



The Deep Atmosphere Extension of the Non-hydrostatic HOMME Dynamical Core

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Abstract. We introduce an extension of the Higher Order Methods Modeling Environment (HOMME) dynamical core that solves the non-hydrostatic deep-atmosphere equations of motion. We call this extension NHD HOMME. Our extension satisfies analogs of the mimetic properties obeyed by the operational non-hydrostatic shallow-atmosphere (NHS) configuration of HOMME. Consequently, we demonstrate that our extension conserves energy under perfect temporal integration. We validate the software implementation of NHD HOMME using several idealized test cases. Small-earth steady-state experiments with a baroclinically unstable steady-state test case show that the horizontal discretizations in NHS HOMME and NHD HOMME are of comparable quality. Small-planet baroclinic wave tests match results from other deep-atmosphere dynamical cores in the literature. We find further evidence that baroclinic wave structure is sensitive to differences in the steady-state in different equation sets. In addition, we find that NHD HOMME produces small-planet Held-Suarez simulations that show an easterly equatorial zonal wind bias induced by discarding the Non-traditional Coriolis Terms (NCTs). These match results observed in other deep-atmosphere dynamical cores. Finally, simulations of the Matsuno-Gill idealized tropical heating forcing show that discarding the NCTs induces systematic errors in the linear response to large-scale heating on planets the size of earth. NHD HOMME will be provided as a switch-on option in the Energy Exascale Earth System Model.

1 Introduction

Atmospheric dynamical cores are components of numerical earth system models. They simulate atmospheric fluid motion on scales that can be resolved on the atmospheric computational grid. Dynamical cores for global models solve approximated versions of the Euler equations in a rotating reference frame. As the resolution of global climate simulations has increased, dynamical cores have transitioned away from assuming that the atmosphere is in hydrostatic balance (Ullrich et al., 2017). Dynamical cores designed for high-resolution simulation typically transition from solving the Hydrostatic Primitive Equations to solving the non-hydrostatic shallow-atmosphere (NHS) equations (see White et al. (2005) for a detailed description of the approximations made in this equation set). However, the NHS equations retain two dynamically significant approximations: the so-called shallow atmosphere (SA) and traditional (T) approximations. We refer to the equations of motion without these two approximations as the non-hydrostatic deep (NHD) equations. The SA and T approximations have historically been treated as physically plausible for simulations with grid spacings exceeding 100 km and model tops in the stratosphere. A growing body



of evidence indicates that these approximations induce systematic biases in equatorial flow, even in simulations with coarse grid spacing. Consequently, many atmospheric dynamical cores have removed these two approximations. The purpose of our work is to remove these approximations from the Higher Order Methods Modeling Environment (HOMME) dynamical core (Taylor et al., 2020). HOMME is the atmospheric dynamical core used in the U.S. Department of Energy's Energy Exascale Earth System Model (E3SM) (Golaz et al., 2022, 2026) and E3SM's convection-allowing configuration, the Simple Cloud-Resolving E3SM Atmosphere Model SCREAM (Donahue et al., 2024).

The SA and T approximations are closely related to each other. Under the shallow atmosphere approximation, the distance between a point in the atmosphere and the center of the planet is assumed to be the constant radius of the Earth. Consequently, the strength of gravity is assumed to be a constant value. The traditional approximation discards the so-called Non-traditional Coriolis terms (NCTs). These are the terms of the Coriolis force that are proportional to the cosine of latitude. When both approximations are made together, the continuum equations of motion obey the conservation of angular momentum. Consequently, the SA and T approximations are typically treated as a pair. However, several authors (Tort and Dubos, 2014; Staniforth, 2022) have identified ways to approximate angular momentum to derive shallow atmosphere equations that retain the non-traditional Coriolis force. In mid-latitudes, the SA and T approximations have historically been justified by scale analyses (White and Bromley, 1995). These scale analyses assume that vertical velocity is small over the length and time scales on which synoptic weather systems evolve. However, White and Bromley (1995) found that these approximations may not be justified at the equator in simulations with grid spacings on the order of hundreds of kilometers. Indeed, a growing body of theoretical and computational evidence across spatial scales suggests that removing the SA and T approximations from a dynamical core increases its physical accuracy.

Linearized studies predict significant dynamical changes in atmospheric flow when the NCTs are discarded under the T approximation. Thuburn et al. (2002) and Kasahara (2003) showed that the elimination of NCTs changes the frequencies of acoustic modes in stratified isothermal atmospheres by several percent. Thuburn et al. (2002) also found changes to the frequencies of the Kelvin and external Rossby modes. Changes in the Kelvin mode frequency were larger when the vertical variation of gravity was included in the analysis. However, the study concluded that the changes in frequency are unlikely to be energetically significant in the atmosphere. Larger biases were evident in linearized studies of cumulus heating near the equator. Hayashi and Itoh (2012) showed that the discarding of NCTs caused pronounced circulation changes induced by idealized unsteady cumulus heating in a quasi-hydrostatic (see, e.g., White et al. (2005)) linearized beta-plane model. Ong and Roundy (2019) studied a similar set of equations with zonally symmetric forcing. They found that the removal of the NCTs induced an easterly zonal wind bias on the order of 10% of the total wind magnitude in the resulting circulation. Together, these findings support the analysis in White and Bromley (1995) that the omission of NCTs can induce errors of comparable magnitude to those induced by making the hydrostatic approximation at the equator. Together, these idealized studies provide evidence that model fidelity can be improved in large-scale simulations by restoring the NCTs.

