaches to solve the SSA equations. We calculate the conditions for different numerical schemes to remain stable and accurate when solving the linear 1D SSA equations for a simple geometry. The comparison spans central difference and upwind schemes, explicit and semi-implicit multistep schemes, and explicit or implicit monolithic schemes. Also considered are schemes time stepping using the Θ -method, where the value of Θ is used to form a weighted combination of the forward and backward Euler methods. We find that monolithic schemes remain stable at all tested time steps for $\Theta \geq 0.5$ whilst multistep schemes become unstable at large time steps. Of the tested schemes the monolithic finite element scheme with $\Theta = 0.5$ remains accurate for the largest range of time steps.

1 Introduction

In recent decades there has been rapid progress in the number and capabilities of ice flow models (e.g. Stokes et al. (2015); Goelzer et al. (2017); Pattyn et al. (2017)). All models must choose how to approximate the ice dynamics and which numerical technique to use to solve the relevant equations. The choice of equations and numerical technique varies widely across models, but a shared requirement of all numerical schemes is that they must be stable, accurate and consistent. In this paper we consider some of the numerical schemes used to solve the equations derived from the shallow-shelf approximation (SSA). The goal of this paper is to assess the conditions under which these numerical schemes remain stable and accurate when solving the SSA equations for a simple geometry and to provide insight into the errors that might be introduced into models when using these schemes for more complex geometries.

Numerical schemes can employ a monolithic or sequential approach to solve the relevant equations at each time step. A scheme is defined to be explicit if the thickness at the next time step is calculated only using the velocity and thickness at the current time step (the scheme is explicit with respect to thickness and velocity). A scheme is defined to be semi-implicit if the thickness at the next time step is calculated by solving the equation for conservation of mass using the velocity at the current time step and the thickness and the next time step. The velocity at the next time step is then calculated by solving the



25 equations for conservation of momentum using the new thickness (the scheme is implicit with respect to thickness and explicit
with respect to velocity). A scheme is fully implicit if the equations for momentum and mass conservation are solved simul-
taneously using the velocity and thickness at the next time step (the equations are fully implicit with respect to both thickness
and velocity). Therefore, to be fully implicit, the scheme must employ a monolithic approach to solve the relevant equations
(Markert et al., 2010). A final time-stepping approach is to use the Θ -method (Stuart and Peplow, 1991) which involves using
30 a weighted average of the state of the system at the current time step and the next time step.

One way to compare different numerical schemes is with inter-comparison projects such as Pattyn et al. (2012, 2013);
Goelzer et al. (2018); Seroussi et al. (2019); Cornford et al. (2020); Goelzer et al. (2020); Seroussi et al. (2020). These inter-
comparison projects have been used to evaluate how results depend on the choice of numerical scheme, numerical setup, and
35 equations to approximate the ice dynamics. Projects involving idealised settings with simple forcings (Pattyn et al., 2012, 2013)
have been used to compare the results to semi-analytical solutions. These projects give some indication of how well the chosen
numerical scheme solves the partial differential equations on which the model is based. However, as the model results are not
compared to an exact solution, questions remain about the accuracy of each scheme.

40 A number of studies have focused on determining the conditions for different schemes to remain stable. The stability of im-
plicit and explicit schemes solving the shallow-ice approximation (SIA) or using a Full-Stokes model is considered in Bueler
(2023), with the benefits of the large stable time step allowed by implicit schemes being highlighted. A simple geometry has
been used to find the conditions for the SIA to be stable under different numerical schemes (Cheng et al., 2017). In this ge-
ometry a perturbation is applied to uniform flow over an infinite inclined plane of uniform ice thickness. This same geometry
45 has been used to find the requirement for different depth-integrated solvers to remain stable under an explicit upwind scheme
(Robinson et al., 2022). However, little analytical progress has been made on determining the conditions under which these
depth-integrated solvers remain stable for other numerical schemes. Further, the time evolution of the perturbation is not com-
pared to an analytical solution, meaning the accuracy of the scheme can only be determined by using a numerical method to
estimate the discretisation error.

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In other settings, global spectral analysis (GSA) has been demonstrated to be a useful method for calculating the accuracy
of numerical schemes as it can be used to assess the dispersion and dissipation errors for all spatial and temporal time scales
(Sagaut et al., 2023; Sengupta et al., 2022). In this study we use GSA to assess the conditions for a numerical scheme to
remain stable, accurate and consistent. We focus on the shallow-shelf approximation (SSA) (MacAyeal, 1989). The SSA is
55 widely used (Seroussi et al., 2020; Goelzer et al., 2020) to model the evolution of fast-flowing ice that is either floating or
sliding rapidly over the bed, resulting in plug flow (Robinson et al., 2022). Given the growing interest in using semi-implicit
or implicit time-stepping schemes (Bueler, 2023; Berends et al., 2025; Ahlkrone et al., 2026), we compare numerical schemes
which employ explicit, semi-implicit and implicit time-stepping as well as schemes which use Θ -methods. We also compare
the benefits of using a central difference or upwind spatial discretisation. We assess the conditions for both finite element and



60 finite difference schemes to remain stable, accurate and consistent. To properly assess the accuracy of each numerical scheme we require an exact solution to the SSA equations. Therefore, we use the same geometry as Gudmundsson (2008); Cheng et al. (2017); Robinson et al. (2022).

We begin in Sect. 2 by introducing the different time-stepping algorithms that can be used to solve the SSA equations for all choices of power law exponent and sliding law. Then, in Sect. 3, we introduce the linear 1D SSA equations and the geometry which we shall use to test each of the numerical schemes. We then find the exact solutions for the evolution in ice thickness and velocity over time. In Sect. 4 we introduce the method to determine the accuracy and stability of each scheme. The accuracy and stability tests are first performed for the finite difference schemes introduced in Sect. 5.1. The dispersive and diffusive properties of these schemes are obtained in Sect. 5.2. We find the conditions under which each finite difference scheme remains stable in Sect. 5.3 and accurate in Sect. 5.4. In Sect. 6 we test finite element schemes using the open source ice sheet model *Úa* (Gudmundsson, 2026). We conclude with which is the optimum scheme in Sect. 7.

2 Time-stepping algorithms

We begin with a general description of the different time-stepping algorithms that can be used to solve the SSA equations. Defining h_0 , $\mathbf{u}_0$ as the solution for thickness h , and velocity $\mathbf{u}$ at the current time step and h_1 , $\mathbf{u}_1$ as the solution for h and $\mathbf{u}$ at the next time step, two different approaches can be used to solve for h_1 and $\mathbf{u}_1$.

In a staggered approach the equations for mass and momentum conservation are solved sequentially via an iterative and multistep approach. At each iteration j new expressions for h_1 and $\mathbf{u}_1$ are calculated until the change in $\mathbf{u}_1$ between adjacent iterations is below a chosen tolerance or the maximum number of iterations is reached. For iteration j the steps are as follows:

- 80 1. Solve the equation of mass conservation for h_1^j . The flux terms in the equation are expressed in terms of a weighted combination of the velocity at the current and next time step and the thickness at the current time step and the next time step. The weighting is given by the value Θ , $\rho_i(h_1^j - h_0) + \Delta t(1 - \Theta)\nabla \cdot (\rho_i \mathbf{u}_0 h_0) + \Delta t \Theta \nabla \cdot (\rho_i \mathbf{u}_1^j h_1^j) - \Delta t \rho_i a = 0$.
2. Solve for $\mathbf{u}_1^j$ by expressing the equation for momentum conservation in terms of h_1^j and $\mathbf{u}_1^j$.

When $j = 1$ $\mathbf{u}_0$ instead of $\mathbf{u}_1^j$ is used to calculate h_1^1 . The h solve is linear, while the $\mathbf{u}$ solve is, in general, non-linear with respect to $\mathbf{u}$. If $\Theta = 1$ the h solve is implicit. In this case we can refer to the time-stepping scheme as a semi-implicit staggered approach. If $\Theta = 0$ the h solve is explicit and we refer to the time-stepping scheme as the explicit staggered approach.

Alternatively, in a monolithic time-stepping approach, updates in $\mathbf{u}$ and h are solved for simultaneously. In *Úa*, where this is the default approach and to our knowledge the only approach used by all users in all large-scale modelling done with the model, this is achieved by solving the resulting non-linear system using the Newton-Raphson method with respect to both $\mathbf{u}$



and h and iterating until the equation residuals are below a given tolerance.

These different time stepping schemes can be used to solve the SSA equations for any choice of sliding law and power flow law exponent n . In the analysis that follows we apply these time stepping schemes to the linear 1D SSA equations assuming
95 $n = 1$ and using a Weertman sliding law with sliding exponent $m = 1$.

3 Exact solution to the transient linear 1D SSA equations

We begin by introducing the conditions under which an exact solution can be found for the SSA. Assuming uniform flow and thickness in the y -direction, setting the power flow law exponent, $n = 1$ and assuming a Weertman sliding law with sliding exponent $m = 1$ gives,

$$100 \quad \frac{\partial}{\partial x} \left[\frac{2}{A} \left(\frac{\partial u}{\partial x} \right) h \right] - C^{-1} u = \rho_i g h \frac{\partial s}{\partial x}, \quad (1)$$

$$\rho_i \frac{\partial h}{\partial t} = - \frac{\partial(\rho_i u h)}{\partial x} + \rho_i a. \quad (2)$$

Here A is the temperature-dependent rate factor in Glen's flow law, u is the ice velocity in the x -direction, h is the ice thickness, C is the sliding parameter, ρ_i is the density of ice, g is the acceleration due to gravity, s is the surface elevation and a is the rate of accumulation minus the rate of melting of ice. See Table A1 for a full list of all the parameters used in this
105 study along with their value and units. An exact solution can be found for the geometry in Gudmundsson (2008); Cheng et al. (2017) and Robinson et al. (2022) where ρ_i , A and C are constant and $a = 0$. The ice sheet is of infinite extent and lies on a slope inclined at an angle α to the horizontal. It is assumed that the thickness and velocity can be expressed as the sum of a uniform background thickness/velocity and a small amplitude perturbation, $h(x, t) = H + \delta h(x, t)$, $u(x, t) = U + \delta u(x, t)$. The zeroth order solution represents plug flow down a uniformly inclined plane,

$$110 \quad U = \alpha \rho_i g H C. \quad (3)$$

To first order in the perturbations we find,

$$\frac{2H}{A} \frac{\partial^2(\delta u)}{\partial x^2} - C^{-1} \delta u = \rho_i g H \frac{\partial(\delta h)}{\partial x} - \alpha \rho_i g \delta h, \quad (4)$$

$$\frac{\partial(\delta h)}{\partial t} = -H \frac{\partial(\delta u)}{\partial x} - U \frac{\partial(\delta h)}{\partial x}, \quad (5)$$

where we have set α to be small such that $\tan \alpha \approx \alpha$.

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The exact solution to these equations can be obtained by performing a Fourier-Laplace transform. We set,

$$\delta h(x, t) = \iint \delta \hat{H}(k, \omega) e^{-i\omega t} e^{ikx} dk d\omega, \quad (6)$$

$$\delta u(x, t) = \iint \delta \hat{U}(k, \omega) e^{-i\omega t} e^{ikx} dk d\omega. \quad (7)$$



where $\omega = \omega_R + i\omega_I$. Substituting Eq.(6) and Eq.(7) into Eq.(4) and Eq.(5) and solving for each k, ω we find,

$$120 \quad \delta\hat{U} = \left[\frac{\alpha\rho_i g - ik\rho_i g H}{\frac{2Hk^2}{A} + \frac{1}{C}} \right] \delta\hat{H}, \quad (8)$$

$$\omega_R = \frac{\alpha k H \rho_i g}{\frac{2H}{A} k^2 + \frac{1}{C}} + kU, \quad (9)$$

$$\omega_I = -\frac{k^2 H^2 \rho_i g}{\frac{2H}{A} k^2 + \frac{1}{C}}. \quad (10)$$

We introduce four dimensionless variables which shall be used throughout this paper to compare the numerical and exact solutions,

$$125 \quad \theta = k\Delta x, \quad (11)$$

$$q = \frac{A(\Delta x)^2}{2HC}, \quad (12)$$

$$\beta = \frac{\alpha\Delta x}{2H}, \quad (13)$$

$$P = U \frac{\Delta t}{\Delta x} \quad (14)$$

where Δx is the spacing between nodes and Δt is the time step used by the numerical model. For fixed Δx , increasing q corresponds to increasing basal friction, faster decaying perturbations and perturbations with higher phase velocities. For small values of α , the ice sheet increases H m in elevation approximately over length $L_{ice} = H/\alpha$. Therefore, the dimensionless variable $\beta = \Delta x/L_{ice}$ indicates the spatial resolution of the mesh. Low values of β correspond to a simulation with high spatial resolution and high values of β correspond to a simulation with low spatial resolution. The last dimensionless variable, P is the CFL number.

