



Nonlocal ice strength regularization of the Viscous-Plastic sea ice rheology: resolving ill-posedness and resolution-dependent fracture patterns

Félix St-Denis¹ and L. Bruno Tremblay¹

¹Department of Atmospheric and Oceanic Sciences, McGill University, Montreal, Quebec, Canada

Correspondence: Félix St-Denis (felix.st-denis@mail.mcgill.ca)

Abstract. We implement a nonlocal ice strength regularization in the standard Viscous-Plastic (VP) and Elastic-Viscous-Plastic (EVP) models. To this end, we introduce an inhomogeneous Helmholtz equation for the ice strength that provides a physically based regularization of the ill-posed VP and EVP formulations while preserving the rate-independence of the plastic regime and the material instabilities that give rise to narrow deformation bands. A linear stability analysis of the new system of equations shows that the nonlocal regularization imposes an upper bound on the growth rates of perturbations, a necessary condition for well-posed problems. Those results are confirmed in idealized divergent numerical experiments: the standard (E)VP models yield resolution-dependent solutions with an increasing number of fractures as the spatial resolution is increased, whereas the nonlocal models produce a resolution-independent solution with a single fracture at the center of the ice plate. Numerical experiments demonstrating the ill-posedness of the hyperbolic tangent viscosity regularization previously identified through a linear stability analysis are also presented. With the advent of kilometer-scale sea ice simulations ($\Delta x \leq 2-3$ km), a well-posed rheology is essential to ensure that fracture statistics reflect physical deformation processes rather than numerical instabilities.

1 Introduction

One of the most challenging aspects of numerically modeling a physical phenomenon is to distinguish whether the observed features in the solution arise from the underlying physics or from numerical artifacts. Hadamard (1902) was the first to articulate that mathematical models describing time-dependent physical processes should satisfy three essential criteria: (i) the existence of the solution for any arbitrary initial conditions, (ii) the uniqueness of the solution for a given set of initial conditions, and (iii) the continuous dependence of the solution on the initial conditions. A model satisfying these criteria is referred to as well-posed; if not, it is referred to as ill-posed. Mathematically, these conditions can be stated as follows: if the growth rates ω of plane wave solutions have a real part that is positive and increasing without bound with wavenumber, the model is said to be ill-posed (Joseph and Saut, 1990). A necessary and sufficient condition for well-posedness is therefore the existence of a constant q such that

$$\text{Re}\{\omega\} < q, \tag{1}$$



for all ω (Gray, 1999; Strikwerda, 2004). A characteristic feature of ill-posed problems is that the solution changes as the spatial resolution increases and higher wavenumbers are resolved, or, stated differently, that there is no convergence to a unique solution as the spatial and temporal resolution increase. It is therefore important to distinguish between instability and ill-posedness. A problem may be well-posed yet have physically unstable solutions. In contrast, an ill-posed problem is inherently unstable and loses physical relevance, as it does not have predictive skill (Guba et al., 2013). For instance, sea ice models are inherently unstable in divergence because of the coupling between the sea ice compressive strength and the sea ice concentration or thickness. In divergent motion, sea ice strength decreases with concentration and thickness, leading to thinning and weakening of sea ice, promoting further divergence (Gray, 1999). The physical relevance of these instabilities, or the well-posedness of sea ice models, is, however, still a matter of debate (Schulkes, 1996; Gray, 1999; Guba et al., 2013).

Ill-posed problems are often the result of missing model physics. For instance, Kelvin–Helmholtz, Rayleigh–Taylor, and phase-change instabilities can be regularized by incorporating viscous effects, by accounting for surface tension, and by including viscosity and capillary effects, respectively (Joseph and Saut, 1990). In all these cases, the instabilities are present in the original set of equations, but the uniqueness of the solution is achieved by including new physics. Regularization techniques typically increase the power of existing derivatives or add new higher-order derivatives in the system of Partial Differential Equations (PDEs), effectively adding a bound to the growth rate. For instance, Coulombic granular incompressible flow using a Von Mises yield surface is always ill-posed (Schaeffer, 1987). However, introducing a dependency of the flow rule on the state of stress regularizes the granular flow (Pitman and Schaeffer, 1987).

The momentum equations describing the motion of sea ice are highly non-linear, raising questions about the existence and uniqueness of their solutions. Sea ice moves under the action of surface winds and ocean currents. In winter, this movement occurs along well-defined localized fracture lines called Linear Kinematic Features (LKFs Kwok, 2001): bands of high deformation, where sea ice slides (shear), ridges (convergence) or moves apart (divergence). Similar types of granular materials, involving quasi-static deformations, also exhibit this behavior (e.g. clay, rock, avalanches, Desrues et al., 2007; Schall and Van Hecke, 2010; Kristensen et al., 2013). Observations of sea ice deformation from the RADARSAT Geophysical Processor System (RGPS) across multiple spatial scales have revealed correlation lengths comparable to the scale of the Arctic basin (Weiss and Dansereau, 2017). As discussed by Weiss and Dansereau (2017), mechanisms such as long-range elastic wave propagation can amplify a short material-scale correlation length into a large mechanical correlation length on the order of thousands of kilometers. The existence of such an intrinsic length scale is indicative of cooperative behavior within the sea ice pack, meaning that stress can be redistributed (by elastic interactions) at a certain distance in the neighbouring sea ice field (Weiss and Dansereau, 2017). This characteristic should therefore also be reflected in the rheological framework.

In most large-scale models, sea ice is treated as a continuum. The rheology describes how sea ice deforms under applied stresses: this enters the equation of motion as the divergence of the vertically integrated internal ice stresses, which is linked to the strain rates via constitutive relations. In the first comprehensive model based on observations from the Arctic Ice Dynamics Joint EXperiment (AIDJEX), Coon et al. (1974) considered the sea ice as an elastic-plastic (EP) material where small stresses lead to small elastic reversible deformations and large stresses, that exceed thresholds in tension, shear or compression (defined by a yield curve), lead to large plastic, permanent and rate-independent deformations. This model, however, was computation-



ally expensive due to the memory in the system associated with the reversible elastic deformations and the need to store the history of previous strain rates. Later, Hibler (1979) proposed to treat small reversible deformations as a highly viscous (creep-
60 ing) flow, and post-fracture large irreversible deformations as plastic with an elliptical yield curve and a normal flow rule. The new viscous-plastic (VP) model was computationally efficient and quickly became the model of choice (Keen et al., 2021).

The standard VP rheology, however, suffers from small-scale linear instabilities that are the result of an ill-posed set of PDEs in the plastic regime (Gray and Killworth, 1995). Schulkes (1996) and later Dukowicz (1997) argued that the ill-posedness was a nonissue since the total energy of the system was bounded; i.e., the VP rheology was dissipative and remained asymptotically
65 stable, and the nonlinearities would prevent the unbounded growth of the instabilities. This points to an apparent contradiction between the concept of linear and global (energy) stability of a nonlinear system of PDEs. In response to Schulkes (1996) and Dukowicz (1997), Gray and Killworth (1997) argued that the VP model presents global instability in the viscous regime where an increase in kinetic energy is possible as potential energy is lost. Later, Gray (1999) demonstrated in simple 1D experiments (one spatial dimension and time) that the set of PDEs in the standard VP rheology was hyperbolic in convergence – characteris-
70 tic of initial value problems – and mixed elliptic hyperbolic in divergence in the plastic regime. Elliptic PDEs in time and space are ill-posed by definition since solutions of elliptic equations are set by the boundary conditions, and therefore, the future can influence the past. These instabilities were demonstrated numerically, appearing as fingers of low ice concentration and thickness that grow over time and result in fracture. Increasing the spatial resolution in the same simulations allowed for the resolution of more wavenumbers, which, in turn, increased the number of ice fingers, providing evidence of the ill-posedness
75 of the VP model (Gray, 1999). More recently, Guba et al. (2013) resolved the apparent contradiction between linear and global stability analysis, and demonstrated that some systems of equations remain ill-posed even when their energy is bounded. Guba et al. (2013) argued that the small perturbations simulated by Gray (1999) do grow but not indefinitely when there is no energy source at the boundary, i.e., when there is open water. The fact that a rheological model requires the domain to be bounded for stability is, however, concerning (Dukowicz, 1997).