Computational studies also indicate that discarding the NCTs induces biases at convection-permitting length scales. Igel and Biello (2020) found that omitting the NCTs induced systematic differences in idealized convection-permitting simulations of tropical convection. Radiative-convective equilibrium simulations showed a change of 2 m s^{-1} in mean convective intensity



when the NCTs were restored. Dry simulations showed changes of up to 8 m s^{-1} in upper-level zonal wind. Marsico et al. (2024) used the theoretical framework developed in Marsico et al. (2023) to conclude that NCTs should not be discarded when simulating moist convecting systems with weak stratification, especially in the presence of strong radiative forcing. Liang and Chan (2005) found that the restoration of the NCTs to a simulation of a tropical cyclone caused a systematic displacement of the storm track to the southwest. Furthermore, the differences between precipitation with and without NCTs were between 10%-20% of total rainfall. These studies of convective systems demonstrate that biases induced by the T approximation cannot be ignored in convection permitting simulations.

An increasing number of dynamical cores can solve the NHD equations. The majority of these are formulated in vertical height coordinates. Height-based models that solve the NHD equations include the Met Office's Unified Model (Wood et al., 2014), the Non-hydrostatic Icosahedral Atmospheric Model (NICAM) (Tomita and Satoh, 2004), the Model for Prediction Across Scales (Skamarock et al., 2021), the ICOSahedral Non-hydrostatic (ICON) model (Borchert et al., 2019), an experimental version of the ICON hexagonal model (Balsmeier, 2021), the Ocean-Land-Atmosphere Model (OLAM) (Walko and Avissar, 2008), the Non-hydrostatic Unified Model of the Atmosphere (Kelly et al., 2026), and MCore (Ullrich and Jablonowski, 2012). In dynamical cores that use a vertical height coordinate, quantities used to calculate metric terms that are simplified under the SA approximation can be derived directly from the vertical coordinate. Recent work has used the strategy described in Wood and Staniforth (2003) to solve the NHD equations in dynamical cores using a mass coordinate. These models include the European Centre for Medium Range Forecasting's IFS/ARPEGE (Yessad and Wedi, 2011), and the Yin-He Global Spectral Model (Yang et al., 2017).

The dynamical core in E3SM, HOMME, cannot currently solve the NHD equations of motion. HOMME is based on the mimetic Continuous Galerkin Spectral Finite Element method described in Taylor and Fournier (2010), which is combined with the Lorenz-staggered mimetic vertical discretization derived in Taylor et al. (2020) to either solve the NHS or Hydrostatic Primitive Equations of motion. The combination of horizontal and vertical discretization conserves energy under exact time integration, and mass is conserved to machine precision. Although discretizations are derived for several vertical coordinates in Taylor et al. (2020), operational E3SM runs use only the semi-Lagrangian hybrid terrain-following mass-based coordinate from Lin (2004). We refer to the version of HOMME that solves the NHS equations in this vertical coordinate as NHS HOMME. We use the strategy described in Wood and Staniforth (2003) to develop a switch-on option to solve the NHD equations that obeys the same discrete conservation properties as the NHS formulation from Taylor et al. (2020). We refer to our augmented deep-atmosphere version of the HOMME model as NHD HOMME.

The remainder of this paper is structured as follows. Section 2 explains the formulation of the continuum NHD equations that NHD HOMME solves. We show that the equations retain the energy consistency of NHS HOMME in the continuum, but defer treatment of discrete quantities to Appendix A. In Section 3, we present results from idealized test cases that verify that NHD HOMME correctly solves the NHD equations of motion. The results of these idealized tests show that adopting the NHD equation set induces measurable changes in equatorial dynamics in earth-like atmospheres.



2 Methods

2.1 Coordinate transforms

NHS HOMME mimetically discretizes a quasi-Hamiltonian form of the NHS equations using a spectral finite element discretization on unstructured quadrilateral grids in the horizontal, with vertical differencing based on Simmons and Burridge (1981) using a Lorenz (1960) vertical grid staggering. Energy consistency in HOMME derives from the fact that the discrete analogs of Equations 23-27 in Taylor et al. (2020) are satisfied to machine precision. This relies on a combination of the mimetic properties of the horizontal differential operators derived in Taylor and Fournier (2010) and the vertical summation-by-parts identities derived in Taylor et al. (2020).

The differential operators in the Taylor and Fournier (2010) numerical scheme are formulated with respect to a metric tensor defined on the unit sphere. Discrete differential operators and horizontal integrals are scaled by a constant nominal radius. Restoring dependence on the vertical dimension to this metric would require substantial changes to the software implementation of the dynamical core. In this section, we present a method for adapting HOMME to the deep atmosphere equations while still calculating discrete horizontal derivatives on the unit sphere, as in NHS HOMME. In this section, we let s be a terrain-following vertical coordinate satisfying $s = 1$ at the surface, and $s = 0$ at the top of the atmosphere.