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The expressions for ω_I and ω_R can be used to calculate three metrics which are often used in GSA to determine the accuracy of a numerical scheme (Suman et al., 2017; Sengupta et al., 2023; Sagaut et al., 2023). These are the

- phase velocity, $c_{ex} = \frac{\omega_R}{k} = U \left[\frac{2q+\theta^2}{q+\theta^2} \right],$
- group velocity, $v_{g,ex} = \frac{\partial\omega_R}{\partial k} = U \left[\frac{2q^2+\theta^2q+\theta^4}{(q+\theta^2)^2} \right],$
- 140 – dispersion coefficient (the decay in amplitude in time Δt), $\Gamma_{ex} = \exp(\omega_I \Delta t) = \exp \left[- \left(\frac{P\theta^2 q}{2\beta(\theta^2+q)} \right) \right].$

We compare these metrics to the equivalent metrics for each numerical scheme at each Fourier mode. Therefore, we are using GSA to determine the accuracy of each numerical scheme. The method to calculate these metrics for each numerical scheme is introduced in Sect. 4.



4 Determining the stability and accuracy of numerical schemes

145 In both finite difference and finite element schemes the shape of the velocity and thickness perturbations are calculated at discrete times ($t = t_n \equiv n\Delta t$, where n is an integer) and positions ($x = x_j \equiv j\Delta x$, where j is an integer). We perform a Fourier-Laplace transform of a numerical thickness perturbation and the numerical velocity perturbation,

$$\delta h_{\text{num}}(x_j, t_n) = \delta h_j^n = \iint \delta \hat{H}_{\text{num}}(k_{\text{num}}, \omega_{\text{num}}) e^{-i\omega_{\text{num}} n \Delta t} e^{ik_{\text{num}} j \Delta x} dk_{\text{num}} d\omega_{\text{num}}, \quad (15)$$

$$\delta u_{\text{num}}(x_j, t_n) = \delta u_j^n = \iint \delta \hat{U}_{\text{num}}(k_{\text{num}}, \omega_{\text{num}}) e^{-i\omega_{\text{num}} n \Delta t} e^{ik_{\text{num}} j \Delta x} dk_{\text{num}} d\omega_{\text{num}}. \quad (16)$$

150 The numerical angular frequency ω_{num} is composed of a real and complex part, $\omega_{\text{num}} = \omega_{\text{num},R} + i\omega_{\text{num},I}$. The numerical phase and group velocities are given respectively by $c_{\text{num}} = \omega_{\text{num},R}/k$ and $v_{g,\text{num}} = \partial\omega_{\text{num},R}/\partial k$. We also define the numerical dispersion coefficient $\Gamma_{\text{num}}(k_{\text{num}}, \Delta t) = \exp[\omega_{\text{num},I}(k_{\text{num}})\Delta t]$.

These metrics can be used to calculate the accuracy of each scheme. Following Suman et al. (2017); Sengupta et al. (2023)
 155 the dispersion error for each scheme is calculated from the ratio of the numerical to the exact phase velocity, $c_{\text{num}}/c_{\text{ex}}$ and ratio of the numerical to the exact group velocity, $v_{g,\text{num}}/v_{g,\text{ex}}$. The diffusion error for each scheme is calculated from the ratio of the numerical to the exact dispersion coefficients, $\Gamma_{\text{num}}/\Gamma_{\text{ex}}$. To ensure a fair comparison, the ratios are compared when $k_{\text{num}} = k$.

If the numerical scheme perfectly preserved the dispersion relation, all the above ratios would be equal to one. Therefore,
 160 we find the range of time steps where the error metrics defined above are all close to 1. Following Sengupta et al. (2022, 2020) we define a dispersion relation preserving region as a region with maximum of 1% deviation tolerance at all wavenumbers i.e. the ratio of the numerical to the exact solution is between 0.99 and 1.01.

We also determine the conditions under which each numerical scheme remains stable. If $\Gamma_{\text{num}} \geq 1$ for any value of k_{num} or
 165 equivalently if $\omega_{\text{num},I} > 0$ for any value of k_{num} the the perturbation will grow over time and the numerical scheme is unstable.

5 Stability and accuracy analysis of finite difference schemes

We shall first perform a stability and accuracy analysis for finite difference numerical schemes before considering finite element schemes. The finite difference schemes are introduced in Sect. 5.1.

5.1 Finite difference numerical schemes considered in this paper

170 We compare explicit, semi-implicit, implicit, and Θ -methods for advancing thickness and velocity to the next time step. We consider spatial discretisation using an upwind and a central difference scheme. To calculate the dispersion relation for each scheme analytically we consider only finite difference schemes on a regular grid. A summary of the numerical schemes that



we consider is listed in Table 1.

Acronym	time stepping	spatial Discretisation	Equations
$uh\Theta - UW$	Θ -method for thickness and velocity	Upwind	Eq.(17), Eq.(21)
$uh\Theta - CD$	Θ -method for thickness and velocity	Central difference	Eq.(18), Eq.(21)
$u0h\Theta - UW$	Θ -method for thickness, explicit for velocity	Upwind	Eq.(19), Eq.(21)
$u0h\Theta - CD$	Θ -method for thickness, explicit for velocity	Central difference	Eq.(20), Eq.(21)

Table 1. Finite difference numerical schemes considered in this study. For all schemes we consider $0 \leq \Theta \leq 1$.

175 Following Robinson et al. (2022), for the finite difference scheme, the δh nodes are at position $x(\delta h_i) = iL/N = i\Delta x$ and the δu nodes at $x(\delta u_i) = \Delta x(i + 1/2)$ where L is the length of the domain and N is the total number of nodes. Robinson et al. (2022) assessed the stability of an explicit upwind scheme. We extend their analysis to consider the schemes mentioned above. Under an upwind scheme, time-stepping using the Θ -method for both velocity and thickness, Eq.(5) is discretised as

$$180 \quad \delta h_j^{n+1} = \delta h_j^n + (1 - \Theta) \left[-\frac{\Delta t}{\Delta x} H(\delta u_j^n - \delta u_{j-1}^n) - \frac{\Delta t}{\Delta x} U(\delta h_j^n - \delta h_{j-1}^n) \right] + \Theta \left[-\frac{\Delta t}{\Delta x} H(\delta u_j^{n+1} - \delta u_{j-1}^{n+1}) - \frac{\Delta t}{\Delta x} U(\delta h_j^{n+1} - \delta h_{j-1}^{n+1}) \right]. \quad (17)$$

Under an central difference scheme, time-stepping using the Θ -method for both velocity and thickness, Eq.(5) is discretised as

$$\delta h_j^{n+1} = \delta h_j^n + (1 - \Theta) \left[-\frac{\Delta t}{\Delta x} H(\delta u_j^n - \delta u_{j-1}^n) - \frac{\Delta t}{2\Delta x} U(\delta h_{j+1}^n - \delta h_{j-1}^n) \right] + \Theta \left[-\frac{\Delta t}{\Delta x} H(\delta u_j^{n+1} - \delta u_{j-1}^{n+1}) - \frac{\Delta t}{2\Delta x} U(\delta h_{j+1}^{n+1} - \delta h_{j-1}^{n+1}) \right]. \quad (18)$$

If $\Theta = 0$ the scheme is explicit. If $\Theta = 1$ the scheme is implicit. If $\Theta = 0.5$ the scheme is known as the Lax-Wendroff method.

185 We also define semi-implicit schemes for the upwind Eq.(19) and central difference methods Eq.(20) which are explicit with respect to velocity and apply the Θ -method with respect to thickness .

$$\delta h_j^{n+1} = \delta h_j^n - \frac{\Delta t}{\Delta x} H(\delta u_j^n - \delta u_{j-1}^n) + \Theta \left[-\frac{\Delta t}{\Delta x} U(\delta h_j^{n+1} - \delta h_{j-1}^{n+1}) \right] + (1 - \Theta) \left[-\frac{\Delta t}{\Delta x} U(\delta h_j^n - \delta h_{j-1}^n) \right]. \quad (19)$$

$$\delta h_j^{n+1} = \delta h_j^n - \frac{\Delta t}{\Delta x} H(\delta u_j^n - \delta u_{j-1}^n) + \Theta \left[-\frac{\Delta t}{2\Delta x} U(\delta h_{j+1}^{n+1} - \delta h_{j-1}^{n+1}) \right] + (1 - \Theta) \left[-\frac{\Delta t}{2\Delta x} U(\delta h_{j+1}^n - \delta h_{j-1}^n) \right]. \quad (20)$$

190 Following Robinson et al. (2022) we write Eq.(4) as

$$\frac{2H}{A(\Delta x)^2} (\delta u_{j+1} - 2\delta u_j + \delta u_{j-1}) - \frac{\delta u_j}{C} = \frac{\rho_i g H}{\Delta x} (\delta h_{j+1} - \delta h_j) - \alpha \rho_i g \delta h_j. \quad (21)$$

We begin by obtaining the numerical dispersion relation for each of these schemes.



5.2 Dispersive and diffusive properties of each finite difference scheme

Following, Robinson et al. (2022) substituting Eq.(15) and Eq.(16) into Eq.(21) approximately gives,

$$195 \quad \delta \hat{U}_{\text{num}}(k_{\text{num}}, \omega_{\text{num}}) = -\zeta \delta \hat{H}(k_{\text{num}}, \omega_{\text{num}}), \quad (22)$$

where

$$\zeta = \frac{qU\phi}{2\beta H} \{ (e^{i\theta} - 1) - \beta (e^{i\theta} + 1) \}, \quad (23)$$

$$\phi = \frac{1}{2 + q - 2 \cos \theta}. \quad (24)$$

The relation between the velocity and thickness perturbation, Eq.(22) can be derived from equation 43 in Robinson et al. (2022)
 200 by substituting in $U = \alpha \rho_i g H C$ and noting that to convert from the notation in Robinson et al. (2022) set $3\beta/(3 + \eta) \mapsto 1/C$
 and $\mu \mapsto 1/(2A)$.

Substituting in Eq.(15)-Eq.(16) into Eq.(17)-Eq.(20) we find an expression for $\exp(-i\omega_{\text{num}}\Delta t)$ for each numerical scheme.
 Using the symbolic toolbox in Matlab, we find expressions for $\psi = \omega_{\text{num},R}\Delta t$ and $\Gamma_{\text{num}} = \exp(\omega_{\text{num},I}\Delta t)$, which are given in
 205 Appendix B. Noting that $\psi = \omega_{\text{num}}\Delta t$ and $\theta = k_{\text{num}}\Delta x$, again using the symbolic toolbox in Matlab we calculate the phase
 velocity as $c_{\text{num}} = \psi/(k_{\text{num}}\Delta t)$ and the group velocity as $v_{g,\text{num}} = (\Delta x/\Delta t)(\partial\psi/\partial\theta)$. Using the expressions for Γ_{num} and ψ
 we can calculate the conditions for each scheme to be stable and accurate. We begin by finding the conditions for each scheme
 to be stable.

5.3 Stability analysis of the finite difference schemes

210 In order for a numerical scheme to be stable the dispersion coefficient must be less than or equal to one at all values of θ ,
 $\max |\Gamma_{\text{num}}(\theta)| < 1$ for all θ . We find the range of Δt where each scheme remains stable. Detailed analysis for each scheme is
 found in Appendix C. Here we quote the key results.

The $uh\Theta - UW$ scheme is stable for

$$215 \quad \Delta t < \left(\frac{1}{1 - 2\Theta} \right) \left(\frac{\Delta x}{U} \right) \left[\frac{2 \left(1 + \frac{q\phi}{\beta} \right)}{(1 - \cos \theta) \left(1 + \frac{q\phi}{\beta} \right)^2 + (1 + \cos \theta) (1 + q\phi)^2} \right]. \quad (25)$$

The $uh\Theta - CD$ scheme is stable for

$$\Delta t < \frac{1}{1 - 2\Theta} \left(\frac{\Delta x}{U} \right) \left[\frac{2 \left(\frac{q\phi}{\beta} \right)}{(1 - \cos \theta) \left(\frac{q\phi}{\beta} \right)^2 + (1 + \cos \theta) (1 + q\phi)^2} \right]. \quad (26)$$

All $uh\Theta$ schemes are unconditionally stable when $\Theta \geq 0.5$. Analytical expressions for the condition on P , for both $uh\Theta$
 schemes, in the limit of small and large β are given in Appendix C.