80 The connection between ill-posedness and deformation localization is well established in the broader literature on rate-independent fluids, viscoplastic solids, and granular materials where linear stability analyses have shown that loss of ellipticity of the governing equations describing the quasi-static regime is a necessary condition for the onset of strain localization into deformation bands (e.g. Hill, 1962; Rudnicki and Rice, 1975; Needleman and Tvergaard, 1992; Gray, 1999; Spiegelman et al., 2016). This onset of localization is understood as a material instability, in the sense that it arises directly from the constitutive
85 response of the material (Li and Henann, 2019). The standard Viscous-Plastic rheology has been shown to exhibit this instability, which is considered desirable given its potential to reproduce observed strain localization (Gray, 1999). However, because the standard VP rheology is local, it possesses no intrinsic length scale, and the model is ill-posed. As a result, shear bands are sharp and numerical solutions are resolution-dependent (Gray, 1999).

Notwithstanding whether the instability represents a real physical behavior, we stress that (i) solving the equations with a
90 finite spatial resolution thereby limiting the range of wavenumbers resolved, (ii) incorporating a replacement pressure scheme (Guba et al., 2013), or (iii) only adding nonlinear drag laws between the atmosphere, oceans, and ice, does not regularize the VP rheology. Instead, it hides intrinsic problems that can still lead to different results with different spatial resolution



(e.g., Hunke, 2001; Lemieux and Tremblay, 2009). Much like viscosity, capillarity, and surface tension regularize the Kelvin-Helmholtz, Rayleigh-Taylor, and phase change instabilities (Joseph and Saut, 1990), the introduction of a regularization on the Δ parameter of the VP rheology, i.e., $\Delta_r = \sqrt{\Delta^2 + \delta}$, leads to a well-posed problem (Gray, 1999; Brandt et al., 2022; Liu et al., 2022). The hyperbolic tangent (HT) viscous regularization of Lemieux and Tremblay (2009) has been shown to be well-posed in one dimension under the following specific assumptions: smooth initial conditions and periodic boundary conditions. This property breaks down, however, in two dimensions when background flow gradients are unbounded (Chatta et al., 2023; Chatta and Khouider, 2026). The inclusion of water drag restores well-posedness in two dimensions regardless of the background flow gradients (Chatta and Khouider, 2026). However, since the well-posedness of the HT regularization is contingent on solution smoothness, numerical schemes that do not guarantee this, such as finite difference methods, may still exhibit resolution-dependent solutions for both one and two dimensions (Chatta and Khouider, 2026). These regularizations, furthermore, introduce a rate-dependency (fluid-like behavior) that extends into the quasi-static regime, the regime that we want to correctly simulate, with some stress states potentially exceeding the plastic yield limit (Chatta et al., 2023).

In recent years, the sea ice community largely moved away from the standard VP model and adopted instead the Elastic-Viscous-Plastic (EVP) sea ice model (Keen et al., 2021). In this model, a new prognostic equation for the internal ice stress is introduced, supporting artificial elastic waves that increase the rate at which information is communicated within the domain, improves convergence (Hunke, 2001), and more importantly, allows for the usage of an explicit numerical scheme, making parallelization of the code simple (the key reason for the transition). Recently, Boutros et al. (2026) quickly showed that the standard EVP model is also ill-posed, but that the underlying mechanism driving these instabilities differs from the one in the VP rheology. How exactly the instability differs is not discussed.

Several nonlocal regularizations incorporating cooperative effects have been proposed in the granular media literature to suppress the unbounded instability while preserving the model's ability to capture strain localization (Li and Henann, 2019). Bouzid (2013) proposed incorporating gradients of state variables into the rheology, thereby introducing an intrinsic length scale; this formulation, however, was subsequently shown to be ill-posed by Li and Henann (2019). Later, Goddard and Lee (2017) included velocity gradients in the stress tensor, and showed that this approach preserves the material instability and is well-posed. Its applicability to sea ice modelling, specifically with respect to numerical convergence, remains an open question. In parallel, a nonlocal granular fluidity (NGF) model was developed to describe a wide range of granular flows including sea ice (Kamrin and Koval, 2012; Henann and Kamrin, 2013; Kamrin and L. Henann, 2015). The model is built around the granular fluidity g , a pressure-weighted inverse viscosity field that allows for nonlocal effects. The fluidity quantifies the degree of flow activation driven by local fluctuations in the material, and is governed by an inhomogeneous Helmholtz-type equation that couples contributions from both the local fluidity field ($g - g_{loc}$) and that of neighbouring points via a Laplacian term $l^2 \nabla^2 g$, where l is a length scale defining the nonlocality (Kamrin, 2019). This approach effectively captures strain localization without resolution-dependent instabilities and keeps the plastic state rate-independent (Li and Henann, 2019).

In this study, we assess the linear stability of the Viscous-Plastic and Elastic-Viscous-Plastic rheologies both analytically and numerically, following the approach of Gray (1999). We first confirm their ill-posedness by showing that their solution is resolution-dependent. We then introduce a new nonlocal formulation for the ice strength that is derived from an inhomogeneous



(steady-state) Helmholtz-type equation. In this case, the pressure perturbations ($P - P_{loc}$) are referenced by the local source term P_{loc} given by the standard ice strength parametrization of Hibler (1979). A linear stability analysis of the resulting system of equations confirms that this physically motivated regularization leads to the well-posedness of the (E)VP rheologies while preserving the rate-independence in the plastic regime and the ability to simulate axial-strain ice strength instabilities in divergent flow. The analytical results are validated in idealized one-dimensional numerical experiments conducted at increasingly high spatial resolutions. As model spatial resolutions continue to increase, allowing for the resolution of LKFs and interest in the statistical characteristics of LKFs grows, it becomes essential to ensure that simulated fractures arise from physical mechanisms rather than numerical artifacts.

The paper is structured as follows. Section 2 presents the sea ice model and the new ice strength equation. Section 3 presents both the stability analysis and the numerical experiments. The results and their discussion are presented in Sections 4 and 5. Finally, the main conclusions are presented in Section 6.

2 Model

2.1 Sea Ice Momentum and Continuity Equations

The two-dimensional sea ice momentum equation, resulting from integrating the 3D momentum equation in the vertical, can be written as:

$$m \left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -m f \hat{\mathbf{z}} \times \mathbf{u} + \boldsymbol{\tau}_a - \boldsymbol{\tau}_w - m g \nabla H_d + \nabla \cdot \boldsymbol{\sigma}, \quad (2)$$

where $m(= \rho_i h)$ is the sea ice mass per unit area, ρ_i is the ice density, h is the mean ice thickness, $\mathbf{u}(= [u, v])$, is the horizontal ice velocity vector, f is the Coriolis parameter, $\boldsymbol{\tau}_a$ is the surface wind stress, $\boldsymbol{\tau}_w$ is the water drag, g is the gravitational acceleration, H_d is the sea surface dynamic height and $\boldsymbol{\sigma}$ is the vertically integrated internal stress tensor. The nonlinear advection term $\mathbf{u} \cdot \nabla \mathbf{u}$ does not enter the linear stability analysis for a zero-velocity base state and is therefore neglected (Gray, 1999). The surface wind stress and the water drag can be written using the standard bulk formulation (e.g., McPhee 1975; Brown 1979) as,

$$\begin{aligned} \boldsymbol{\tau}_a &= \rho_a C_a |\mathbf{u}_a^g| \mathbf{u}_a^g, \\ \boldsymbol{\tau}_w &= \rho_w C_w |\mathbf{u} - \mathbf{u}_w^g| [(\mathbf{u} - \mathbf{u}_w^g)], \end{aligned}$$

where ρ_a and ρ_w are the air and water densities, $\mathbf{u}_a^g$ and $\mathbf{u}_w^g$ are the horizontal surface winds and ocean currents, and C_{da} and C_{dw} are the air and water drag coefficients.

The continuity equations for sea ice concentration (A) and mean ice thickness (h) can be written as:

$$\frac{\partial A}{\partial t} + \nabla \cdot (\mathbf{u} A) = S_A, \quad (4a)$$



$$\frac{\partial h}{\partial t} + \nabla \cdot (\mathbf{u}h) = S_h, \quad (4b)$$

where S_A and S_h are thermodynamic sink and source terms.