The prognostic equations are specified in spherical coordinates, and the mapping from a reference element coordinates can be decomposed into three parts, $(x', y', s) \mapsto (x, y, z) \mapsto (\lambda, \varphi, r)$. Consequently, the prognostic equations are formulated in spherical coordinates. The Jacobian of these mappings can be decomposed as

$$g_\alpha = \begin{pmatrix} \bar{r} & 0 & 0 \\ 0 & \bar{r} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial z}{\partial x'} & \frac{\partial z}{\partial y'} & \frac{\partial z}{\partial s} \end{pmatrix} \quad (1)$$

with

$$D = \begin{pmatrix} \cos \phi & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial(\lambda, \theta)}{\partial(x, y)}.$$

Here, $\bar{r}$ represents the distance between a point and the center of the earth and g_α denotes the covariant transform. Recall that the transformation into contravariant coordinates can be calculated as $g^\beta = (g_\alpha)^{-1}$. We make the definition $J = |\det D|$. Under the shallow atmosphere approximation, $\bar{r} \equiv a$, where a is the nominal radius of the earth. In the NHD equations, $\bar{r} = r = a + z$. Energy consistency is formulated with respect to integrals of the form

$$\iint \xi \bar{r}^2 J \frac{\partial z}{\partial s} dA ds,$$

for arbitrary quantities ξ . Throughout this work, integrals with respect to area on the unit sphere are denoted by integration with respect to the unitless area element dA . Here $\bar{r}^2$ accounts for the quadratic increase in spherical area as height increases. In NHD HOMME, the terms $\bar{r}^2 \partial_s z$ are grouped together to incorporate the spherical geometry correction term into the vertical coordinate.



2.1.1 The horizontal discretization

In the horizontal dimension, the mimetic vector calculus operators in NHS HOMME are calculated as follows. For a function
 125 f on the reference element $[-1, 1]^2$, let $\partial_{x_{\text{ref}}^k} f$ for $k = 1, 2$ be derivatives calculated to machine precision in the spectral basis
 with respect to the k th reference coordinate. The calculation of the horizontal gradient of f in physical coordinates along a
 constant vertical level (denoted by the subscript s) can be computed as

$$\nabla_s f = D^{-\top} \left[\frac{1}{\bar{r}} \begin{pmatrix} \frac{\partial f}{\partial x_{\text{ref}}^1} \\ \frac{\partial f}{\partial x_{\text{ref}}^2} \end{pmatrix} \right]. \quad (2)$$

For a horizontal velocity vector $\mathbf{u}$ in spherical coordinates with the zonal wind component u and meridional wind component
 130 v , first let its contravariant formulation be

$$\mathbf{u}_{\text{contra}} = D^{-1} \left(\frac{1}{\bar{r}} \mathbf{u} \right).$$

The horizontal divergence is then given by

$$\nabla_s \cdot \mathbf{u} = \frac{1}{J} \sum_{k=1}^2 \frac{\partial}{\partial x_{\text{ref}}^k} (J(\mathbf{u}_{\text{contra}})^k). \quad (3)$$

Finally, if we express the covariant velocity vector as $\mathbf{u}_{\text{cov}} = D^{\top} \mathbf{u}$, then vorticity is calculated as

$$135 \quad \nabla_s \times \mathbf{u} = \frac{1}{\bar{r}J} \left(\frac{\partial}{\partial x_{\text{ref}}^1} (\mathbf{u}_{\text{cov}})_2 - \frac{\partial}{\partial x_{\text{ref}}^2} (\mathbf{u}_{\text{cov}})_1 \right). \quad (4)$$

As a result, we can define $\hat{r} = r/a$ so that

$$\nabla_s^{(\text{deep})} f = \frac{1}{\hat{r}} \nabla_s f \quad (5)$$

$$\nabla_s^{(\text{deep})} \cdot \mathbf{u} = \nabla_s \cdot \left(\frac{1}{\hat{r}} \mathbf{u} \right) \quad (6)$$

$$\nabla_s^{(\text{deep})} \times \mathbf{u} = \frac{1}{\hat{r}} \nabla_s \times \mathbf{u}. \quad (7)$$

140 Therefore, the spherical differential operators can be adjusted for the deep-atmosphere equations by incorporating the correc-
 tion factor $\hat{r}$ into the equations of motion.

2.1.2 The vertical coordinate

The only vertical coordinate used operationally in E3SM simulations is the terrain-following semi-Lagrangian mass coordinate.
 This article only describes the Deep Atmosphere extension of HOMME for this choice of vertical coordinate. We follow the
 145 method suggested in Wood and Staniforth (2003) for incorporating the spherical geometry of the deep atmosphere into the
 way mass is calculated in an atmospheric column. We define geopotential by $\partial_z \phi \equiv g$, which can be rewritten in another
 coordinate as $\partial_s \phi = g \partial_s z$. The symbol g denotes gravity. In this work, g may stand for a constant reference value $g \equiv g_0$, but



the derivations do not depend on the form of $g(z)$. Therefore, energy conservation holds for the usual definition of gravitational acceleration under the spherical geopotential approximation, $g = \hat{r}^{-2}g(a)$.