Scheme	Stability condition	Limit when $\Delta x \rightarrow 0$	Limit when $\Delta x \rightarrow \infty$
$uh\Theta - UW$	(25)	$\Delta t < \frac{1}{1-2\Theta} \left(\frac{\Delta x}{U} \right)$	$\Delta t < \frac{1}{1-2\Theta} \left(\frac{\Delta x}{U} \right) \left(\frac{1}{4} \right)$
$uh\Theta - CD$	(26)	$\Delta t < \begin{cases} \frac{1}{1-2\Theta} \left(\frac{4C\alpha}{AU} \right), & \alpha \ll \frac{1}{2} \sqrt{\frac{AH}{2C}}, \\ \frac{1}{1-2\Theta} \left[\frac{A(\Delta x)^2}{\alpha CU} \right], & \alpha \gg \frac{1}{2} \sqrt{\frac{AH}{2C}}, \end{cases}$	$\Delta t < \frac{1}{1-2\Theta} \left(\frac{H}{2\alpha U} \right)$
$u0h\Theta - UW$	(27)	$\Delta t < \frac{1}{1-2\Theta} \left(\frac{\Delta x}{U} \right)$	$\Delta t < \frac{1}{1-\Theta} \left(\frac{\Delta x}{U} \right) \left(\frac{1}{4} \right)$
$u0h\Theta - CD$	(28)	$\Delta t < \begin{cases} \frac{4C\alpha}{AU}, & \alpha \ll \frac{1}{2} \sqrt{\frac{AH}{2C(1-\Theta)}}, \\ \frac{1}{1-\Theta} \left[\frac{A(\Delta x)^2}{\alpha CU} \right], & \alpha \gg \frac{1}{2} \sqrt{\frac{AH}{2C(1-\Theta)}}, \end{cases}$	$\Delta t < \frac{1}{1-\Theta} \left(\frac{H}{2\alpha U} \right)$

Table 2. Allowed range of time steps, Δt under which each finite difference scheme remains stable for all wavelengths. See Table 1 for acronym definitions.

The $u0h\Theta - UW$ scheme is stable for

$$\Delta t < \left(\frac{\Delta x}{U} \right) \frac{2 \left(1 + \frac{q\phi}{\beta} \right)}{(1 - \cos\theta) \left[\frac{q\phi}{\beta} + (1 - \Theta) \right]^2 + (1 + \cos\theta) [q\phi + (1 - \Theta)]^2 - 2\Theta^2}. \quad (27)$$

If $\Theta = 1$ and $\beta > \sqrt{q/(q+8)}$, the scheme is unconditionally stable.

225 The $u0h\Theta - CD$ scheme is stable for

$$\Delta t < \left(\frac{\Delta x}{U} \right) \frac{2 \left(\frac{q\phi}{\beta} \right)}{(1 - \cos\theta) \left(\frac{q\phi}{\beta} \right)^2 + (1 + \cos\theta) \{ [q\phi + (1 - \Theta)]^2 - \Theta^2 \}}. \quad (28)$$

For each scheme we find the allowed range of Δt in the limit of $\Delta x \rightarrow 0$ and $\Delta x \rightarrow \infty$. The limits on Δt are shown in Table 2. Both upwind schemes exhibits a CFL-type stability constraint in the limit of $\Delta x \rightarrow 0$, whilst the central difference schemes tend to a constant value of Δt in the limit when the slope of the bed is small.

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Figure 1 shows the maximum value of $P = U\Delta t/\Delta x$ for which each scheme remains stable as a function of $\beta = \alpha\Delta x/(2H)$. Each plot is for fixed Θ and $q = A(\Delta x)^2/(2HC)$ with q increasing from left to right and Θ increasing from top to bottom. Note that for $\Theta \geq 0.5$ the schemes that use the Θ -method to time step both thickness and velocity are unconditionally stable. For $\Theta = 0$, the schemes give the same result for the same spatial discretisation. For $\Theta < 0.5$ and small values of β , correspond-
 235 ing to $\alpha \ll H/\Delta x$, the maximum value of P is similar across all schemes, with the central difference scheme giving a slightly higher value of P than the upwind scheme. For larger values of β , upwind schemes are stable for much larger values of P than central difference schemes.

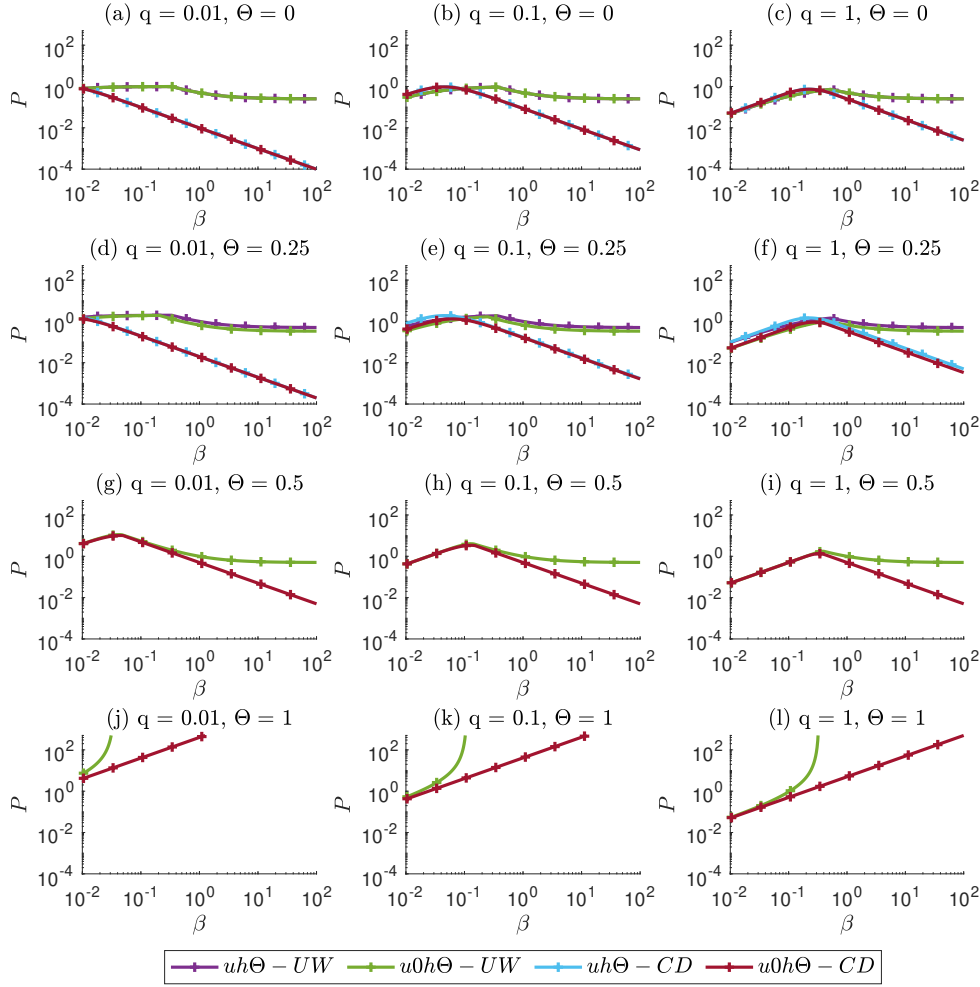


Figure 1. Maximum value of $P = U\Delta t/\Delta x$ for which each numerical scheme stays stable as a function of $\beta = \alpha\Delta x/(2H)$. Each plot is for fixed $q = A(\Delta x)^2/(2HC)$ and Θ with q increasing from left to right and Θ increasing from top to bottom. Each numerical scheme is plotted with different coloured lines. See Table 1 for acronym definitions. Formulas for the maximum allowed value of P are given in Sect. 5.3.

5.4 Accuracy analysis of the finite difference schemes

Having obtained a prediction for the conditions under which each of the numerical schemes remain stable we now assess the
 240 accuracy of each scheme.

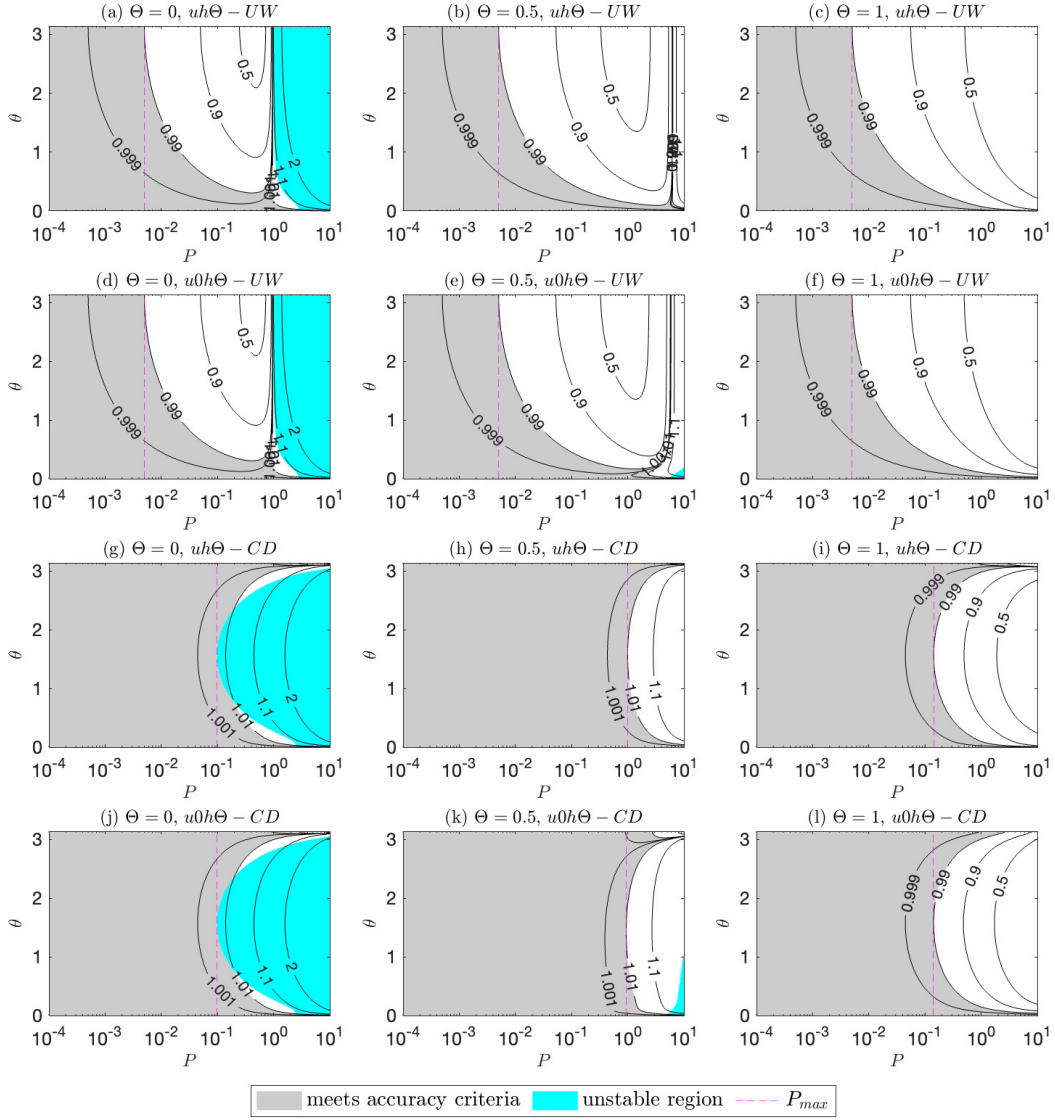


Figure 2. Ratio of the numerical and exact dispersion coefficients, $\Gamma_{\text{num}}/\Gamma_{\text{ex}}$ for the four different numerical schemes considered for $\beta = 0.1$ and $q = 0.01$. Regions where $0.99 \leq \Gamma_{\text{num}}/\Gamma_{\text{ex}} \leq 1.01$ are shaded grey and regions where the scheme is unstable are shaded in blue. The maximum value of P , P_{max} where the scheme is stable and meets the accuracy criteria for all $P \leq P_{\text{max}}$ is shown with the pink dashed line. The value of Θ is the same in each column and the numerical scheme is the same in each row. The numerical scheme and value of Θ are given in the title of each panel. See Table 1 for acronym definitions and (11)-(14) for definitions of θ, q, β and P .