2.2 Viscous-Plastic Sea Ice Rheology

In order to close the system of equations (Eq. 2 - 4), constitutive relations linking the internal stresses σ and the ice velocities $\mathbf{u}$ are required. Following Hibler (1979) or other compressible (non)-Newtonian viscous flows such as blood, pastes and other Bingham fluids (Ockendon and Ockendon, 1995; Farina et al., 2018), we write the Viscous-Plastic constitutive equation as:

$$\sigma_{ij} = 2\eta\dot{\epsilon}_{ij} + (\zeta - \eta)\dot{\epsilon}_{kk}\delta_{ij} - \frac{P}{2}\delta_{ij}, \quad (5)$$

where η and ζ are the shear and bulk viscosity, $P(=\sigma_I)$ is the internal sea ice strength, σ_I is the first stress invariant, $\dot{\epsilon}_{ij} (= \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}))$ is the strain rate tensor, $\dot{\epsilon}_{kk} (= \dot{\epsilon}_I)$, is the divergence or first strain rate invariant, δ_{ij} is the Kronecker delta and summation over repeated indices is implied. Following Hibler (1979), the shear η and bulk ζ viscosities for an elliptical yield curve and normal flow rule are then written as:

$$\zeta = \frac{P}{2\Delta}, \quad (6)$$

$$\eta = \frac{\zeta}{e^2}, \quad (7)$$

$$\Delta = [(\dot{\epsilon}_{11} + \dot{\epsilon}_{22})^2 + e^{-2}(\dot{\epsilon}_{11} - \dot{\epsilon}_{22})^2 + 4e^{-2}\dot{\epsilon}_{12}^2]^{1/2}, \quad (8)$$

where e is the ellipse aspect ratio. For small deformations ($\Delta \rightarrow 0$), the shear η and the bulk ζ viscosities tend to infinity and the ice behaves as a viscous material. To avoid the singularity as $\Delta \rightarrow 0$, the shear η and bulk ζ viscosities are capped to a maximum value:

$$\zeta = \min \left[\frac{P}{2\Delta}, \zeta_{max} \right], \quad (9)$$

$$\eta = \min \left[\frac{\zeta}{e^2}, \frac{\zeta_{max}}{e^2} \right], \quad (10)$$

where $\zeta_{max} (= 1 \times 10^{12} \text{ Nsm}^{-1})$ is a constant. It is worth noting that other regularization approaches have been employed in sea ice models (Kreyscher et al., 2000; Lemieux and Tremblay, 2009). Following Hibler (1979), the ice strength P is given by

$$P = P^* h e^{-C(1-A)}, \quad (11)$$



where $P^*(= 27.5 \text{ kNm}^{-2})$ is the maximum sea ice compressive strength and the constant $C(= 20)$ defines the dependence of the compressive strength to the open water fraction. The present formulation was chosen specifically to keep the plastic and viscous regimes separated.

2.3 Elastic-Viscous-Plastic Sea Ice Rheology

In the EVP model, a prognostic equation for the ice stress is introduced to relax the time step constraint when using an explicit numerical scheme with the standard VP rheology Hunke and Dukowicz (1997). First inverting the VP constitutive relationship for the strain rates (Eq. 5), and then adding a time dependence to the stress tensor yields:

$$\frac{\partial \sigma_{ij}}{\partial t} = E \left[\dot{\epsilon}_{ij} - \frac{1}{2\eta} \sigma_{ij} - \left(\frac{\eta - \zeta}{4\eta\zeta} \sigma_{kk} + \frac{P}{4\zeta} \right) \delta_{ij} \right], \quad (12)$$

where $E(= \zeta/T)$ is analogous to the Young's modulus and $T(= T_0 \Delta t)$ is the damping timescale of the elastic waves (Hunke and Dukowicz, 1997).

2.4 Nonlocal Ice Strength

Following Henann and Kamrin (2013), we consider an inhomogeneous Helmholtz-type equation for the ice strength P :

$$\nabla^2 P = \frac{1}{l^2} (P - P_H), \quad (13)$$

where P_H is the local (Hibler, Eq. 11) ice strength and $l(= l_c A)$ is the cooperative length scale for plastic rearrangements is assumed to be a function of the sea ice concentration A , and $l_c(= 10 \text{ km})$ is the cooperative length scale for a fully compact ($A = 1$) sea ice cover. We assumed a linear relationship between the cooperative length scale and concentration on the basis that higher stress or strain leads to larger redistributions through a higher concentration. Note that this assumption makes both the viscous and plastic regimes nonlocal and that in the absence of ice thickness or concentration gradients ($\nabla^2 P \approx 0$), the system reduces to the local law of Hibler (1979):

$$P = P_H.$$

In the NGF model, the analogous fluidity equation is grounded in the kinetic elasto-plastic (KEP) theory for soft glassy materials (Bocquet et al., 2009), in which interactions between neighboring regions arise from elastic perturbations induced by localized yield events (Henann and Kamrin, 2013). An inhomogeneous Helmholtz-type equation structurally similar to Eq. 13 can be derived within this framework (Bocquet et al., 2009). A localized zone of plastic deformation can therefore induce nonlocal stress fluctuations extending over a distance of order l . These fluctuations, accentuated by the linear operator $(\nabla^2 - 1/l^2)$, combine with the applied stresses, potentially triggering deformation in a neighboring material element that would otherwise remain undeformed. This mechanism is clearly evident in sea ice, where deformation-induced secondary cracks link to form sliding lineaments. These localized fractures, which occasionally develop as conjugate pairs, arise under



far-field compressive stress states and are geometrically analogous to strike-slip fault systems observed in the Earth's crust (SYLVESTER, 1988; Schulson, 2004; Marsan and Weiss, 2010). The scale invariance exhibited by these deformation structures further supports the presence of nonlocal interactions within the sea ice cover (Schulson, 2004; Marsan and Weiss, 2010; Weiss and Dansereau, 2017).

210 It should be noted that within the KEP framework, the cooperative length scale l is not only a function of state variables (i.e., the grain diameter) but is also a specific function of the local stress state and deformations (i.e., l increases when the yield curve is reached, Bocquet et al., 2009). A more physically complete parametrization of nonlocality where the cooperative length l depends on the stress and strain rates is left for future work.

3 Methods

215 3.1 Linear Stability Analysis

We perform a 1D linear stability analysis for the plastic regime of the standard and nonlocal (E)VP models (referred to as nVP and nEVP in the following) to study the well-posedness of the system. Following Gray (1999), we consider horizontal deformations along the x -axis assuming that all variables are independent of the y coordinate ($\frac{\partial}{\partial y} = 0$), the y -component of the velocity is zero ($v = 0$), and we consider no external forcing, and a base state ($P_0, h_0, A_0, \sigma_0 = \text{constants}$ and $u_0 = 0$) that
220 satisfies the conservation equations.

In the plastic regime ($\zeta \neq \zeta_{max}$), the stress tensor σ_{11} in the VP model (Eq. 5) is written as (Gray, 1999; Williams et al., 2017)

$$\sigma_{11} = \lambda_{\pm} P, \quad (14)$$

where $\lambda_{\pm}$

$$225 \quad \lambda_{\pm} = \begin{cases} \frac{\sqrt{1+e^{-2}}-1}{2}, & \frac{\partial u}{\partial x} > 0, \\ \frac{-\sqrt{1+e^{-2}}-1}{2}, & \frac{\partial u}{\partial x} < 0. \end{cases} \quad (15)$$

In the EVP model, the 1D stress equation is obtained by adding the elastic time derivative to the standard VP stress tensor (σ_{11} , Eq. 5 Williams et al., 2017):

$$\frac{\partial \sigma}{\partial t} = E \left(\frac{\partial u}{\partial x} - \frac{\sigma}{\alpha^2 \zeta} - \frac{P}{2\alpha^2 \zeta} \right), \quad (16)$$

where $\alpha = \sqrt{1+e^2}$. Substituting for E ($= \zeta/T$) and ζ (Eq. 6) in Equation 16, we obtain the plastic EVP stress equation:

$$230 \quad \frac{\partial \sigma}{\partial t} = \frac{-1}{\alpha^2 T} (\sigma - \lambda_{\pm} P). \quad (17)$$



We study the linear stability of the models by adding a perturbation $(u', h', A', P', \sigma')$ to the base states with $u_0 = 0$ and A_0, h_0, P_0 and σ_0 as constants:

$$u = u',$$

$$h = h_0 + h',$$

$$235 \quad A = A_0 + A',$$

$$P = P_0 + P',$$

$$\sigma = \sigma_0 + \sigma'.$$

Linearizing the governing equations for the (n)VP and (n)EVP models in the plastic regime, and neglecting second-order terms, we write:

240 mass

$$\frac{\partial h'}{\partial t} + h_0 \frac{\partial u'}{\partial x} = 0, \quad (18)$$

concentration

$$\frac{\partial A'}{\partial t} + A_0 \frac{\partial u'}{\partial x} = 0, \quad (19)$$

momentum

$$245 \quad \rho_i h_0 \frac{\partial u'}{\partial t} = \begin{cases} \lambda_{\pm} P^{**} \left(\frac{\partial h'}{\partial x} + C h_0 \frac{\partial A'}{\partial x} \right), & \text{VP,} \\ \lambda_{\pm} \frac{\partial P'}{\partial x}, & \text{nVP,} \\ \frac{\partial \sigma'_{11}}{\partial x}, & \text{(n)EVP,} \end{cases} \quad (20)$$

Stress

$$\frac{\partial \sigma'_{11}}{\partial t} = \frac{-1}{\alpha T} (\sigma'_{11} - P'), \quad \text{(n)EVP,} \quad (21)$$

Pressure

$$\frac{\partial^2 P'}{\partial x^2} = \frac{1}{l_0^2} (P' - P^{**} (h' + C h_0 A')) - \gamma A', \quad \text{n(E)VP,} \quad (22)$$

250 where $l_0 (= l_c A_0)$, $P^{**} (= P^* e^{-C(1-A_0)})$ and $\gamma (= 2(P_0 - P^{**} h_0)/(l_0^2 A_0))$ are constants.

We substitute plane wave solutions $u', h', A', P', \sigma' = [\hat{u}, \hat{h}, \hat{A}, \hat{P}, \hat{\sigma}] e^{ikx + \omega t}$ in the system of linearized equations and solve for the growth rate ω for the plastic regime and all models (VP and EVP). The well-posedness of the system of equations is assessed based on the boundedness and the sign of the real part of ω (see Eq. 1).

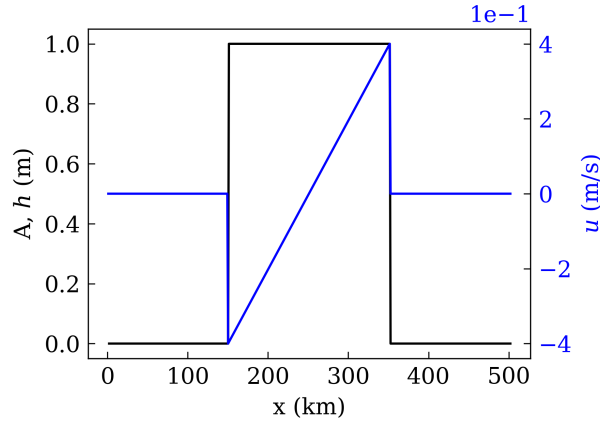


Figure 1. Sea ice thickness (black), concentration (black) and velocity (blue), initial conditions for all numerical experiments. The ice plate is 200 km and located in the center of the domain.

3.2 Numerical Approaches

255 The nVP and nEVP models are implemented in the 1D McGill Sea Ice Model (Lemieux et al., 2014; Auclair et al., 2017) to validate the stability analysis numerically. The governing equations are discretized in space on an Arakawa C grid, with tracers defined at cell centers and velocities at cell vertices. The VP momentum equation is discretized implicitly in time using a backward Euler scheme. The linearized momentum equation (Eq. 2) is solved iteratively using a Successive Over-Relaxation (SOR) scheme with a relaxation parameter $\omega = 1.5$, followed by an update of the nonlinear bulk and shear viscosities and
 260 the water drag coefficient. This procedure is repeated until convergence ($\gamma_{nl} = 10^{-4}$), or until the maximum number of outer iterations ($NL_{max} = 500$) is reached. The continuity equations for the ice thickness h and concentration A are integrated outside the Picard loop using a simple mass-conserving upwind scheme (Tremblay and Mysak, 1997; Lemieux et al., 2014) and the local ice strength P_H is updated. Finally, the pressure equation – a simple tri-diagonal matrix in 1D with known boundary conditions $P = P_H = 0$ – is solved using the Thomas algorithm (Süli and Mayers, 2003). The EVP momentum
 265 equation is solved explicitly using 1200 subcycles.

The time step ($\Delta t = 15$ s) for both (n)VP and (n)EVP models is chosen such that it respects the advective CFL criteria by one order of magnitude, i.e.

$$\Delta t \leq \frac{\Delta x}{2 \max(u)}. \quad (23)$$

To ensure that the elastic waves are adequately resolved in the EVP model, the damping time scale ($T_0 = 0.36$) is chosen such
 270 that the viscous CFL condition (Hunke and Dukowicz, 1997) is respected:

$$\sqrt{\frac{\zeta_{max}}{\rho_i h T_0}} \frac{\sqrt{\Delta t}}{N} < \Delta x. \quad (24)$$



In the following, three spatial resolutions are considered (1000 m, 500 m, and 250 m) in order to test the resolution dependence of the solution. In all experiments, the Coriolis force, the sea surface height gradient, and the surface wind stress are set to zero for consistency with the linear stability analysis. The linearized water drag term, however, is set to a very small constant to avoid a singularity in the linear solver ($C_w \approx 10^{-31}$). We set the maximum viscosity ζ_{max} to a constant value ($= 10^{12} \text{ Nsm}^{-1}$) approximately equal to that suggested by Hibler (1979) ($\zeta_{max} = 2.5 \times 10^8 P$).

The model domain is 500 km for all simulations, with a sea ice slab of length L ($= 200 \text{ km}$), constant thickness ($h = 1 \text{ m}$) and concentration ($A = 1$) located at the center of the domain (Fig. 1). Dirichlet boundary conditions are applied for the velocity ($u = 0$ at the boundaries). The presence of water on both sides of the plate ensures that boundary conditions do not influence the solution in the domain. The initial condition for the sea ice velocity is a zero-mean, divergent field symmetrical with respect to the center of the floe (Fig. 1):

$$u(x, 0) = \begin{cases} \frac{u_{max}}{100 \times 10^3} (x - 250 \times 10^3), & 150 \text{ km} \leq x \leq 350 \text{ km}, \\ 0, & \text{otherwise,} \end{cases}$$

where $u_{max} (= 0.4 \text{ ms}^{-1})$ is the maximum speed at the edges of the ice plate. The corresponding axial strain ($2u_{max}/L = 4 \times 10^{-6} \text{ s}^{-1}$) is such that the ice is in the plastic regime where the linear stability indicates instabilities should develop.

285 4 Results

Well-posed sea ice rheologies can support instabilities; their growth rate is bounded, and the solution converges as the spatial resolution increases. Conversely, ill-posed rheologies are necessarily unstable, the instability is unbounded, and the solution changes as the spatial resolution increases. Below, we present a linear stability analysis for the standard (n)VP and (n)EVP rheologies in the plastic regime (Sect. 4.1), and confirm numerically that they exhibit the expected signature of ill- and well-posedness (Sect. 4.2.1). The well-posedness of the viscous regime was shown previously (Gray and Killworth, 1995) and is not repeated here. Finally, the response of the nonlocal regularization is compared numerically with that of the rate-dependent HT regularization of Lemieux and Tremblay (2009), previously shown to cure the linear 1D ill-posedness of the standard rheology in the absence of large background velocity gradients (Sect. 4.2.2 and Chatta et al., 2023).