150 The vertical coordinate of HOMME is constructed using the monotonic quantity $\pi_{\text{shallow}}(s) = \int_z^\infty \rho g dz$. $\pi(s)$ takes units of pressure and coincides with hydrostatic pressure in hydrostatic dynamical cores. However, in non-hydrostatic models $\pi(s)$ can also be seen as column-integrated density (taking units of, e.g., kg m^{-2}) scaled by the gravity. The expansion of control volumes at model levels above the surface can be incorporated into this vertical coordinate by defining $\pi_{\text{deep}}(z) = \int_z^\infty \hat{r}^2 \rho g dz$. In the continuum, this can be equivalently rewritten as $\partial_z \pi_{\text{deep}} = -\rho g \hat{r}^2$. In a terrain following coordinate, this reads as

$$155 \quad \frac{\partial \pi}{\partial s} = -\rho g \hat{r}^2 \frac{\partial z}{\partial s} = -\rho \hat{r}^2 \frac{\partial \phi}{\partial s}. \quad (8)$$

This allows us to relate physical density, ρ , to the so-called pseudo-density $\partial_s \pi$ by

$$\rho = \hat{r}^{-2} \frac{\partial \pi}{\partial s} \bigg/ \frac{\partial \phi}{\partial s}. \quad (9)$$

Wood and Staniforth (2003) shows that these lead to a well-posed formulation of the deep atmosphere equations that satisfy the same conservation identities as the standard deep atmosphere equations in height coordinates. A mass weighted vertical
 160 integral of an arbitrary quantity, ξ , in HOMME is approximated as $\int \xi \partial_s \pi ds \equiv \sum_{k=1}^N \xi_k \Delta \pi_k$. Under the new mass coordinate, this should be discretized as

$$\int \xi \hat{r}^2 \frac{\partial \pi}{\partial s} ds \equiv \sum_{k=1}^N \xi_k \hat{r}_{\text{avg},k}^2 \Delta \pi_k,$$

where

$$\hat{r}_{\text{avg}}^2 = \frac{1}{3}(\hat{r}_{k+1/2}^2 + \hat{r}_{k-1/2}^2 + \hat{r}_{k+1/2} \hat{r}_{k-1/2}).$$

165 N symbolizes the number of full model levels. Note that $\hat{r}_{\text{avg},k}^2$ is the average value of $\hat{r}^2$ computed in height coordinates.

A hybrid terrain-following η mass coordinate (Arawaka and Lamb, 1977; Simmons and Burridge, 1981) can then be defined as $\pi(s) = A(s)\pi_0 + B(s)\pi_{\text{surf}}$, with $A(1) = 0$, $B(0) = 1$. In NHS HOMME and NHD HOMME, π_{surf} stands for $\pi_{\text{shallow}}(1)$ and $\pi_{\text{deep}}(1)$, respectively. $A(s)$ and $B(s)$ are typically referred to as hybrid coefficients, where $\eta(s) = A(s) + B(s)$. In this coordinate, π_0 is commonly called a “reference pressure” that is constant over the surface of the earth, and π_{surf} is called the
 170 “surface pressure”. Note that within a column with a hydrostatically balanced state, $\pi_{\text{shallow}}(1)$ will match the thermodynamic pressure at the surface, but $\pi_{\text{deep}}(1)$ will exceed the thermodynamic pressure. The discrepancy between $\pi_{\text{deep}}(1)$ and hydrostatic surface pressure is quite small on planets the size of earth, but becomes larger as the planetary radius decreases.

2.1.3 The resulting equations

The continuum equations that NHD HOMME solves are analogous to the quasi-Hamiltonian form of the NHS equations
 175 given in Taylor et al. (2020). We include boxes around the specific terms that are introduced in the NHD equations. These equations are formulated in terms of the following prognostic variables. Velocity is decomposed as $\mathbf{u} = (u, v)$, which is the



horizontal velocity vector tangent to surfaces of constant z , and $w = \dot{z} = Dz/Dt$, which represents the vertical velocity in height coordinates. The scalar prognostic variables are geopotential, ϕ , thermodynamic density, $\Theta = \theta_v \partial_s \pi$, and pseudo-density $\partial_s \pi$. θ_v symbolizes the virtual potential temperature. HOMME's thermodynamics assume that for adiabatic flow, $D\theta_v/Dt = 0$, and makes the so-called constant- κ approximation (Eldred et al., 2022), where we assume that the ideal gas constant of dry air R and the specific heat capacity of dry air at constant pressure c_p are constants. Combining these definitions with the relation between density and pseudo-density, the ideal gas law is written as

$$\frac{\partial \phi}{\partial s} = -\hat{r}^{-2} R \Theta \frac{\Pi}{p}, \quad (10)$$

where $\Pi = T_v/\theta_v = (p/p_0)^{R/c_p}$ is the Exner function with the virtual temperature T_v , the pressure p , and the reference pressure $p_0 = 10^5$ Pa. HOMME uses a 2-D vector invariant formulation of the equations of fluid motion. The work in this paper only covers the semi-Lagrangian mass coordinate for which $\dot{s} = 0$, and we omit vertical transport terms in the equations that follow.