5.4.1 Diffusion error: ratio of the dispersion coefficients

We first compare the ratio of the dispersion coefficients, $\Gamma_{\text{ex}}/\Gamma_{\text{num}}$. Figure 2 shows $\Gamma_{\text{num}}/\Gamma_{\text{ex}}$ for the four different numerical schemes considered for $\beta = 0.1$ and $q = 0.01$. Regions where the scheme is accurate ($0.99 \leq \Gamma/\Gamma_{\text{ex}} \leq 1.01$) are shaded grey and regions where the scheme is unstable are shaded in blue. The maximum value of P , P_{max} where the scheme is stable and
 245 meets the accuracy criteria for all $P \leq P_{\text{max}}$ is shown with the pink dashed line.

Excessive damping is observed if $\Gamma_{\text{num}}/\Gamma_{\text{ex}} < 1$ and under-diffusion is observed if $\Gamma_{\text{num}}/\Gamma_{\text{ex}} > 1$. For $\beta = 0.1$ and $q = 0.01$, the upwind schemes exhibit under-diffusion in the stable region, whilst the central difference schemes exhibit excessive damp-
 ing in the stable region for $\Theta = [0, 0.5]$ and under-diffusion for $\Theta = 1$. Comparing down the columns is clear that for $\beta = 0.1$
 250 and $q = 0.01$ that central difference schemes remain accurate for larger values of P than upwind schemes. There seems to be little variation in P_{max} across the different values of Θ for the upwind schemes, whilst the central difference schemes show larger values of P_{max} at $\Theta = 0.5$. Therefore, for $\beta = 0.1$ and $q = 0.01$ the central difference schemes with $\Theta = 0.5$ are accurate for the largest range of CFL numbers than the upwind schemes. Now we extend this analysis to a wider range of values of Θ , β and q .

255 We find P_{max} where the scheme remains stable and accurate ($0.99 \leq \Gamma_{\text{num}}/\Gamma_{\text{ex}} \leq 1.01$) when $P \leq P_{\text{max}}$. Figure 3a gives the minimum value of P when the optimum numerical scheme (the scheme with the highest P_{max}) is chosen for each value of q and β . The optimum scheme is shown in Fig. 3b. The optimum choice of Θ is approximately 0.5 for $q < 0.1$ and in most cases the $uh\Theta - CD$ scheme is favoured.

260 To test this conclusion P_{max} is plotted in Fig. 3c for $\Theta = 0.5$, under the $uh\Theta - CD$ scheme. Figure 3d plots P_{max} for $\Theta = 0.5$, under the upwind and $u0h\Theta - CD$ schemes. Figure 3e plots P_{max} for $\Theta = 0$ and Fig. 3f plots P_{max} for $\Theta = 1$. Comparing from Fig. 3c to Fig. 3f the value of P_{max} in Fig. 3c is less than an order of magnitude different to P_{max} in Fig. 3a for all values of β and q . By contrast, P_{max} in Fig. 3e for all schemes with $\Theta = 1$ and Fig. 3f for all schemes with $\Theta = 0$
 265 is at least an order of magnitude smaller for $q \leq 0.01$. Fig. 3d demonstrates that P_{max} is much larger for central difference spatial discretisations when $q < 0.01$. Comparing P_{max} from the $u0h\Theta - CD$ scheme in Fig. 3d to P_{max} from the $uh\Theta - CD$ scheme in Fig. 3c it is clear that the $uh\Theta - CD$ scheme has a much lower value of P_{max} when $\beta \geq 1$ or $q \geq 10^{-3}$.

For $q = [10^{-4}, 10^{-3}]$, the value of P_{max} is similar for all schemes. This is because when P_{max} is small the dispersion
 270 coefficients have similar values for all schemes ($\Gamma_{\text{num}} = 1 - P(1 + q\phi/\beta)(1 - \cos\theta) + \mathcal{O}(P^2)$ for all upwind schemes and $\Gamma_{\text{num}} = 1 - P(q\phi/\beta)(1 - \cos\theta) + \mathcal{O}(P^2)$ for all central difference schemes). Therefore, the value of P_{max} is similar across all schemes if P_{max} is small.

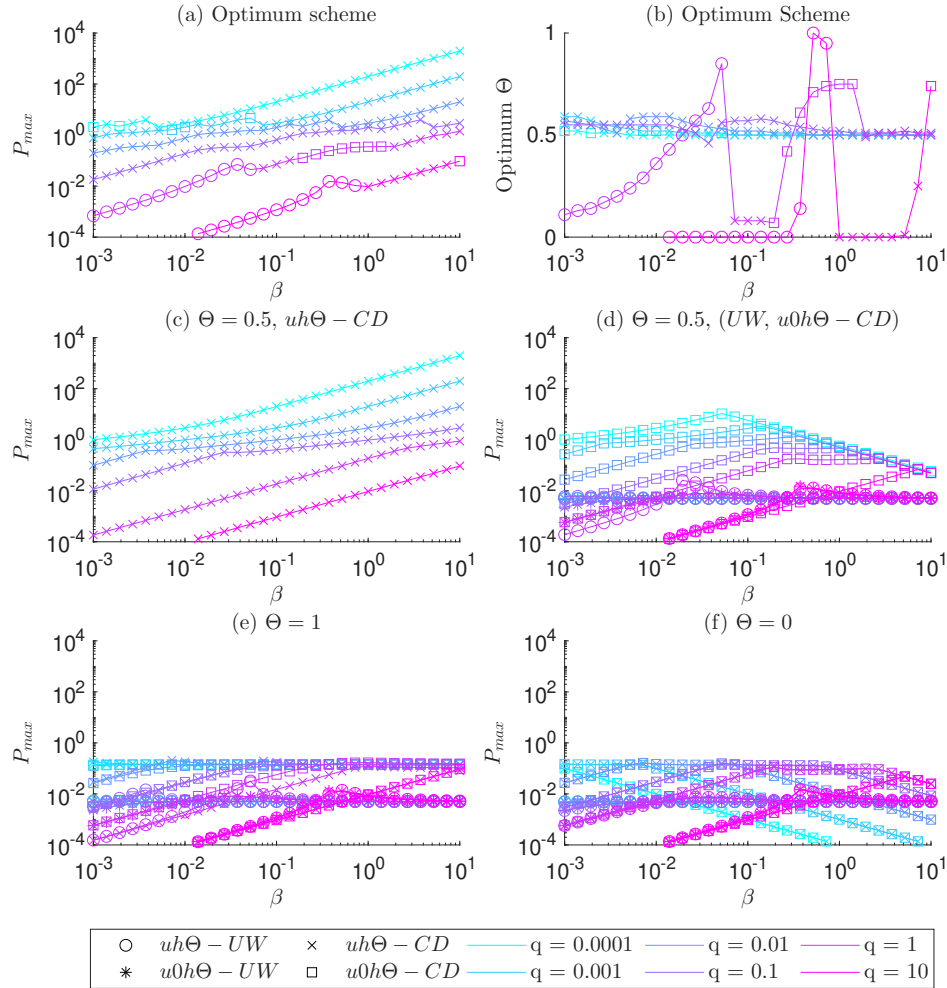


Figure 3. (a) Highest value of the CFL number P where the numerical scheme remains accurate ($0.99 \leq \Gamma_{num}/\Gamma_{ex} \leq 1.01$) and stable for all $P \leq P_{max}$. Different coloured lines correspond to different choices of $q = A(\Delta x)^2/(2HC)$, with the values of q given in the legend in (b). Different markers correspond to different numerical schemes with the name of the scheme also given in the legend. See Table 1 for acronym definitions. (b) value of Θ and the numerical scheme where the maximum value of P is obtained. (c-f) P_{max} for the numerical scheme(s) given in the title. If no schemes are listed, all schemes are included in the plot.

5.4.2 Dispersion error: ratio of the phase and group speeds

Next we compare the ratio of phase speeds, c_{ex}/c_{num} . Figure 4 shows c_{num}/c_{ex} for the four different numerical schemes considered for $\beta = 0.1$ and $q = 0.01$. Regions where $0.99 \leq c_{num}/c_{ex} \leq 1.01$ are shaded grey and regions where the scheme is unstable are shaded in blue. None of the values of P meet the accuracy criteria for all values of θ . For large values of θ the

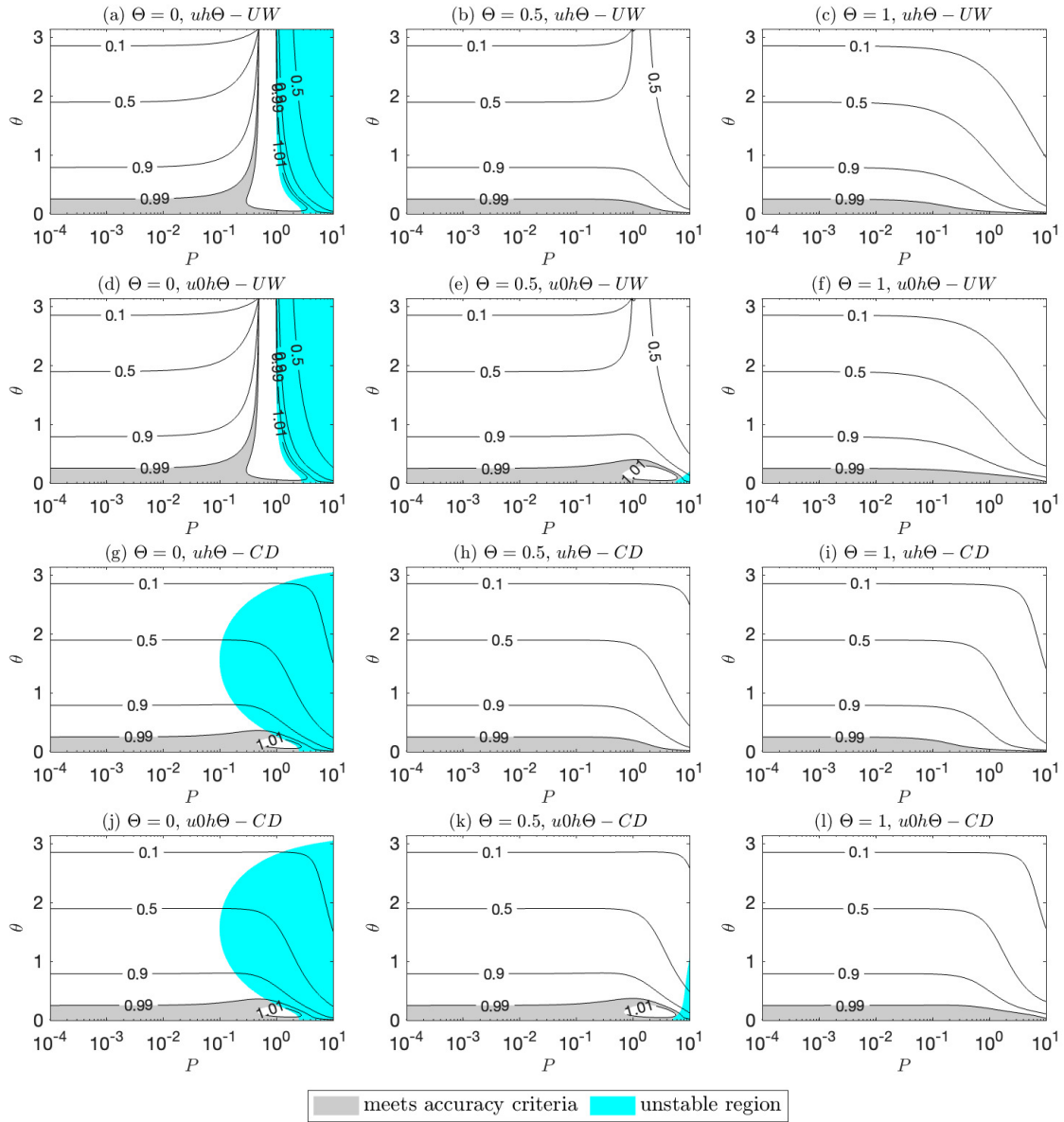


Figure 4. Same as Fig. 2. but for the ratio of the numerical and exact phase velocities, $c_{\text{num}}/c_{\text{ex}}$.



numerical phase speed is always lower than the physical phase speed.