4.1 Linear Stability Analysis

295 4.1.1 Plastic Regime: VP

In the plastic regime, the linearized system of equations for the (n)VP (Eq. 18–20 and 22) and (n)EVP models and (Eq. 18 – 22), where plane wave solutions have been substituted, is given by:

$$\omega \hat{h} + ikh_0 \hat{u} = 0, \quad \text{thickness,} \quad (25)$$

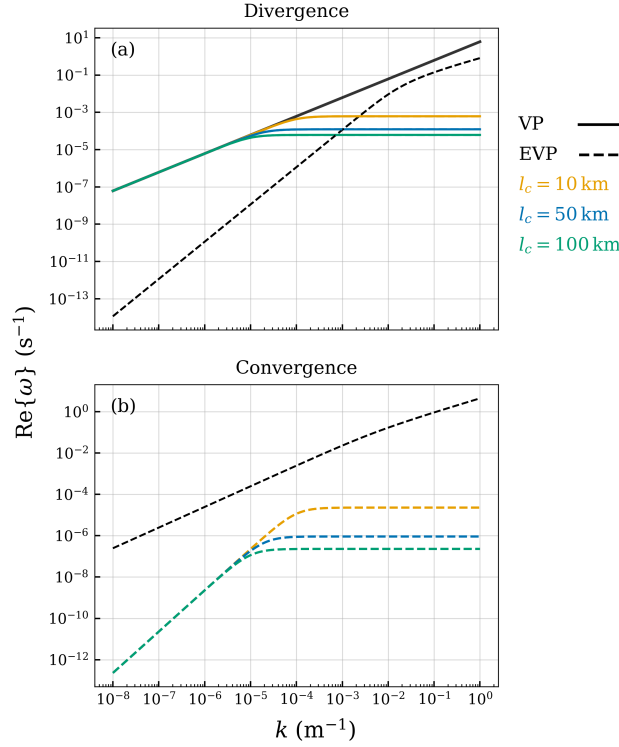


Figure 2. Real part of the growth rate roots, ω , as a function of the wavenumber k for the standard (black) and nonlocal (colors) VP (solid) and EVP (dashed) rheologies in the (a) plastic divergent and (b) plastic convergent regimes, for $h_0 = 1$ m and $A_0 = 1$. For the (n)VP rheology, only the real parts of the positive growth rates ω_+ are presented, while for the (n)EVP rheologies all nonzero and positive real growth rates are shown. Sensitivity to the cooperative length scale l_c is presented for both n(E)VP models. All n(E)VP curves coincide in the top panel. In plastic convergence, the growth rates are imaginary for (n)VP models, hence not shown.

$$\omega \hat{A} + ik A_0 \hat{u} = 0, \quad \text{concentration,} \quad (26)$$

$$300 \quad \rho_i h_0 \omega \hat{u} - i \lambda_{\pm} P^{**} k (\hat{h} + C h_0 \hat{A}) = 0, \quad \text{momentum VP,} \quad (27)$$

$$\rho_i h_0 \omega - i \lambda_{\pm} k \hat{P} = 0, \quad \text{momentum nVP,} \quad (28)$$

$$\rho_i h_0 \omega - ik \hat{\sigma} = 0, \quad \text{momentum (n)EVP,} \quad (29)$$

$$(\omega \alpha^2 T + 1) \hat{\sigma} - \lambda_{\pm} P^{**} (\hat{h} + C h_0 \hat{A}) = 0, \quad \text{stress EVP,} \quad (30)$$

$$(\omega \alpha^2 T + 1) \hat{\sigma} - \lambda_{\pm} \hat{P} = 0, \quad \text{stress nEVP,} \quad (31)$$

$$305 \quad \hat{P} - \frac{1}{k^2 + \frac{1}{l_0^2}} \left(\frac{P^{**}}{l_0^2} (\hat{h} + C h_0 \hat{A}) + \gamma \hat{A} \right) = 0, \quad \text{pressure n(E)VP.} \quad (32)$$



Solving for the dispersion relation ω in the plastic regime for the VP and nVP models (Eq. 25–27, and Eq. 25, 26, 28, and 32), we obtain

$$\omega_{\pm} = \begin{cases} \pm k \sqrt{\frac{\lambda_{\pm}}{\rho_i} P^{**} (1 + C A_0)}, & \text{VP,} \\ \pm k \sqrt{\frac{\lambda_{\pm}}{\rho_i h_0} (2P_0 + (C A_0 - 1) P^{**} h_0) \frac{1}{k^2 l_0^2 + 1}}, & \text{nVP.} \end{cases} \quad (33)$$

Note that the standard VP growth rate $\omega_{\pm}$ can be recovered from the nVP dispersion relation by setting $l_0 = 0$ and $P_0 = P^{**} h_0$.

310 In the standard VP and the nVP models, λ_{-} is negative, and the roots $\omega_{\pm}$ are purely imaginary for convergent motion ($\partial u / \partial x < 0$), resulting in wavelike solutions, i.e., this mimics the propagation of a ridge away from a coastline under compression (Williams et al., 2017). In the nVP model, the growth rate is imaginary except in regions outside of the Marginal Ice Zone ($A_0 < 1/C$ and $P_0 < P^{**} h_0 (C A_0 - 1)/2$) where the growth rate is positive (but still bounded), confirming the well-posedness of the nVP rheology (not shown). In both models in divergence ($\partial u / \partial x > 0$), λ_{+} is positive: the negative root ω_{-} is negative
 315 for both models, leading to exponentially damped solutions (not shown), and the positive root ω_{+} is real and positive, allowing for growing modes (Fig. 2 b).

In both models, the growth rates are equal and increase linearly with k (in log-log space) for low wavenumber ($k < 10^{-5} \text{ m}^{-1}$ or wavelength $> 100 \text{ km}$). For the higher wavenumber ($k > 10^{-5} \text{ m}^{-1}$, wavelength $< 100 \text{ km}$), however, the growth rate remains bounded to $\omega_{+} = \sqrt{\lambda_{+} (P^{**} h_0 (1 + C A_0) + \gamma A_0) / (\rho_i h_0 l_0)}$ for the nonlocal VP up to the highest numerically resolved
 320 wavenumber ($k = 10^{-3} \text{ m}^{-1}$, wavelength of 1 km) and unbounded for the standard VP model. The standard VP model is therefore ill-posed, as shown previously by Gray (1999), and the nVP model is well-posed (Eq. 1). The nonlocal pressure term in the inhomogeneous Helmholtz equation extends pressure perturbations – the curvature in P ($\nabla^2 P$) is proportional to $(P - P_H)$ – over a length scale l_0 preserving the underlying material instability, while regularizing its spatial growth. Note that the wavenumber at which the growth rate plateaus is a function of the cooperative length scale l_c : as l_c increases, the plateau
 325 starts at a lower wavenumber. For a cooperative length smaller than the smallest resolved wavenumber ($k = 2\pi / \Delta x$), the nVP rheology reduces to the ill-posed standard VP model (Fig. 2b).

Finally, the positive growth rate changes the nature of the equation from elliptical to hyperbolic, i.e., instabilities in the solutions are now only felt in a closed domain of influence (the fracture) rather than throughout the whole domain, an indication of the ability to model strain localization. The response of the nVP model is therefore more physical: in divergence, the ice plate
 330 breaks in two, independently of spatial resolution, rather than in several pieces (Figs. 2 and 3 in Gray, 1999, and Fig. 3 below).

4.1.2 Plastic Regime: EVP

Solving for the dispersion relationships ω for the standard EVP and nEVP systems of equations (Eq. 25, 26, 29 and 30, and, Eq. 25, 26, 29, 31 and 32), we obtain cubic polynomials:

$$\alpha^2 T \omega^3 + \omega^2 - \frac{\lambda_{\pm} k^2}{\rho_i} P^{**} (1 + C A_0) = 0, \quad \text{EVP,} \quad (34)$$



$$335 \quad \alpha^2 T \omega^3 + \omega^2 - \frac{\lambda_{\pm} k^2}{\rho_i h_0 (k^2 l_0^2 + 1)} (2P_0 + (CA_0 - 1)P^{**} h_0 = 0, \quad \text{nEVP.} \quad (35)$$

Solving for ω in Eq. 34 numerically yields real, positive, and unbounded roots in both convergence and divergence, indicating that the EVP rheology is ill-posed in both for any sign of the velocity gradient (Fig. 2b,c).

In divergence, the growth rate is smaller in magnitude than that of the VP rheology and depends strongly on the elastic wave damping timescale $T = T_0 \Delta t$ (Fig. 2b). For typical spatial resolutions ($\Delta x = 1\text{--}10$ km), the time step ($\Delta t = 10\text{--}600$ s) and
 340 damping timescale parameter ($T_0 = 0.1\text{--}0.9$) used in sea ice models is such that the unbounded growth occurs at wavenumbers larger than the maximum resolved wavenumber ($k_{max} \propto 1/\Delta x$) making the resolution-dependence non-apparent. This result is consistent with the findings of Williams et al. (2017), who showed that reducing the damping timescale T_0 by an order of magnitude reduces model errors in 1D-EVP simulations. The growth rate is independent of the number of subcycles since the damping timescale depends on the advective timescale only. The number of subcycles can, however, influence the resolution
 345 and damping of elastic waves, i.e., the viscous CFL condition needs to be respected (Eq. 24, Williams et al., 2017). Increased damping of elastic waves can therefore make the ill-posedness of the EVP rheology non-apparent, but does not constitute a regularization.