The resulting equations read

$$\frac{\partial \mathbf{u}}{\partial t} + \left(\frac{1}{\hat{r}} \nabla_s \times \mathbf{u} + f_s \right) \hat{k} \times \mathbf{u} + \frac{1}{2\hat{r}} \nabla_s (\mathbf{u}^2 + w^2) - \frac{1}{\hat{r}} w \nabla_s (w) + \quad (11)$$

$$\left[\frac{\mathbf{u}w}{r} + f_c \begin{pmatrix} w \\ 0 \end{pmatrix} \right] + c_p \theta_v \left[\frac{1}{\hat{r}} \nabla_s (\Pi) + \frac{\mu}{\hat{r}} \nabla_s (\phi) \right] = 0 \quad (12)$$

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \left[\frac{1}{\hat{r}} \nabla_s (w) \right] - \frac{\mathbf{u}^2}{r} - f_c u + g - \mu g = 0 \quad (13)$$

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \left[\frac{1}{\hat{r}} \nabla_s (\phi) \right] - gw = 0 \quad (14)$$

$$\frac{\partial \Theta}{\partial t} + \theta_v \nabla_s \cdot \left(\left[\frac{1}{\hat{r}} \frac{\partial \pi}{\partial s} \mathbf{u} \right] + \left[\frac{1}{\hat{r}} \frac{\partial \pi}{\partial s} \mathbf{u} \cdot \nabla_s (\theta_v) \right] \right) = 0 \quad (15)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial \pi}{\partial s} \right) + \nabla_s \cdot \left(\left[\frac{1}{\hat{r}} \frac{\partial \pi}{\partial s} \mathbf{u} \right] \right) = 0 \quad (16)$$

$$\frac{\partial}{\partial t} \left(q_i \frac{\partial \pi}{\partial s} \right) + \nabla_s \cdot \left(\left[\frac{1}{\hat{r}} q_i \frac{\partial \pi}{\partial s} \mathbf{u} \right] \right) = 0, \quad (17)$$

where q_i are tracers, $\hat{k}$ symbolizes the vertical unit vector, and we have redefined the vertical non-hydrostatic pressure gradient

$$\mu = \left[\hat{r}^2 \frac{\partial p}{\partial s} \left(\frac{\partial \pi}{\partial s} \right)^{-1} \right]. \quad (18)$$

The two Coriolis parameters are defined as $f_s = 2\Omega \sin(\varphi)$ and $f_c = 2\Omega \cos(\varphi)$, where φ is latitude and Ω denotes the angular frequency of Earth's rotation.

The operational version of NHD HOMME will omit the vertical variation of gravity. The inverse quadratic variation of gravity is important for deep-atmosphere dynamical cores that are designed to simulate the upper atmosphere (Skamarock et al., 2021; Borchert et al., 2019). There is little evidence to suggest that assuming gravity is constant induces measurable biases in



flow in the lower atmosphere. Since NHS HOMME and E3SM are primarily used to simulate tropospheric and stratospheric processes, we elected to retain this simplification. We developed an experimental implementation of NHD HOMME with variable gravity and present several simulations from this model in Section 3. We refer to this configuration as NHD(g) HOMME. We later show that the energy consistency of the equation sets holds regardless of the treatment of gravity. Consequently, the continuum equations are dynamically consistent in both model configurations.

If only the non-traditional Coriolis terms,

$$f_c \begin{pmatrix} w \\ 0 \end{pmatrix} \text{ and } -f_c u,$$

are restored to the NHS equations, the resulting equations conserve energy. We call this the NHNT equation set, but we make no claim that this equation set can be derived from a variational perspective. Tort and Dubos (2014) describe a variational derivation of non-traditional shallow atmosphere equations that do conserve angular momentum, which could likely be applied to HOMME’s equations. However, NHS HOMME does not conserve discrete angular momentum, so it is unclear whether the additional complexity would be warranted for operational simulations. The NHNT equation set alleviates the biases induced by the traditional approximation with minimal computational cost compared to the NHS equations. Consequently, this equation set could be nearly equivalent to the full NHD equations for tropospheric simulation.

2.2 Boundary conditions

Two upper boundary conditions are suggested in Wood and Staniforth (2003) for the deep mass coordinate: a rigid upper surface where the geopotential at the top interface does not vary over time, and an isobaric upper surface where $p_{\text{top}} = \pi_{\text{top}}$. However, they note that this list is not exhaustive. In NHD HOMME, we use a modified version of the elastic lid in which $p_{\text{top}} = \pi_{\text{top}}/\hat{r}_{\text{top}}^2$. As will be shown later, this incorporates a correction for spherical geometry into the pressure work term in the expression for internal energy. Like the rigid and elastic lids, our choice of the upper boundary does not resemble a physically realistic boundary surface in the atmosphere. In practice, the differences between our scaled boundary condition and a standard isobaric upper boundary are extremely small. In operational simulations, explicit artificial diffusion is added to eliminate the spurious reflection of waves off the artificial boundary. Any error induced by our ad hoc strategy for adjusting the upper boundary for the deep atmosphere equations will be negligible compared to the existing artificial fixers.