For $P \ll 1$ the contours of constant $c_{\text{ex}}/c_{\text{num}}$ are at almost identical values of θ . Expanding Eq.(B2), Eq.(B4), Eq.(B6) and Eq.(B8) for small P we find that in all cases

$$\frac{c_{\text{num}}}{c_{\text{ex}}} = \frac{\sin \theta (1 + q\phi)(q + \theta^2)}{\theta(2q + \theta^2)} + \mathcal{O}(P), \quad (29)$$

where the $\mathcal{O}(P)$ term differs across schemes. Expanding the $\mathcal{O}(P^0)$ term in Eq.(29) in the limit of small and large q we find

$$\lim_{q \rightarrow 0} \left\{ \frac{\sin \theta (1 + q\phi)(q + \theta^2)}{\theta(2q + \theta^2)} \right\} = \frac{\sin \theta}{\theta} + \mathcal{O}(q), \quad (30)$$

$$\lim_{q \rightarrow \infty} \left\{ \frac{\sin \theta (1 + q\phi)(q + \theta^2)}{\theta(2q + \theta^2)} \right\} = \frac{\sin \theta}{\theta} + \mathcal{O}(q^{-1}). \quad (31)$$

Therefore, as q changes, there is little variation in the value of θ where $c_{\text{num}}/c_{\text{ex}} = 0.99$, which occurs close to $\theta = 0.24$ (Fig. 4). This means that wavelengths greater than $26\Delta x$ will propagate with phase speeds significantly lower than the physical phase speed.

Finally, we compare the ratio of the group speeds. Figure 5 shows $v_{g,\text{num}}/v_{g,\text{ex}}$ for the four different numerical schemes considered with $\beta = 0.1$ and $q = 0.01$. Regions where $0.99 \leq v_{g,\text{num}}/v_{g,\text{ex}} \leq 1.01$ are shaded grey and regions where the scheme is unstable are shaded in blue. None of the values of P meet the accuracy criteria for all values of θ . The numerical group velocity is always lower than the physical group velocity.

For $P \ll 1$ the contours of constant $v_{g,\text{num}}/v_{g,\text{ex}}$ are at almost identical values of θ . Expanding Eq.(B2), Eq.(B4), Eq.(B6) and Eq.(B8) for small P we find that in all cases

$$\frac{v_{g,\text{num}}}{v_{g,\text{ex}}} = \frac{2(2\cos \theta - q - 2q\cos^2 \theta + q^2\cos \theta - 4\cos^2 \theta + 2\cos^3 \theta + 3q\cos \theta)(q + \theta^2)^2}{(q - 2\cos \theta + 2)^2(2q^2 + \theta^2q + \theta^4)} + \mathcal{O}(P), \quad (32)$$

where the $\mathcal{O}(P)$ term differs across schemes. Expanding the $\mathcal{O}(P^0)$ term in Eq.(32) in the limit of small and large q we find

$$\lim_{q \rightarrow 0} \left\{ \frac{2(2\cos \theta - q - 2q\cos^2 \theta + q^2\cos \theta - 4\cos^2 \theta + 2\cos^3 \theta + 3q\cos \theta)(q + \theta^2)^2}{(q - 2\cos \theta + 2)^2(2q^2 + \theta^2q + \theta^4)} \right\} = \cos \theta + \mathcal{O}(q), \quad (33)$$

$$\lim_{q \rightarrow \infty} \left\{ \frac{2(2\cos \theta - q - 2q\cos^2 \theta + q^2\cos \theta - 4\cos^2 \theta + 2\cos^3 \theta + 3q\cos \theta)(q + \theta^2)^2}{(q - 2\cos \theta + 2)^2(2q^2 + \theta^2q + \theta^4)} \right\} = \cos \theta + \mathcal{O}(q). \quad (34)$$

Therefore, as q changes, there is little variation in the value of θ where $v_{g,\text{num}}/v_{g,\text{ex}} = 0.99$, which occurs close to $\theta = 0.14$. Further, for $\theta \geq \pi/2$ the numerical group velocity is negative. For these wavelengths we observe upstream propagating, non-physical waves, potentially resulting in non-physical flow events (Sengupta et al., 2022).

Given that no finite difference numerical scheme meets the accuracy criteria for the dispersion relation for all values of θ , it is hard to determine which finite difference scheme is optimum based on its dispersive properties. Instead, we have found

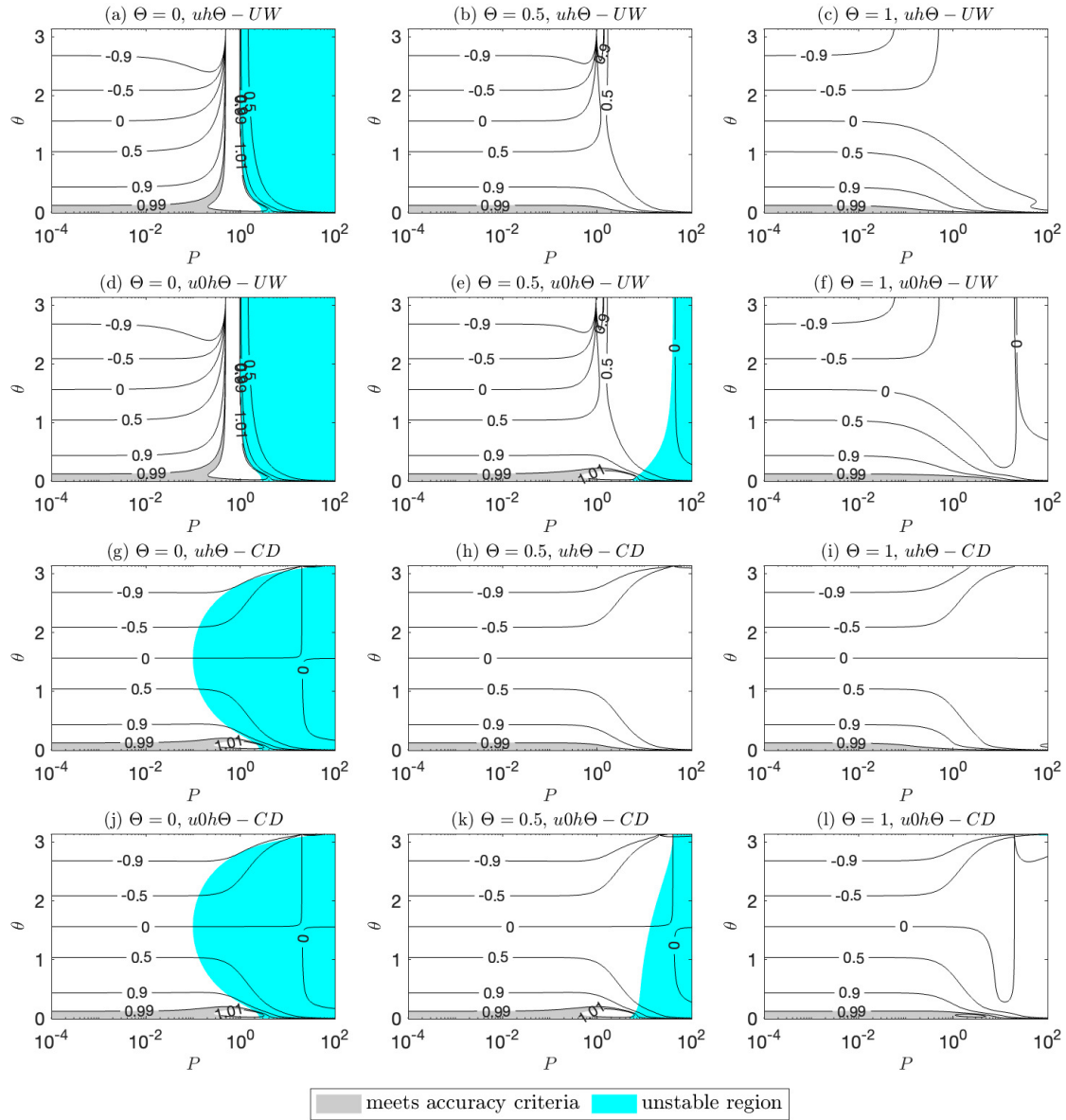


Figure 5. Same as Fig. 2. but for ratio of the numerical and exact group velocities, $v_{g,num}/v_{g,ex}$.



that for $\theta \gtrsim 0.14$ each numerical scheme fails to meet the accuracy criteria. Therefore, it is beneficial to run simulations where there is significant damping at values of θ where the schemes do not meet the accuracy criteria. Various solutions for this have been proposed in the literature such as introducing an artificial high-wavenumber viscosity to damp these nodes (Cook and Cabot, 2004) or by spectral filtering (Lele, 1992). A full discussion of the optimum approach to damp high-wavenumber Fourier modes is beyond the scope of this paper.

6 Testing with a finite element model

We now perform a stability and accuracy analysis using $\hat{U}a$, a finite-element ice-flow model, which is written in Matlab (Gudmundsson, 2026). In $\hat{U}a$ the equations for conservation of momentum and conservation of mass are expressed in their weak form. For grounded ice with power flow law exponent $n = 1$, uniform flow in the y -direction and assuming a Weertman sliding law with sliding exponent $m = 1$. The equations are,

$$\iint dS \left[\rho_i \frac{h_1 - h_0}{\Delta t} + (1 - \Theta) \partial_x (\rho_i h_0 u_0) + \Theta \partial_x (\rho_i h_1 u_1) - \rho_i a \right] \left[\Phi + \frac{1}{2} \tau u_0 \partial_x \Phi \right] = 0, \quad (35)$$

$$\iint dS \left[\partial_x \left(\frac{h_1}{2A} \partial_x u_1 \right) - C^{-1} u_1 - \frac{1}{2} g \partial_x (\rho_i h_1^2) - g \rho_i h_1 \partial_x B \right] \Phi = 0. \quad (36)$$

where $\Phi = \Phi(x, y)$ is the test function, B is the depth of the bedrock and equivalently the depth of the ice for grounded ice, $h_1 = h_1(x, y)$ is the thickness at time $t + \Delta t$, $h_0 = h_0(x, y)$ is the thickness at time t , and likewise with u_1 and u_0 . In this case we have assumed that the density and the rate of accumulation minus the rate of melting of ice remain constant over time but $\hat{U}a$ is capable of solving the equations when this scenario does not hold. The equation for conservation of mass has a streamline upwind Petrov–Galerkin (SUPG) stabilisation term (Brooks and Hughes, 1982; Jia and Esmaily, 2023), which is weighted by τ . Therefore, the equation for conservation of mass has an upwind weighting. The default in $\hat{U}a$ is to set $\tau = \sqrt{2A_{ele}} / (2\sqrt{u_0^2 + v_0^2 + 10^{-8}} + 10^{-4})$ where A_{ele} is the area of each element and v_0 is the velocity in the y -direction at time t .

In $\hat{U}a$ the default approach is the monolithic, uh , scheme in which the equations for mass (35) and momentum (36) conservation are solved simultaneously for h_1 and u_1 . Users can choose the value of Θ in (35), with the default value of Θ being 0.5. If Θ is set to one, the monolithic approach can be thought of as a fully implicit approach. A multistep, $u - h$, approach is also available and was used here for the purposes of comparison. In the multistep approach (35) and (36) are solved separately. The steps in the multistep approach are as follows,

1. Assume $u_1 = u_0$ and solve (35) for h_1 .
2. Using h_1 from step 1 solve (36) for u_1 .
3. Repeat step 1 but this time using u_1 from step 2.



335 4. Repeat steps 1-3 until the change in u_1 is below a certain threshold or until a maximum number of iterations has been reached.

We find the conditions for the finite element model to remain accurate when $\Theta = [0, 0.5, 1]$. We test both the monolithic and the multistep approaches. To closely match the $u_0 h \Theta$ finite difference approach, for the multistep approach we only perform steps 1-2 such that h_1 only depends on u_0 and we have a semi-implicit approach. We perform tests with $\tau = 0$ and with the
 340 default value of τ to assess the impact of adding the upwind weighting. A summary of the numerical schemes that we consider is listed in Table 3.

Acronym	time stepping	τ in (35)
$uh(SUPG)$	Monolithic	$\sqrt{2A_{ele}} / (2\sqrt{u_0^2 + v_0^2 + 10^{-8} + 10^{-4}})$
$u - h(SUPG)$	Multistep	$\sqrt{2A_{ele}} / (2\sqrt{u_0^2 + v_0^2 + 10^{-8} + 10^{-4}})$
$uh(\tau = 0)$	Monolithic	0
$u - h(\tau = 0)$	Multistep	0

Table 3. Finite element numerical schemes considered in this study. For all schemes we consider $\Theta = [0, 0.5, 1]$.