Similarly, solving for ω in Eq. 35 numerically yields real, positive, and bounded roots in both convergence and divergence, indicating that nonlocality also regularizes the EVP rheology. For default parameters, the growth rates in the nVP and nEVP
 350 models reach similar values. Variations in the time step and damping timescale do not alter the qualitative behavior of the nEVP model, but do affect the asymptotic values of the growth rates (Fig. 2b). Notably, at low wavenumbers, the nEVP growth rate does not converge to the EVP values. Instead, it converges to the VP model solution in divergence, while in convergence it decays toward zero more rapidly than the EVP growth rate (Fig. 2b,c).

4.2 Numerical Simulations

355 4.2.1 Linear Stability Validation

The numerical simulations using the (n)VP and the (n)EVP rheologies are consistent with their ill- and well- posedness. In the standard VP rheology, the ice plate fractures at several locations under tension, and the number of fracture points changes as the resolution increases (Fig. 3a–c). The fracture onset occurs at the same time for the three models (~ 2 hours, Fig. 3). The necks (or strain localization) eventually lead to a fracture of the plate at multiple locations (Fig. 3a). While the velocity
 360 is damped because of the dissipative nature of the plastic stress tensor (Schulkes, 1996), small perturbations in the solution from numerical discretization and rounding errors seed instabilities that are amplified at the onset of the simulation (recall the positive growth rate, Eq. 33). The instabilities are then further amplified through a positive feedback mechanism: 1) reduced ice strength, (2) reduced tensile stress, (3) enhanced divergence, (4) further decrease in thickness and concentration, and therefore (5) ice strength (Gray, 1999). At the fracture locations, we observe a step in the ice velocity field akin to salt fingering
 365 instability (Fig. 4d). Note the similarity with the governing equations for the ocean presented in Radko (2003). As the spatial resolution increases ($\Delta x = 500$ m and 250 m), instabilities appear earlier and at more locations compared to the low-resolution

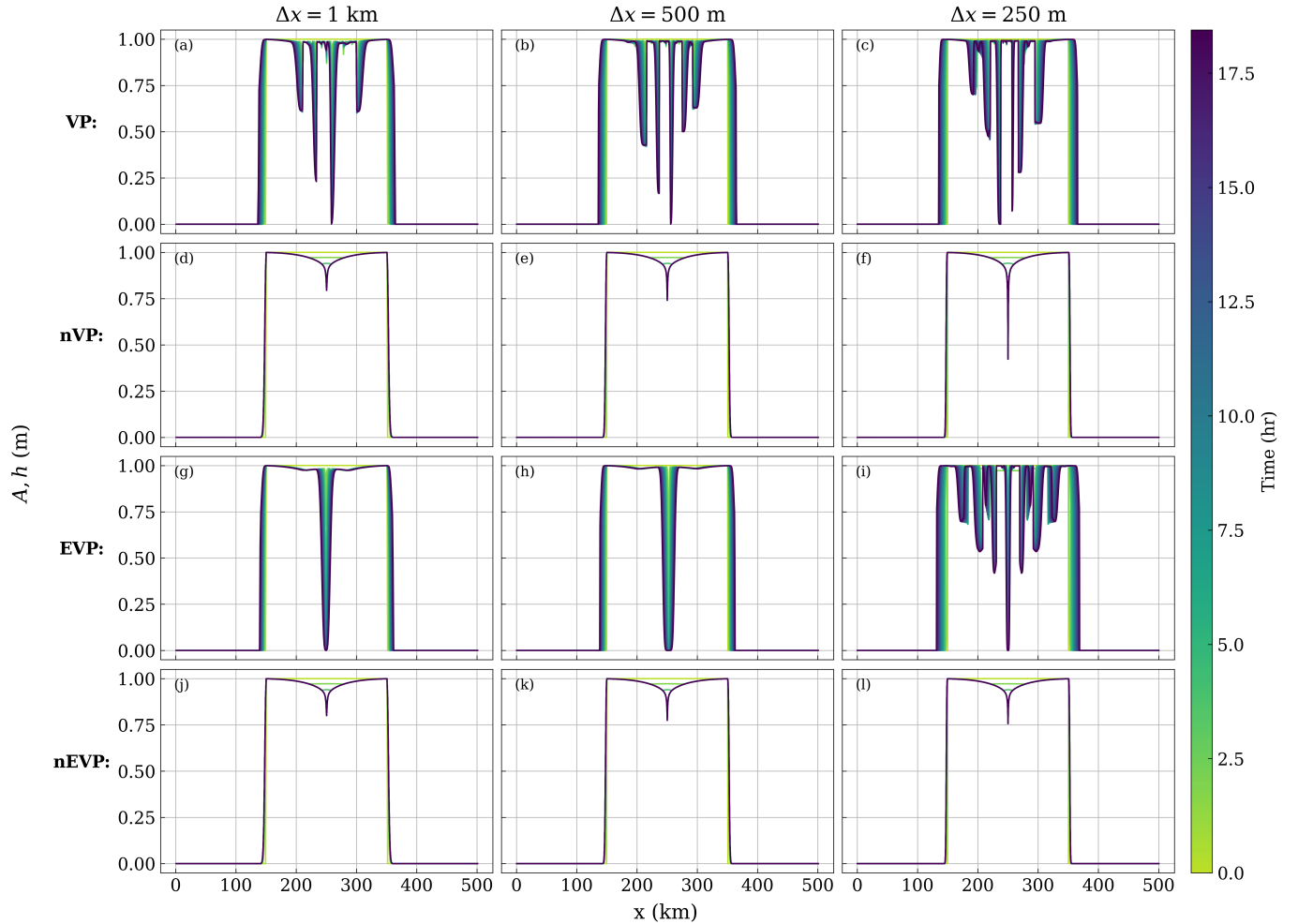


Figure 3. Ice thickness and concentration (A, h (m)) from divergent-flow experiments using the standard VP (first row), nVP (second row), EVP (third row) and nEVP (fourth row) rheologies at three spatial resolutions: 1 km (first column), 500 m (second column) and 250 m (third column). Each panel shows the evolution over 17.5 hours of simulation time, with color representing time (hours). All of the velocity curves past 7.5 hours coincide at $u = 0$ for both n(E)VP models.

simulation, i.e. one additional fracture is present each time the resolution is increased by a factor of two (Fig. 3b,c). Lastly, the growth rate of the ice fractures is asymmetric for all spatial resolutions (Fig. 3). The asymmetry and the number of fractures are characteristic features of ill-posed problems (Gray, 1999; Li and Henann, 2019).

370 In the EVP model, a single fracture develops at the center of the domain (Fig. 3g,h) at the coarse and intermediate resolutions (1000 and 500 m). As the spatial resolution increases further ($\Delta x = 250$ m), however, an increasingly large number of fractures characterized by a step in the velocity are present (Fig. 3i and 4i). The same strain-weakening feedback drives the instabilities, but the inherent damping associated with the elastic waves requires a finer resolution — compared to the standard VP model — for the same number of fracture locations.



375 In the nVP and nEVP models, a single fracture is simulated at the center of the plate independently of the spatial resolution (Fig. 3d-f and j-l). The nonlocal operator acts as a low-pass filter and suppresses small-scale instabilities (wavelength $\ll l$). As the fracture grows, the ice enters the viscous regime, resulting in an increase in the damping by almost three orders of magnitude compared to earlier parts of the simulation, and the positive strain weakening feedback mechanism is suppressed. As a result, the finger does not completely fracture the plate (Fig. 3d,j). An analytical solution (P) of the inhomogeneous
380 Helmholtz equation (Eq. 13), using Green's function – a weighted average of the local ice strength P_H , with an exponentially decaying kernel with length scale $l = l_c A$ (solution not shown) – shows that the ice plate must weaken over a length scale l before fracture starts, minimizing the effect of the positive feedback mechanism. Once a finger starts to develop, its growth rate is reduced relative to the standard VP model, since the stresses are released by axial strain (elongation) spanning a larger distance. When forced with sustained divergent winds, the ice fractures completely (result not shown). As the spatial resolution
385 (Δx) increases ($\Delta x = 500$ m and 250 m), the solution remains qualitatively unchanged, with a single fracture developing at the center of the plate and the maximum velocity reaching the same value as in the lower-resolution experiment (Fig. 3d-f and j-l). The fracture depth and localization, however, increase with resolution, as the same divergence in velocity is acting on a smaller area ($h\Delta x$). The change in concentration and thickness of the neck roughly scales as Δx with a decrease of 10% at 500 km and 40% at 250 m when compared with the 1km resolution simulation (Fig. 3d–f). The same behavior is observed for
390 the nEVP model, but the change in concentration and thickness of the neck is less pronounced, roughly 20% at $\Delta x = 250$ m. Moreover, a localized deformation band is more visible compared to the single fracture produced by the standard EVP model at $\Delta x = 1$ km and $\Delta x = 500$ m (Fig. 3g,h,j,k). The prominence of the fracture reduces to a single grid cell for the nEVP model, while the fracture width increases with resolution, until multiple fractures appear for the standard EVP model. In summary, this grid-independence of the solution is consistent with the well-posedness of both the nVP and nEVP rheologies (Li and Henann,
395 2019).