Mass conservation requires a no-flux condition at the lower boundary, stated in the continuum as $w = (g\hat{r})^{-1} \nabla_s \phi_{\text{surf}} \cdot \mathbf{u}$. Discretely, this is written as $w = (g\hat{r})^{-1} \mathbf{A}[\nabla_s \phi_{\text{surf}}] \cdot \mathbf{u}_{\text{surf}}$, where $\mathbf{A}[\cdot]$ is the spectral element Direct Stiffness Summation operator (see, e.g., Kopriva (2009)), and $\mathbf{u}_{\text{surf}}$ is the horizontal wind extrapolated to the lower boundary. Due to how discrete extrapolation occurs, $\mathbf{u}_{\text{surf}}$ is equivalent to the wind at the lowest full model level. Let $\mathbf{u}_n, w_n$ be the state at time level n , and let $\delta \mathbf{u}, \delta w$ be the calculated (explicit) time tendencies calculated under the assumption that $\mu_{\text{surf}} \equiv 1$. The boundary condition is enforced by taking an approximate predictor step and then deriving a correction to the surface pressure such that the boundary



condition is satisfied. Formally, the predictor step defines

$$\mathbf{u}^* - \mathbf{u}_n = \Delta t \mathbf{A}[\delta \mathbf{u}]$$

$$w^* - w_n = \Delta t \mathbf{A}[\delta w].$$

235 Since the $g(1 - \mu_{\text{surf}})$ term was discarded in the predictor step, the conditions that must be satisfied by μ_{surf} is

$$\delta w_{\text{corr}} = w_{n+1} - w^* = \Delta t g(\mu_{\text{surf}} - 1)$$

$$\delta \mathbf{u}_{\text{corr}} = \mathbf{u}_{n+1} - \mathbf{u}^* = -\frac{\Delta t}{\hat{r}}(\mu_{\text{surf}} - 1) \mathbf{A}[\nabla_s \phi_{\text{surf}}].$$

This system can be solved by requiring that $\mathbf{u}_{n+1}$ and w_{n+1} satisfy the boundary condition exactly, which gives an equation that μ_{surf} must satisfy,

$$240 \quad \Delta t \left(g + \frac{1}{2g\hat{r}^2} \mathbf{A}[\nabla_s \phi_{\text{surf}}]^2 \right) (\mu_{\text{surf}} - 1) = \frac{1}{g\hat{r}} \mathbf{u}^* \cdot \mathbf{A}[\nabla_s \phi_{\text{surf}}] - w^*. \quad (19)$$

Since spectral element assembly is applied to time tendencies before advancing the state, this process guarantees that the total explicit time step satisfies the lower boundary condition to machine precision.

2.3 Energy conservation

In this section, we show that the time rate of change of total energy is zero in the continuum, assuming exact time integration and identifying which terms constrain how averaging must be performed in the mimetic vertical discretization. The description of the chosen discrete averages can be found in Appendix A. The definitions of kinetic energy

$$K = \frac{1}{2} \frac{\partial \pi}{\partial s} (\mathbf{u}^2 + w^2) \quad (20)$$

and potential energy

$$P = \frac{\partial \pi}{\partial s} \phi \quad (21)$$

250 given in Taylor et al. (2020) still hold, although the quantities have different definitions. From the continuum definition

$$I = c_p \Theta \Pi - \frac{\partial \pi}{\partial s} \frac{1}{\rho} p + \pi_{\text{top}} \phi_{\text{top}}$$

and the choice of the upper boundary condition, we can rewrite internal energy as

$$I = c_p \Theta \Pi + \hat{r}^2 \frac{\partial \phi}{\partial s} p + \hat{r}_{\text{top}}^2 p_{\text{top}} \phi_{\text{top}}. \quad (22)$$

As before, the subscript “top” denotes the interface position at the model top. The work in this article covers only a semi-Lagrangian coordinate, and energy conservation is considered only during the floating Lagrangian integration. In the integral

$$\iint \frac{\partial}{\partial t} P = \iint \left(-\phi \nabla_s \cdot \left(\hat{r}^{-1} \frac{\partial \pi}{\partial s} \mathbf{u} \right) - \frac{\partial \pi}{\partial s} \frac{\mathbf{u}}{\hat{r}} \cdot \nabla_s \phi + \frac{\partial \pi}{\partial s} g w \right) dA ds, \quad (23)$$



The first two terms cancel by invoking the divergence theorem for the horizontal mimetic operators over the entire sphere and have no dependence on vertical averaging. The remaining term will cancel with a term in the kinetic energy tendency.