6.1 Finite element model setup

We use a square mesh with periodic boundary conditions in x and y . We set the velocity in the y -direction to be zero at the boundary. Noting that the elements are triangular, we define Δx as the shortest distance between nodes in the x -direction such
 345 that the area of each element is given by $(\Delta x)^2$. The length L of the mesh in the x -direction is 8000km and $4\Delta x$ in the y -direction. We apply a perturbation $\delta h = 0.01H \cos(kx)$. We fix g, C, H, α and ρ_i with the values given in Table A1. Changing the resolution Δx alters the dimensionless parameters β and q . Therefore, in order to obtain comparable values of q to those tested for the finite difference schemes, we also change A . We seek perturbations that satisfy the periodic boundary conditions. Therefore, we require an integer number of wavelengths to fit exactly between the boundaries. This is met for $kL = 2p\pi$ where
 350 p is a positive integer. Therefore, we choose $1 \leq p \leq L/(8\Delta x)$ such that $2\pi\Delta x/L \leq \theta \leq \pi/4$. Following Donea and Huerta (2003) the lower limit on θ is chosen because above this value the perturbation is poorly resolved as there are fewer than eight elements per wavelength.

6.2 Stability and accuracy analysis of the finite element schemes

6.2.1 Diffusion error: ratio of the dispersion and diffusion coefficients

355 We first compare the ratio of the dispersion and diffusion coefficients for $q = 0.01$ and $\beta = 0.1$. Figure 6 shows $\Gamma_{\text{num}}/\Gamma_{\text{ex}}$ for the different time stepping schemes listed in Table 3. There is a region of instability at large values of P when $\Theta = 0$ for all schemes with the unstable region being the largest for the $u - h(\tau = 0)$ scheme. As is expected from the implicit nature of the scheme, the monolithic scheme is stable at all values of P when $\Theta = [0.5, 1]$. The multistep scheme is only stable for $\Theta = 1$.

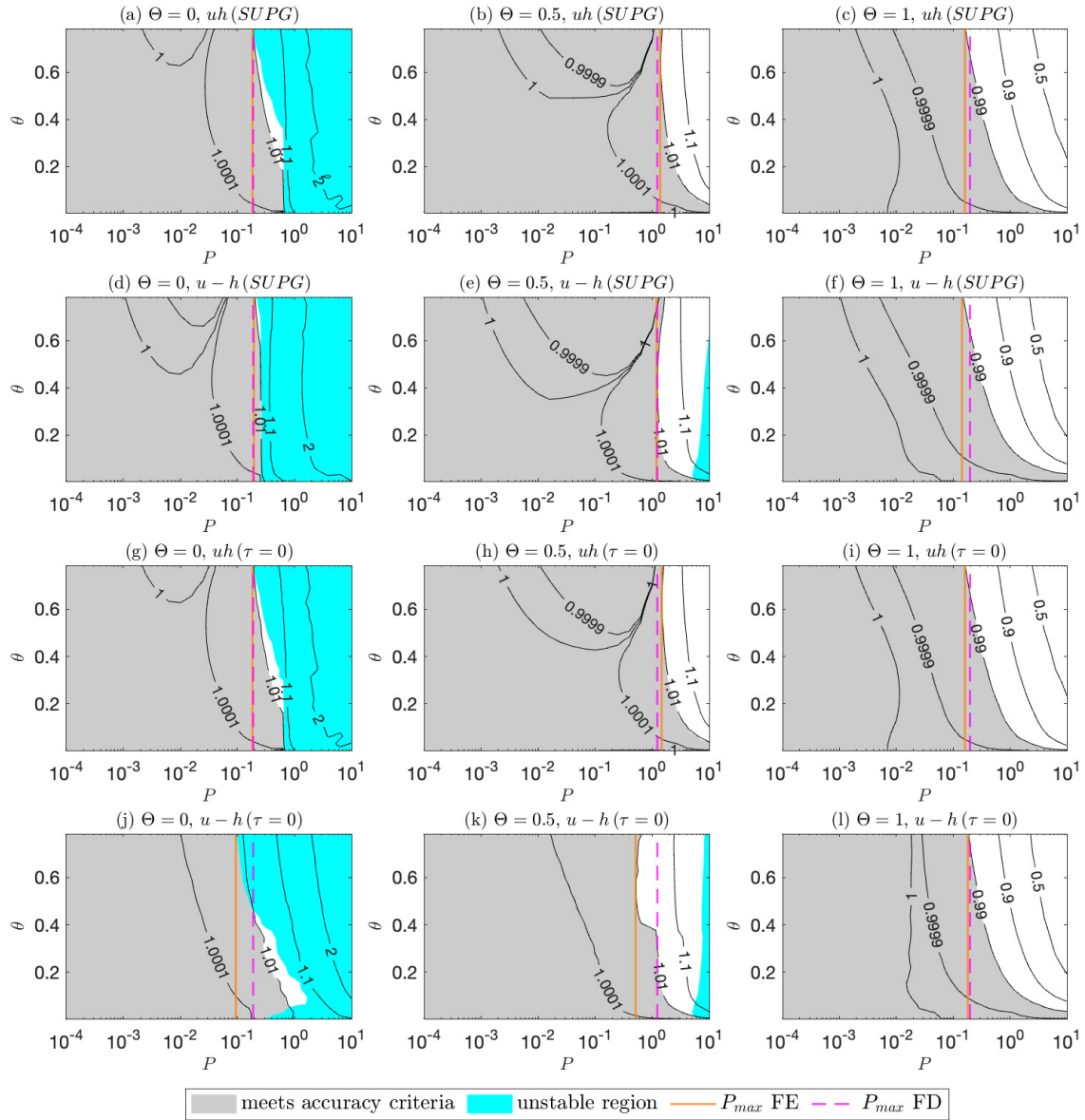


Figure 6. Ratio of the numerical and exact dispersion coefficients, Γ_{num}/Γ_{ex} for the two different finite element numerical schemes considered with $\beta = 0.1$ and $q = 0.01$. Regions where $0.99 \leq \Gamma/\Gamma_{ex} \leq 1.01$ are shaded grey and regions where the scheme is unstable are shaded in blue. The maximum value of P , P_{max} where the finite element scheme is stable and meets the accuracy criteria for all $P \leq P_{max}$ is shown with the yellow solid line. For each value of Θ , the highest value of P_{max} across the four finite difference schemes is shown with the pink dashed line. The value of Θ is the same in each column and the numerical scheme is the same in each row. The numerical scheme and value of Θ are given in the title of each panel. See Table 3 for acronym definitions and (11)-(14) for definitions of θ, q, β and P .



The multistep scheme shows more sensitivity to the removal of the SUPG stabilisation term. For the same value of Θ , the
 360 monolithic and multistep schemes are accurate for very similar ranges of CFL numbers. The exception is the $u - h$ ($\tau = 0$)
 scheme which is stable and accurate for a small range of CFL numbers than the other schemes at $\Theta = 0.5$.

As with the finite difference schemes, Fig. 6 demonstrates that the schemes meet the accuracy criteria for the ratio of the
 dispersion coefficients for the largest range of CFL numbers when $\Theta = 0.5$. To compare the values of P_{max} obtained for the
 365 the finite element and finite difference schemes, we restrict the range to $\theta \leq \pi/4$ for all schemes. Comparing the positions of
 the pinked dashed lines with the yellow solid lines, we note that P_{max} is very similar for the monolithic finite element and
 central difference finite difference schemes.

6.2.2 Dispersion error: ratio of the phase speeds

Before extending our analysis to consider a larger range of values of β and q , we calculate the dispersion error. We only cal-
 370 culate the ratio of the phase speeds as the group velocity is a derived quantity. Figure 7 shows c_{num}/c_{ex} for the different time
 stepping schemes listed in Table 3. Noting that c_{num}/c_{ex} does not meet the accuracy criteria for all values of P , we instead find
 P_{min} , the first value of P where $0.99 \leq c_{num}/c_{ex} \leq 1.01$ at all tested values of θ and P_{max} , the first value of where P does not
 meet the accuracy criteria for $P \leq P_{max}$. The values of P_{min} and P_{max} are marked with the pink dashed lines.

375 At the same value of Θ , all but the $u - h$ ($\tau = 0$) scheme show similar behaviour. Except for when $P \lesssim 10^{-3}$ the $u - h$ ($\tau = 0$)
 is accurate for a smaller range of values of θ than the other schemes. The monolithic scheme is accurate for the largest range
 of values of θ and P , with $\Theta = 0.5$ again being the optimum choice of Θ .

Comparing Fig. 4 and Fig. 7 we note that the finite element schemes meet the accuracy criteria for a much larger range of
 380 values of P and θ than the finite difference schemes. All schemes tend to underestimate the phase velocity for $P \gtrsim 1$. The finite
 difference schemes tend to also underestimate the phase velocity for $P \lesssim 10^{-3}$ whilst the finite element schemes overestimate
 the phase velocity in the same region.

6.3 Optimum finite element scheme

Finally, we find the range of values of P where each finite element scheme is stable and meets the accuracy criteria for the ratio
 385 of the phase velocities and the ratio of the dispersion coefficients ($0.99 \leq c_{num}/c_{ex} \leq 1.01$ and $0.99 \leq \Gamma_{num}/\Gamma_{ex} \leq 1.01$). The
 range of values is shown in Fig. 8 with the scheme with the largest range of P at each value of β and q highlighted in each panel.
 In all cases the optimum scheme uses SUPG stabilisation. For $q \leq 10^{-3}$ the uh scheme with $\Theta = 0.5$ is optimum for all tested
 values of β . As q increases with β fixed (corresponding to increasing basal friction) the optimum scheme is instead the $u - h$
 scheme with Θ either 0.5 or 1. For $\beta = 0.01$ and $q = 0.1$ (corresponding to high spatial resolution simulations with high basal
 390 friction) no scheme is able to meet the accuracy and stability criteria and we found that the $u - h$ schemes became unstable
 for sufficiently large P . These difficulties in meeting the stability and accuracy criteria can be solved by increasing the mesh

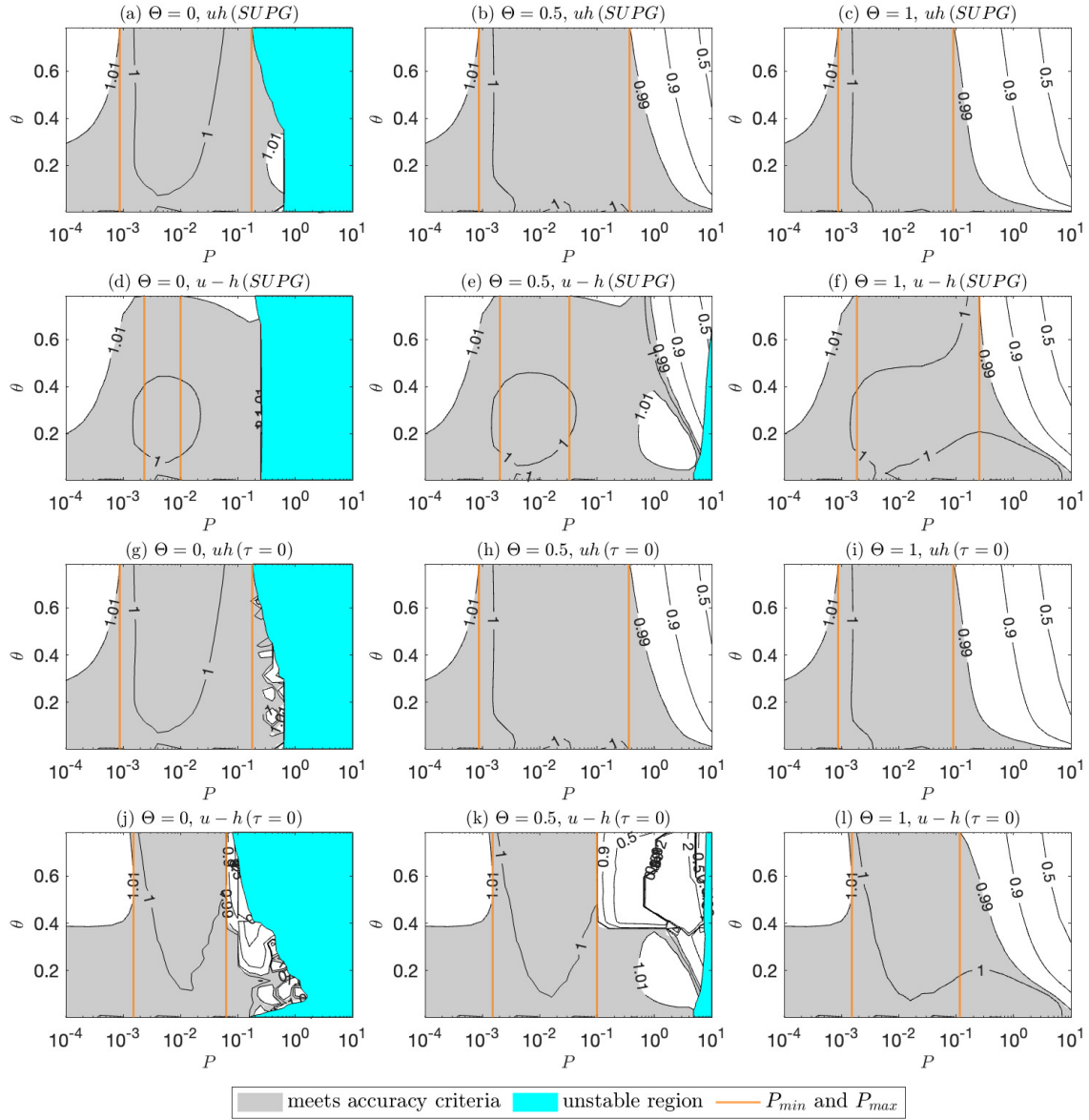


Figure 7. Same as Fig. 6 but for but for the ratio of the numerical and exact phase velocities $c_{\text{num}}/c_{\text{ex}}$. The yellow lines give the range of CFL numbers there the scheme remains accurate for all tested values of θ .