4.2.2 Effects of the viscous regularization

The HT regularization of Lemieux and Tremblay (2009), with "sufficiently smooth" initial conditions and periodic boundary conditions, was shown to be well-posed in 1D. (Chatta et al., 2023) This result, however, was not confirmed numerically (Chatta et al., 2023; Chatta and Khouider, 2026). We repeat the same divergent-flow experiments using the HT viscous regularization

$$400 \quad \zeta = P \frac{\Delta}{\Delta_{min}} \tanh\left(\frac{\Delta_{min}}{\Delta}\right), \quad (36)$$

where $\Delta_{min} (= 2 \times 10^{-9})$ is a constant. The simulation results show that the HT regularization exhibits resolution-dependent behavior qualitatively similar to that of the standard VP rheology when large gradients in thickness, concentration, and velocity are present initially (Fig. 5). When using a divergent velocity with smaller u_{max} as initial conditions, the fracture location is no longer dependent on initial conditions (results not shown). It appears that smooth enough initial conditions imply a plastic
405 initial state with sufficient viscous effect from the hyperbolic tangent for the model to be well-posed. Specifically, the number of fractures increases with spatial resolution, with one additional fracture appearing each time the resolution is refined by a

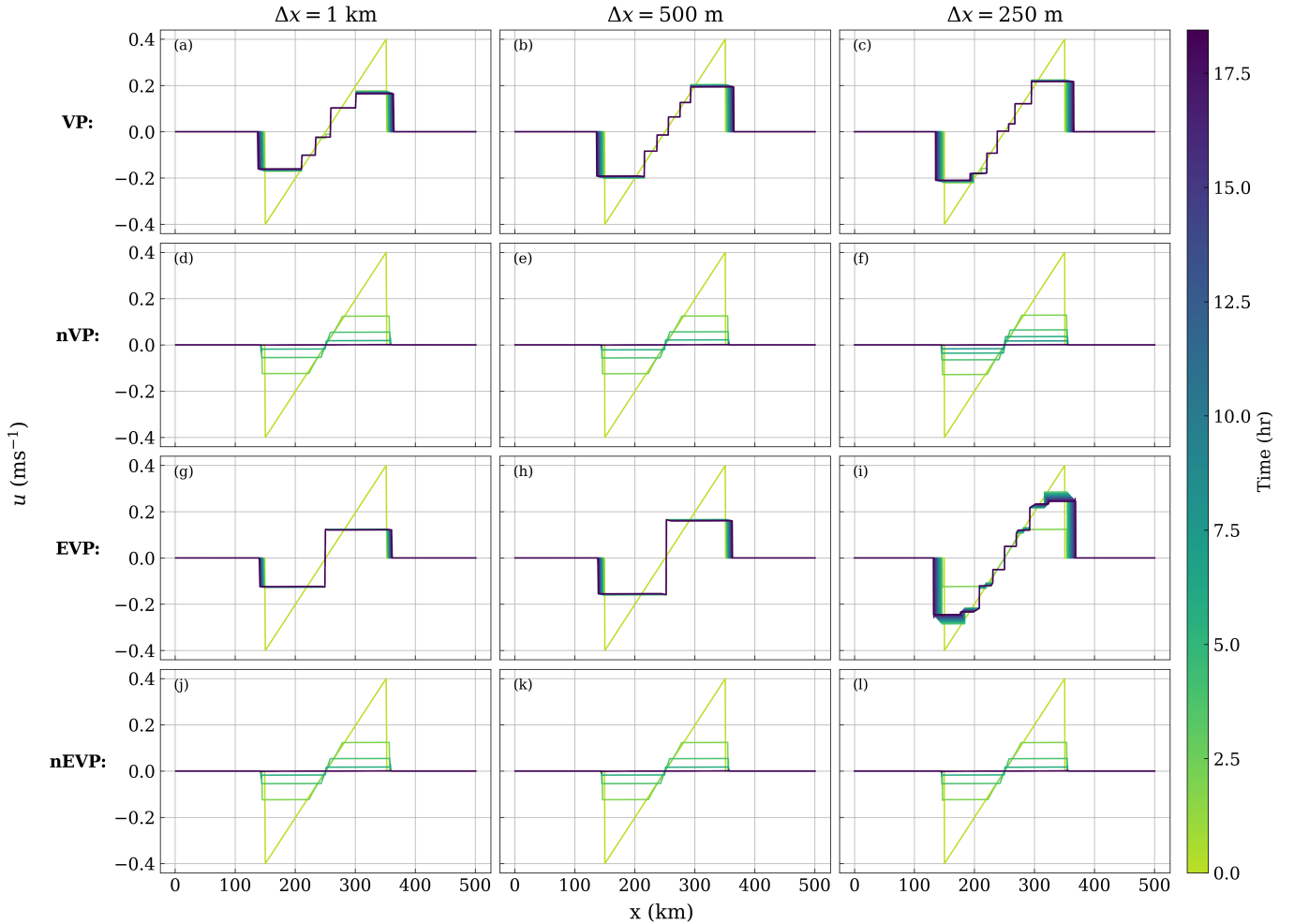


Figure 4. Velocity from divergent-flow experiments using the standard VP (first row), nVP (second row), EVP (third row) and nEVP (fourth row) rheologies at three spatial resolutions: 1 km (first column), 500 m (second column) and 250 m (third column). Each panel shows the evolution over 17.5 hours of simulation time, with color representing time (hours).

factor of two (Fig. 5). In summary, nonlocality regularizes the ill-posedness of the standard VP rheology without dependence on the initial conditions and viscous effects affecting the plastic regime as in the HT formation.

5 Discussion

410 The inclusion of nonlocality in the Viscous-Plastic rheology has three main effects: (i) it regularizes the standard VP rheology without extending the viscous regime well into the plastic regime as for existing rate-dependent regularizations (e.g. Kreyscher et al., 2000; Lemieux and Tremblay, 2009), (ii) it allows for localized (resolved) deformation developing into well-defined fractures, and (iii) the number of fracture simulated does not change with resolution. The nonlocal formulation takes the form