260 The simplification of $\partial_t I$ performed in Taylor et al. (2020) must be done carefully with our augmented definition of internal energy. Assuming exact time integration, we find

$$\frac{\partial I}{\partial t} = c_p \Pi \frac{\partial \Theta}{\partial t} - c_p \Theta \frac{\partial \Pi}{\partial t} + \frac{\partial}{\partial t} \left(p \hat{r}^2 \frac{\partial \phi}{\partial s} \right) + \frac{\partial}{\partial t} (p_{\text{top}} \hat{r}^2 \phi_{\text{top}}).$$

By invoking $p_{\text{top}} = \hat{r}^2 \frac{\pi_{\text{top}}}{\hat{r}^2}$, where π_{top} is a constant, we find that the final term can be simplified to $\hat{r}^2 p_{\text{top}} \partial_t \phi_{\text{top}}$. We expand the third term using the product rule. Noting that $\hat{r}^2 \partial_s \phi \partial_t p = c_p \Theta \partial_t \Pi$, we can simplify the above as

$$\frac{\partial I}{\partial t} = c_p \Pi \frac{\partial \Theta}{\partial t} - p \frac{\partial}{\partial s} \hat{r}^2 \frac{\partial \phi}{\partial t} + p_{\text{top}} \hat{r}^2 \frac{\partial}{\partial t} \phi_{\text{top}}.$$

265 Invoking integration by parts in the vertical and noting that $\partial_t(\phi_{\text{surf}}) = 0$, we conclude

$$\frac{\partial I}{\partial t} = c_p \Pi \frac{\partial \Theta}{\partial t} - \hat{r}^2 \frac{\partial p}{\partial s} \frac{\partial \phi}{\partial t}. \quad (24)$$

By taking Π times the Θ tendency equation and $\partial_s p$ times the ϕ tendency equation, and then integrating horizontally, we conclude

$$\iint \frac{\partial}{\partial t} I = \iint \left(-c_p \Pi \nabla_s \cdot (\hat{r}^{-1} \Theta \mathbf{u}) + \mu \frac{\partial \pi}{\partial s} \frac{\mathbf{u}}{\hat{r}} \cdot \nabla_s \phi - \mu \frac{\partial \pi}{\partial s} g w \right) dA ds. \quad (25)$$

270 These terms will cancel with those in the kinetic energy equation.

For kinetic energy, we can remove vertical transport terms and terms that cancel pointwise to find

$$\begin{aligned} \iint \frac{\partial K}{\partial t} = \iint & \left(-\frac{1}{2} \frac{\partial \pi}{\partial s} \frac{\mathbf{u}}{\hat{r}} \cdot \nabla_s \mathbf{u}^2 - \frac{1}{2} \mathbf{u}^2 \nabla_s \cdot \hat{r}^{-1} \frac{\partial \pi}{\partial s} \mathbf{u} - \frac{\partial \pi}{\partial s} \frac{\mathbf{u}}{\hat{r}} \cdot \nabla_s \frac{1}{2} w^2 - \frac{1}{2} w^2 \nabla_s \cdot \hat{r}^{-1} \frac{\partial \pi}{\partial s} \mathbf{u} \right. \\ & \left. - g w \frac{\partial \pi}{\partial s} + g w \frac{\partial \pi}{\partial s} \mu - \frac{\mathbf{u}}{\hat{r}} \cdot \frac{\partial \pi}{\partial s} (c_p \theta_v \nabla_s \Pi + \mu \nabla_s \phi) \right) dA ds. \end{aligned} \quad (26)$$

The first four terms cancel through integration by parts in the horizontal. The fifth term cancels with the third term in Eq. (23).

275 The sixth term cancels with the third term in Eq. (25), since $g w \partial_s \pi \mu = \hat{r}^2 g w \partial_s p$. The seventh term, through integration by parts, cancels with the first term in Eq. (25) via integration by parts in the horizontal. The final term cancels pointwise with the second term in Eq. (25) by noting that $\mu \partial_s \pi = \hat{r}^2 \partial_s p$.

Discrete energy consistency relies on the Lorenz vertical staggering, in which model variables are either located on full model levels or their surrounding interface levels, as detailed below. If we use a hybrid coordinate specification with N full
280 model levels, then model interfaces $\pi_{k+\frac{1}{2}}$ are indexed from $\pi_{\frac{1}{2}}$, which is the interface at the top of the atmosphere, to $\pi_{N+\frac{1}{2}}$, which is the surface. Recall that the vertical coordinate at model levels can be computed as

$$\pi_k = \frac{1}{2} (\pi_{k+\frac{1}{2}} + \pi_{k-\frac{1}{2}}).$$

The quantities located on full model levels are $\mathbf{u}, \Theta, \theta_v, \partial \pi / \partial s, \partial \phi / \partial s, \hat{r}_{\text{avg}}^2, p$, and Π . The quantities located on model interfaces are ϕ, w, π, μ , and $\hat{r}$. Appendix A specifies the choices of averaging between model and interface levels that guarantee spatial
285 energy consistency for terms that depend on deep-atmosphere geometric corrections.



2.4 IMEX and tracer consistency

Operational NHS HOMME simulations are performed with an Implicit-Explicit (IMEX) Runge-Kutta time stepping scheme, which uses operator splitting to evolve vertical acoustic modes using implicit time stepping (Steyer et al., 2019). The internal stages used to evolve u , v , Θ_v , and $\partial_s \pi$ are decoupled from the implicitly treated terms. However, the method for computing
 290 the internal stages for ϕ and w must be modified in NHD HOMME. Steyer et al. (2019) derived a single non-linear equation (Equation 4.8 in their article) that can be solved to find the internal stages for the implicitly treated variables w and ϕ . This equation is solved using Newton's method, and the resulting Jacobian is block tri-diagonal within the vertical columns of the model. Therefore, each column can be solved independently. The entries of this matrix become more complicated under the NHD equation of state and are derived in Appendix B. NHD HOMME retains the columnar independence of NHS HOMME.
 295 We omit vertical variation in gravity in the IMEX subroutines since the discarded terms are proportional to the vertical gradient of gravity and are therefore negligible on earth-sized planets.