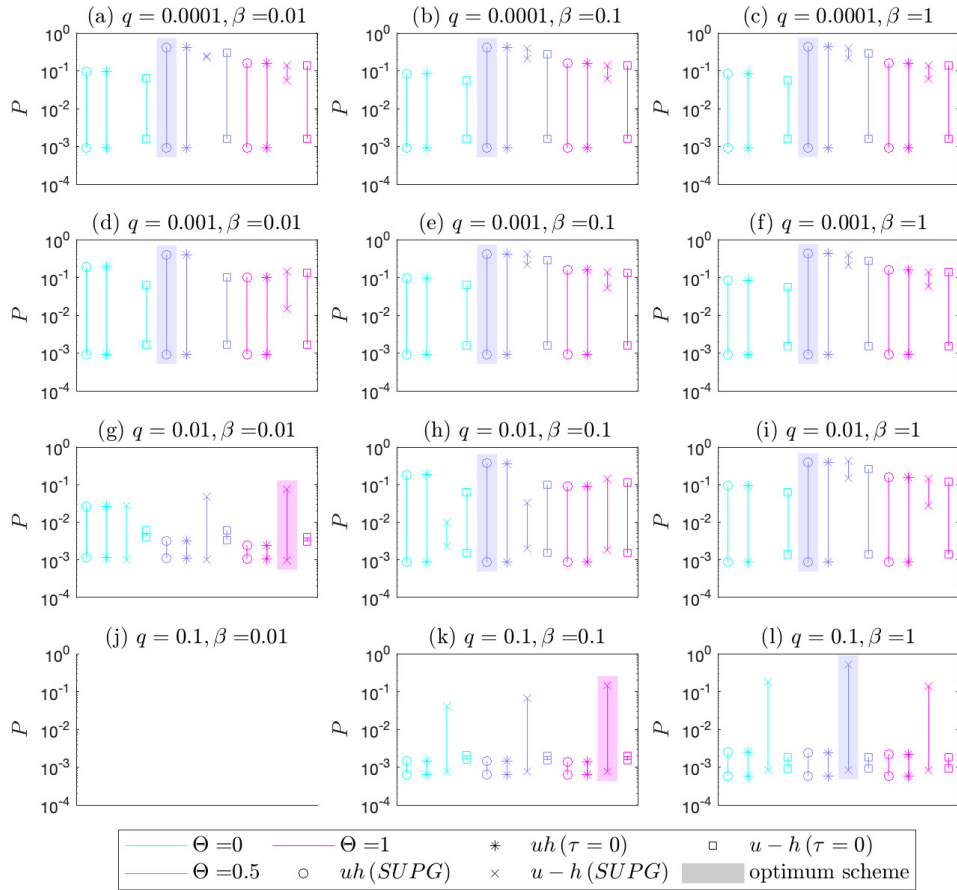


Figure 8. Range of CFL numbers, P where each finite element scheme is stable ($\Gamma_{\text{num}} \leq 1$) and meets the accuracy criteria for the ratio of the phase velocities and the ratio of the dispersion coefficients ($0.99 \leq c_{\text{num}}/c_{\text{ex}} \leq 1.01$ and $0.99 \leq \Gamma_{\text{num}}/\Gamma_{\text{ex}} \leq 1.01$). Each panel is for a different value of q and β with the value of q and β given in the panel title. The colour of each line indicates the value of Θ and the marker indicates the numerical scheme used to obtain the data. The scheme with the largest range of CFL numbers for each panel is highlighted. In cases where the scheme fails to meet the accuracy criteria, the plot is left blank. See Table 3 for acronym definitions and (11)-(14) for definitions of θ, q, β and P .



resolution and hence decreasing q . Notably, for $q \leq 0.01$, the $u - h(SUPG)$ scheme fulfils the accuracy and stability criteria for a significantly smaller range of values of P than the $uh(SUPG)$ scheme suggesting that inaccuracies may be introduced when this scheme is chosen.

395 7 Conclusions

We have found the criteria under which different numerical schemes remain stable and accurate when they are used to solve the equations derived from the shallow-shelf approximation. We used a simple geometry in which an exact solution can be found for the evolution of the ice sheet: a small amplitude perturbation was applied to an ice sheet of uniform thickness, density and viscosity. We investigated finite difference and finite element schemes. For each finite difference numerical scheme an equation
 400 for the time-dependent evolution of the perturbation was obtained. From this equation we calculated how the phase velocity, group velocity and dispersion coefficient depended on the wavelength of the perturbation. For each finite element scheme we numerically calculated the phase velocity and dispersion coefficient for a discrete range of wavelengths. Schemes where the dispersion coefficient was greater than one for any wavelength were deemed unstable.

405 For the finite difference schemes investigated, we found that schemes employing explicit or semi-implicit time-stepping were unstable at large time steps where, as the distance between nodes approaches zero, the criteria for an upwind scheme to remain stable approaches a CFL-type stability constraint. Schemes which were fully implicit or used the Θ -method of time-stepping for both velocity and thickness were unconditionally stable for $\Theta \geq 0.5$. For the finite element schemes investigated, the monolithic schemes were stable at all tested time steps for $\Theta \geq 0.5$, whilst the semi-implicit multistep schemes were un-
 410 stable for large time steps even with $\Theta = 1$.

In general, both the finite element and finite difference schemes met the accuracy criteria for the dispersion coefficient for the largest range of time steps when using the Lax-Wendroff method for time-stepping ($\Theta = 0.5$). Unlike the finite element schemes, the finite difference schemes failed to meet the accuracy criteria for the phase velocity at all wavelengths. For all
 415 tested finite difference schemes, the group velocity became negative for $k\Delta x \geq \pi/2$, potentially leading to non-physical flow events. The tested finite element schemes were able to meet the accuracy criteria at all tested wavelengths, with the monolithic finite element scheme with $\Theta = 0.5$ remaining accurate for the greatest range of time steps. This suggests that from the tested schemes, this scheme is optimum.

420 The purpose of this paper was to assess the conditions under which each scheme is stable and accurate. Therefore, we did not consider how the CPU time depends on the numerical scheme. Further work would seek to make this comparison and also extend this work to larger-amplitude perturbations involving possible grounding-line migration. It is likely that the differences between numerical schemes will be amplified in this scenario.



Appendix A: Nomenclature

425 Table A1 gives a list of all the parameters used in this study along with their value and units.

Table A1: Nomenclature. The value given to C, H, α, ρ_i is the value we use in the numerical simulations in Sect. 6.

Physical meaning	Symbol	Value/Equation	Unit
Temperature-dependent rate factor in Glen's flow law	A		mskg^{-1}
Sliding law parameter	C	$1/(\alpha \rho_i g)$	$\text{m}^2 \text{kg}^{-1} \text{s}^{-1}$
Background thickness	H	1000	m
CFL number	P	$U \Delta t / \Delta x$	
Background velocity	U	$\alpha \rho_i g H C$	m
Net accumulation minus net melting of ice	a	0	ms^{-1}
Exact phase velocity	c_{ex}	$U(2q + \theta^2)/(q + \theta^2)$	ms^{-1}
Numerical phase velocity	c_{num}	$\psi/(k_{\text{num}} \Delta t)$	ms^{-1}
Acceleration due to gravity	g	9.81	ms^{-2}
Ice thickness	h	$H + \delta h$	m
Wavenumber of the exact perturbation	k		m^{-1}
Wavenumber of the numerical perturbation	k_{num}		m^{-1}
Ice velocity	u	$U + \delta u$	ms^{-1}
	q	$A(\Delta x)^2/(2HC)$	
Exact group velocity	$v_{g,\text{ex}}$	$U(2q^2 + \theta^2 q + \theta^4)/(q + \theta^2)^2$	ms^{-1}
Numerical group velocity	$v_{g,\text{num}}$	$(\Delta x / \Delta t)(\partial \psi / \partial t)$	ms^{-1}
	β	$\alpha \Delta x / (2H)$	
Slope of bed	α	0.05	radians
Exact diffusion coefficient	α_{ex}	$qCH^2 \rho_i g / (q + \theta^2)$	$\text{m}^2 \text{s}^{-1}$
Numerical diffusion coefficient	α_{num}	Eq.()	$\text{m}^2 \text{s}^{-1}$
Exact thickness perturbation	δh		m
Numerical thickness perturbation	δh_{num}		m
Exact velocity perturbation	δu		m
Numerical velocity perturbation	δu_{num}		m
Numerical time step	Δt		s
Distance between adjacent nodes	Δx		m

Continued on next page



Physical meaning	Symbol	Value/Equation	Unit
Decay in amplitude of exact perturbation per time step	Γ_{ex}	$\exp\{-(P\theta^2 q)/[2\beta(\theta^2 + q)]\}$	
Decay in amplitude of numerical perturbation per time step	Γ_{num}	Eq.(B1), Eq.(B3), Eq.(B5), Eq.(B7)	
Change in phase of the numerical perturbation per time step	ϕ ψ	$(2 + q - 2\cos\theta)^{-1}$ Eq.(B2), Eq.(B4), Eq.(B6), Eq.(B8)	
Density of ice	ρ_i	920	kgm^{-3}
Dimensionless wavenumber of the perturbation	θ	$k\Delta x$	
Theta in Θ -method	Θ	$0 \leq \Theta \leq 1$	
Angular frequency of the exact perturbation	ω	Eq. (10), Eq.(9)	s^{-1}
Angular frequency of the numerical perturbation	ω_{num}		s^{-1}

Appendix B: Expressions for the dispersion coefficient and angular frequency for each finite difference numerical scheme

In this appendix we give expressions for the dispersion coefficient and angular frequency for each finite difference numerical scheme. The expressions are obtained by substituting in Eq.(15)-Eq.(13) into Eq.(17)-Eq.(20) to find an expression for $\exp(-i\omega_{\text{num}}\Delta t)$ for each numerical scheme. We find expressions for $\psi = \omega_{\text{num},R}\Delta t$ and $\Gamma_{\text{num}} = \exp(\omega_{\text{num},I}\Delta t)$.