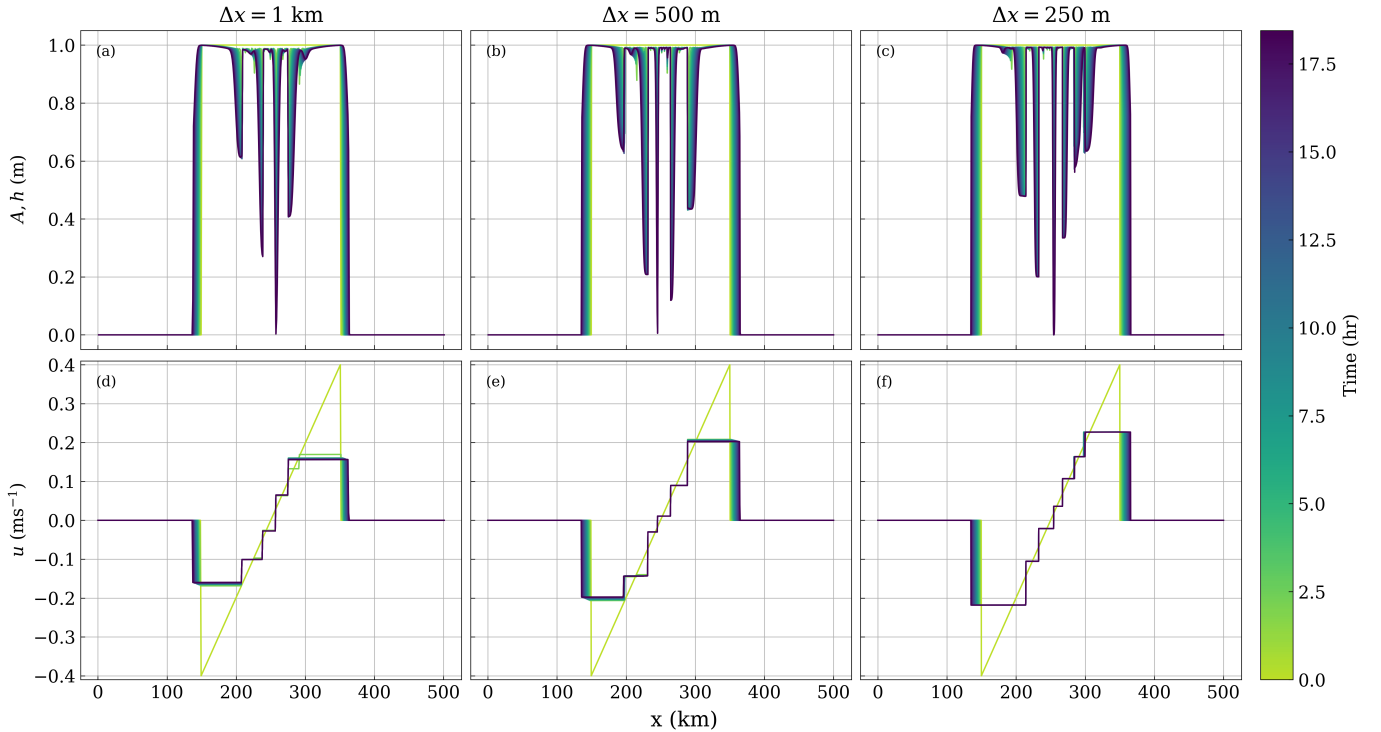


Figure 5. Divergent-flow experiments using the hyperbolic tangent regularization at a resolution of 1 km, 500 m and 250 m. Each panel shows the evolution over 17.5 hours of simulation time, with color representing time (hours). Top row: ice thickness and concentration h (m); bottom row: ice velocity u (ms^{-1}).

of an inhomogeneous Helmholtz equation for the ice strength, inspired by the NGF model of Kamrin and Koval (2012), in
 415 which the local source term is given by Hibler’s ice strength parametrization. This equation introduces a finite cap on the
 plastic growth rate, rendering the nVP model well-posed. In the idealized divergent-flow experiments presented above, the ice
 fractures at the center into two pieces of equal length, as expected physically, rather than into several pieces, the number of
 which increases with resolution.

The results from the experiments using nVP and nEVP are robust to the choice of solver or solver parameters – ω_{SOR} or
 420 the subcycling time step for the SOR or explicit solvers, unlike those of the standard VP and EVP models (results not shown).
 This sensitivity to solver parameters is a direct symptom of the ill-posedness of the standard (E)VP rheologies that has been
 reported in pan-Arctic simulations for the VP model (Lemieux and Tremblay, 2009). Replacing the hard-capping scheme with
 a P -dependent variable maximum viscosity (e.g., $\zeta_{max} = 2 \times 10^8 P$, Hibler, 1979) will affect the viscous regime but retain
 the well-posedness of the nVP model, and the plastic regime remains unaffected by this choice. A reduction in ice strength
 425 and the associated decrease in maximum viscosity will, however, lead to less well-defined (localized) deformation bands. The
 results from the divergent-flow experiments are also robust to the inclusion of the replacement pressure scheme (Ip et al., 1991;
 Hibler III and Ip, 1995), which suppresses sea ice drift in the absence of external forcings or in the presence of ice thickness



or concentration (i.e., ice strength) gradients. In divergence, the replacement pressure would have no effect since the velocity perturbations are such that the ice is initially in the plastic regime (results not shown). Note that the standard VP rheology with the replacement pressure scheme has been shown to be ill-posed when deformations exceed a critical threshold (Guba et al., 2013), and mesh-dependent solutions are therefore expected in that regime, consistent with the results shown here. Moreover, introducing atmospheric forcing or water drag only adds diffusion, making instabilities less apparent but does not regularize the system of equations (Chatta and Khouider, 2026).

Beyond robustness, it is worth noting that the nVP model retains the dissipative nature of the standard VP rheology since the nonlocal formulation leaves the stress tensor σ unchanged (Schulkes, 1996; Dukowicz, 1997). The stress dissipation (D_σ), obtained from the rate of change of kinetic energy (Rothrock, 1975), is given

$$D_\sigma = - \int P \left[\lambda_\pm \frac{\partial u}{\partial x} \right] dx. \quad (37)$$

Since the sign of $\lambda_\pm$ is determined by the sign of the deformation $\frac{\partial u}{\partial x}$ and P is always positive, the integrand is strictly non-negative, yielding $D_\sigma \leq 0$, as expected for a plastic material.

Boutros et al. (2026) recently showed that the standard EVP model is globally well-posed when regularized with a Voigt-type inviscid term in which a Laplacian contribution $-\alpha^2 \nabla^2 \sigma$ is added to the time derivative of the stress tensor (Eq. 16). This regularization, however, shares a common limitation with previous formulations: the introduction of an additional temporal stress evolution term eliminates the rate-independent plastic regime (e.g., Chatta et al., 2023; Brandt et al., 2022; Liu et al., 2022). In contrast, we introduce a spatially nonlocal regularization grounded in the granular nature of the sea ice cover, which leaves the quasi-static regime unchanged. Rather than invoking temporal memory, nonlocal effects are represented through finite-range force transmission over a cooperative length scale l , reflecting the propagation of stresses through contact networks, a mechanism more physically consistent with the treatment of sea ice as a granular material.

6 Conclusion

We introduce an inhomogeneous Helmholtz equation for the ice strength in the constitutive relations. This new equation of state communicates information about deformations occurring at distances of order l_c from a given point, introducing a physically motivated intrinsic length scale into the (E)VP framework. Theory-confirmed numerical experiments demonstrate that the n(E)VP models are well-posed yet linearly unstable, in contrast to the standard (E)VP rheologies. I.e., the plastic growth rate of the n(E)VP model is bounded at all wavenumbers, whereas that of the standard (E)VP rheology grows without bound as the wavenumber increases. The material instability retained in the n(E)VP model is essential for the localization of deformation into narrow strain bands, as demonstrated by the idealized divergent flow experiments. The bound on the growth rate ensures that a single fracture develops independently of the spatial resolution, in contrast to the standard VP model, in which the number of fractures depends on the spatial resolution.

The n(E)VP model furthermore offers a physically grounded alternative to existing regularization approaches, as it leaves



the plastic regime unchanged. The numerical implementation of n(E)VP models requires only minor additions to an existing (E)VP solver, namely the solution of an inhomogeneous Helmholtz equation within the outer loop or subcycling solvers. The HT regularization, by contrast, renders the ice inherently more viscous and has been shown to be ill-posed in the presence of large flow gradients, as confirmed by the divergent-flow experiments presented here. The n(E)VP model thus constitutes a physically motivated and computationally efficient regularization strategy that preserves the mechanical response of ice at small deformations while ensuring the localization of deformation into narrow, well-defined strain bands.

Future work of the n(E)VP model will incorporate a time-dependent formulation ($\partial P/\partial t$) to account for the memory of past deformation events over a characteristic timescale, drawing a parallel with damage-based parametrizations of the VP rheology. In addition, a cooperative length scale that depends on the stress, deformation states, or viscosity of the ice, as shown in finite-element simulations of Henann and Kamrin (2013), will be introduced. A final test includes its implementation in a two-dimensional framework, where the influence of nonlocal effects on the characteristics of Linear Kinematic Features (LKFs) — such as their intersection angles — can be systematically investigated through idealized configurations, including uniaxial (e.g., Ringeisen et al., 2019) and shear experiments providing its well-posedness in two dimensions. The introduction of nonlocality into the VP rheology represents a first step toward the development of a nonlocal granular sea ice model, in which the viscosities would additionally depend on floe diameter, following the development of the Nonlocal Granular Fluidity model (Kamrin and Koval, 2012).

Code availability. No datasets were used in this article. All simulation data have been obtained with the 1D-SIM. Model configuration and code modifications are described in detail in the paper. Additionally they are available on GitHub (<https://github.com/Felixstd/1D-SIM>, last access: 03/08/2026). All published code, including the model and analysis code can be found on Zenodo (10.5281/zenodo.21771392).

Author contributions. FSTD and BT designed the experiments. FSTD developed the nonlocal regularization, performed the stability analysis, the numerical simulations, analysed the model output and wrote the manuscript with contributions and review from BT.

Competing interests. The authors declare that they have no conflict of interest.

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