For the $uh\Theta - UW$ scheme,

$$\Gamma_{\text{num}} = \left\{ \frac{1 - 2P(1 - \Theta)(1 - \cos\theta) \left(1 + \frac{q\phi}{\beta}\right) + P^2(1 - \Theta)^2 \left[(1 - \cos\theta)^2 \left(1 + \frac{q\phi}{\beta}\right)^2 + \sin^2\theta (1 + q\phi)^2 \right]}{1 + 2P\Theta(1 - \cos\theta) \left(1 + \frac{q\phi}{\beta}\right) + P^2\Theta^2 \left[(1 - \cos\theta)^2 \left(1 + \frac{q\phi}{\beta}\right)^2 + \sin^2\theta (1 + q\phi)^2 \right]} \right\}^{1/2}, \quad (\text{B1})$$

$$\psi = \arctan \left[\frac{P(1 - \Theta)\sin\theta(1 + q\phi)}{1 - (1 - \Theta)(1 - \cos\theta)P \left(1 + \frac{q\phi}{\beta}\right)} \right] + \arctan \left[\frac{\Theta P \sin\theta(1 + q\phi)}{1 + \Theta(1 - \cos\theta)P \left(1 + \frac{q\phi}{\beta}\right)} \right]. \quad (\text{B2})$$



For the $uh\Theta - CD$ scheme,

$$\Gamma_{\text{num}} = \left\{ \frac{1 - 2P(1 - \Theta)(1 - \cos\theta) \left(\frac{q\phi}{\beta}\right) + P^2(1 - \Theta)^2 \left[(1 - \cos\theta)^2 \left(\frac{q\phi}{\beta}\right)^2 + \sin^2\theta(1 + q\phi)^2 \right]}{1 + 2P\Theta(1 - \cos\theta) \left(\frac{q\phi}{\beta}\right) + P^2\Theta^2 \left[(1 - \cos\theta)^2 \left(\frac{q\phi}{\beta}\right)^2 + \sin^2\theta(1 + q\phi)^2 \right]} \right\}^{1/2}, \quad (\text{B3})$$

$$\psi = \arctan \left[\frac{P(1 - \Theta)\sin\theta(1 + q\phi)}{1 - (1 - \Theta)(1 - \cos\theta)P \left(\frac{q\phi}{\beta}\right)} \right] + \arctan \left[\frac{\Theta P \sin\theta(1 + q\phi)}{1 + \Theta(1 - \cos\theta)P \left(\frac{q\phi}{\beta}\right)} \right]. \quad (\text{B4})$$

For the $u0h\Theta - UW$ scheme,

$$440 \quad \Gamma_{\text{num}} = \left\{ \frac{1 - 2P \left[\frac{q\phi}{\beta} + (1 - \Theta) \right] (1 - \cos\theta) + P^2 \left\{ (1 - \cos\theta)^2 \left[\frac{q\phi}{\beta} + (1 - \Theta) \right]^2 + \sin^2\theta[q\phi + (1 - \Theta)]^2 \right\}}{2(1 - \cos\theta)(P^2\Theta^2 + P\Theta) + 1} \right\}^{1/2}, \quad (\text{B5})$$

$$\psi = \arctan \left[\frac{P \sin\theta(q\phi + 1 - \Theta)}{1 - P(1 - \cos\theta) \left(\frac{q\phi}{\beta} + 1 - \Theta\right)} \right] + \arctan \left[\frac{\Theta P \sin\theta}{1 + \Theta P(1 - \cos\theta)} \right]. \quad (\text{B6})$$

For the $u0h\Theta - CD$ scheme,

$$\Gamma_{\text{num}} = \left\{ \frac{\left[1 - \frac{qP\phi}{\beta}(1 - \cos\theta) \right]^2 + \sin^2\theta P^2 [q\phi + (1 - \Theta)]^2}{P^2\Theta^2 \sin^2\theta + 1} \right\}^{1/2}, \quad (\text{B7})$$

$$\psi = \arctan \left[\frac{P \sin\theta(q\phi + 1 - \Theta)}{1 - \frac{qP\phi}{\beta}(1 - \cos\theta)} \right] + \arctan[\Theta P \sin\theta]. \quad (\text{B8})$$

445 Appendix C: Determining the stability conditions for each of the numerical schemes

In this appendix we determine the criteria for each of the numerical schemes to be stable. A numerical scheme is stable if $\Gamma(\theta) \leq 1$ for all θ .

450 Firstly, we consider the $uh\Theta - UW$ scheme. Solving to find the values of P where the equation for $\Gamma(\theta)^2$, Eq.(B1), is less than one, we obtain

$$P^2(1 - 2\Theta) \left[(1 - \cos\theta)^2 \left(1 + \frac{q\phi}{\beta} \right)^2 + \sin^2\theta(1 + q\phi)^2 \right] - 2P \left(1 + \frac{q\phi}{\beta} \right) (1 - \cos\theta) < 0. \quad (\text{C1})$$

If $\Theta \geq 0.5$ the scheme is stable for $P > 0$ i.e. the scheme is unconditionally stable. If $\Theta < 0.5$ the scheme is stable for

$$P < \frac{1}{1 - 2\Theta} \left[\frac{2 \left(1 + \frac{q\phi}{\beta} \right)}{(1 - \cos\theta) \left(1 + \frac{q\phi}{\beta} \right)^2 + (1 + \cos\theta)(1 + q\phi)^2} \right] \quad (\text{C2})$$



In the limit where $\beta \ll q/[2(q^2 + 4q + 1)^{1/2} - q - 2]$, the minimum value of P for the scheme to remain stable is found at $\theta = \pi$. Therefore, if $\Gamma(\theta = \pi) < 1$, then the scheme is stable for all values of θ . Substituting $\theta = \pi$ into Eq.(C2) and noting that $\phi(\theta = \pi) = 1/(4 + q)$ we find

$$P < \frac{1}{1 - 2\Theta} \left[\frac{1}{1 + \frac{q}{\beta(4+q)}} \right]. \quad (C3)$$

In the limit where $\beta \gg q/[2(q^2 + 4q + 1)^{1/2} - q - 2]$, the minimum value of P for the scheme to remain stable is found at $\theta = 0$. Therefore, if $\Gamma(\theta = 0) < 1$, then the scheme is stable for all values of θ . Substituting $\theta = 0$ into Eq.(C2) and noting that $\phi(\theta = 0) = 1/q$ we find

$$P < \frac{1}{1 - 2\Theta} \left[\frac{(1 + 1/\beta)}{4} \right]. \quad (C4)$$

In the intermediate limit, the maximum stable value of P has to be determined numerically.

Secondly, we consider the $uh\Theta - CD$. Solving to find the values of P where the equation for $\Gamma(\theta)^2$, Eq.(B3) is less than one, we obtain,

$$P^2(1 - 2\Theta) \left[(1 - \cos\theta)^2 \left(\frac{q\phi}{\beta} \right)^2 + \sin^2\theta (1 + q\phi)^2 \right] - 2P \left(\frac{q\phi}{\beta} \right) (1 - \cos\theta) < 0. \quad (C5)$$

If $\Theta \geq 0.5$ the scheme is stable for $P > 0$ i.e. the scheme is unconditionally stable. If $\Theta < 0.5$ the scheme is stable for

$$P < \frac{1}{1 - 2\Theta} \left[\frac{2 \left(\frac{q\phi}{\beta} \right)}{(1 - \cos\theta) \left(\frac{q\phi}{\beta} \right)^2 + (1 + \cos\theta) (1 + q\phi)^2} \right]. \quad (C6)$$

In the limit when $\beta \ll 1/2\sqrt{q/(4 + q)}$, if $\Gamma(\theta = \pi) < 1$, then the scheme is stable for all values of θ . This condition gives

$$P < \frac{1}{1 - 2\Theta} \frac{\beta(q + 4)}{q}. \quad (C7)$$

In the limit when $q \leq -1 + \sqrt{2}$ and $\beta \gg 1/2\sqrt{q/(4 + q)}$, if $\Gamma(\theta = \pi/2) < 1$, then the scheme is stable for all values of θ . This condition gives

$$P < \frac{1}{1 - 2\Theta} \left(\frac{1}{2\beta} \right) \left[\frac{q(2 + q)}{(1 + q)^2} \right]. \quad (C8)$$

In the limit when $q > -1 + \sqrt{2}$ and $\beta \gg 1/2\sqrt{q/(4 + q)}$, if $\Gamma(\theta = 0) < 1$, then the scheme is stable for all values of θ . This condition gives

$$P < \frac{1}{1 - 2\Theta} \left[\frac{1}{4\beta} \right]. \quad (C9)$$

In the intermediate cases the limit on P must be determined numerically.



Thirdly, we consider the $u0h\Theta - UW$. First, we consider the limit of $P \rightarrow \infty$ and $\theta = \pi$. In this limit, the expression for
480 $\Gamma(\theta)$ Eq.(B5) tends to

$$|\Gamma(\pi)| \rightarrow \frac{2}{\Theta} - 1. \quad (C10)$$

Therefore, if $\Theta < 1$, then the scheme is not unconditionally stable. Solving to find the values of P where the equation for $\Gamma(\theta)^2$, Eq.(B5) < 1 we obtain

$$P < \frac{2 \left(1 + \frac{q\phi}{\beta}\right)}{(1 - \cos \theta) \left[\frac{q\phi}{\beta} + (1 - \Theta)\right]^2 + (1 + \cos \theta)[q\phi + (1 - \Theta)]^2 - 2\Theta^2}. \quad (C11)$$

485 In the limit when $\Theta = 0$, we recover the limits for the Θ -method with the upwind scheme for $\Theta = 0$. For $\Theta < 1$, in the limit where $\beta \ll [(2 + q - 4\Theta - \Theta q) + \sqrt{(2 + q - 4\Theta - \Theta q)^2 + q(4 + q)(3 - 2\Theta)}]/[(4 + q)(3 - 2\Theta)]$, if the scheme is stable for $\theta = \pi$, the scheme is stable for all values of θ . We require

$$P < \frac{1}{\frac{q}{\beta(4+q)} + 1 - 2\Theta}. \quad (C12)$$

In the limit where $\beta \gg [(2 + q - 4\Theta - \Theta q) + \sqrt{(2 + q - 4\Theta - \Theta q)^2 + q(4 + q)(3 - 2\Theta)}]/[(4 + q)(3 - 2\Theta)]$, if the scheme is
490 stable for $\theta = 0$, the scheme is stable for all values of θ .

$$P < \frac{1 + \frac{1}{\beta}}{4(1 - \Theta)}. \quad (C13)$$

For the intermediate values of β , the stability criteria must be determined numerically.

In the limit when $\Theta = 1$, the numerical scheme is unconditionally stable when the denominator in Eq.(C11) is negative. This
495 occurs when

$$\beta^2 > \frac{(1 - \cos \theta)q^2}{q^2 + 8q + 8 - \cos \theta(4 + q)^2 + 8\cos^2 \theta}, \quad (C14)$$

where the above expression has been obtained by substituting in Eq.(24) for ϕ and rearranging the denominator to make β the subject. For all values of q the maximum of this expression occurs at $\theta = 0$. Therefore, the condition on β for the scheme to be unconditionally stable is given by,

$$500 \quad \beta > \sqrt{\frac{q}{q + 8}}. \quad (C15)$$

If $\beta \leq \sqrt{q/(q + 8)}$, the conditions on P must be determined numerically.

Fourthly, consider the $u0h\Theta - CD$ scheme. First, we consider the limit of $\theta = \pi$. In this limit, the expression for $\Gamma(\theta)$, Eq.(B7) tends to

$$505 \quad |\Gamma(\pi)| = \left[1 - 2P \left(\frac{q}{\beta(q + 4)}\right)\right]^2. \quad (C16)$$



Therefore, the scheme is never unconditionally stable. For the scheme to be stable we require

$$P < \frac{2 \left(\frac{q\phi}{\beta} \right)}{(1 - \cos \theta) \left(\frac{q\phi}{\beta} \right)^2 + (1 + \cos \theta) \{ [q\phi + (1 - \Theta)]^2 - \Theta^2 \}}. \quad (\text{C17})$$

When $\beta \ll \sqrt{q/[4(1 - \Theta)(q + 4)]}$, if the scheme is stable for $\theta = \pi$, then the scheme is stable for all values of θ . The stability condition is

$$510 \quad P < \frac{\beta(q + 4)}{q}. \quad (\text{C18})$$

In the limit when $q < \frac{-(2 - \Theta) + \sqrt{9\Theta^2 - 16\Theta + 8}}{2(1 - \Theta)}$ and $\beta \gg \sqrt{q/[4(1 - \Theta)(q + 4)]}$, if the scheme is stable for $\theta = \pi/2$, then the scheme is stable for all values of θ . The stability condition is

$$P < \frac{2q(2 + q)\beta}{4(1 + q)^2\beta^2 + q^2 - 4\Theta(1 + q)(2 + q)\beta^2}. \quad (\text{C19})$$

If instead $q \geq \frac{-(2 - \Theta) + \sqrt{9\Theta^2 - 16\Theta + 8}}{2(1 - \Theta)}$ and $\beta \gg \sqrt{q/[4(1 - \Theta)(q + 4)]}$, if the scheme is stable for $\theta = 0$, then the scheme is
 515 stable for all values of θ . The stability condition is

$$P < \frac{1}{4\beta(1 - \Theta)}. \quad (\text{C20})$$

For the intermediate values of β , the conditions on P must be determined numerically.

Code and data availability. Code for the open-source ice-flow model Úa is available via Zenodo at <https://doi.org/10.5281/zenodo.22016902> (Gudmundsson, 2026). The code and data used to reproduce the figures and run the models in this article is available via Zenodo at <https://doi.org/10.5281/zenodo.21977299> (Rhee, 2026).
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