E3SM uses the Interpolation Semi-Lagrangian Element-based Transport (Islet) scheme (Bradley et al., 2022) to solve for the transport of tracer species. Due to the computational expense of tracer transport in climate simulations with potentially hundreds of tracer species, transport uses an operator-split scheme, where the timestep for tracer transport is much larger than
 300 that of dynamics. The Islet scheme solves the transport equations with guaranteed range-preservation and mass conservation for large numbers of tracers while minimizing inter-process communication. Tracer consistency between dynamics and transport in NHS HOMME is imposed by ensuring that time-averaged pseudo-density and wind are treated consistently. In NHD HOMME, the argument for the divergence operator must be scaled by $\hat{r}^{-1}$ in Eq. (16). Once this is done, the arguments in (Bradley et al., 2022) that show consistency and conservation follow immediately for the redefinition of pseudo-density in NHD HOMME.

305 3 Results

To validate the correct performance of our implementation of the NHD HOMME, we simulated several highly idealized test cases designed for dynamical cores. These tests are performed with the version of NHD HOMME that assumes no vertical variation of gravity. The first two tests derive from the baroclinically unstable base state in Ullrich et al. (2014). This paper specifies separate base states for the NHD and NHS equations. Both test cases are performed on a planet with a reduced-
 310 size radius of $a/20$. The dynamical differences between the NHS and NHD equations increase on smaller planets because the approximation $r/a \approx a$ becomes less suitable as a decreases. Consequently, test cases that simulate the circulation on a small-size planet are commonly used to verify previous NHD dynamical cores (Yessad and Wedi, 2011; Skamarock et al., 2021; Ullrich et al., 2014). In the first test, we use the base-state of the atmosphere to compare the ability of NHD HOMME and NHS HOMME to approximate their respective equations of motion. In the second test, we simulate a baroclinic wave and
 315 compare it against other deep-atmosphere dynamical cores in the literature. Next, we perform simulations on a planet with a radius $a/20$ using the Held-Suarez forcing (Held and Suarez, 1994) adapted to small-planet simulations using the method from Wedi and Smolarkiewicz (2009). Finally, we compare the atmospheric circulation induced by an idealized equatorial heating



pattern (Gill, 1980). This test shows a measurable difference between NHS HOMME and NHD HOMME on an earth-sized planet with grid spacings exceeding 100 km.

320 In simulations that are performed on planets smaller than earth, the planetary period is adjusted so that the Rossby number for the small planet is comparable to that of earth. To do this, the rotation rate of the planet is scaled according to $\Omega_{\text{small}} = X\Omega$ for a planet $a_{\text{small}} = a/X$ where X symbolizes the reduction factor. Due to the decreased length scale of horizontal motions, planetary systems evolve more quickly on a reduced-radius planet (Wedi and Smolarkiewicz, 2009). As a result, simulation "days" are calculated based on the planet's period of rotation. To reduce ambiguity, we refer to these as "reduced-radius days"

325 in our plots and discussions.

All the simulations in this paper are performed using quasi-uniform meshes derived from the gnomonic cubed-sphere grid. This is the standard grid for quasi-uniform HOMME simulations. Spectral finite element methods exhibit persistent grid-scale numerical artifacts that are damped using artificial diffusion mechanisms Ullrich et al. (2018). HOMME uses operator-split fourth-order hyperviscosity to damp grid-scale computational modes in the prognostic variables. On quasi-uniform grids, the coefficient of the diffusion term ν is constant across all grid points and has units of $\text{m}^4 \text{s}^{-1}$. It is calculated for a particular

330 quasi-uniform grid with a nominal element width Δx according to

$$\nu = \left(\frac{\Delta x_{\text{ref}}}{\Delta x} \right)^{3.2} 10^{15} \text{ m}^4 \text{ s}^{-1},$$

where Δx_{ref} is the nominal width of an element on an "ne30" (30 elements per cubed-sphere panel edge) grid in HOMME. On an earth-sized planet, this corresponds to an average gridpoint spacing of about 120 km ($\approx 1^\circ$). The ratio of the grid spacings

335 is unitless, so they must be calculated on the same size of sphere.

3.1 Steady-state maintenance

We validated that NHD HOMME correctly solves the NHD equations by simulating a highly idealized steady-state atmosphere for which the temporal evolution of the atmosphere is known analytically. Ullrich et al. (2014) constructed zonally symmetric steady-state initial conditions for the NHD and the NHS equations. When a dynamical core is initialized with this state and

340 simulated forward in time, the deviation from the analytic initial conditions is a quantitative measure of the error introduced by the numerical discretizations in the dynamical core. Because the particular base state in Ullrich et al. (2014) is constructed to be baroclinically unstable, perturbations to the steady-state will grow exponentially. The amount of time between the beginning of a steady-state test and the breakdown of the steady-state is a quantitative measure of how well a dynamical core represents the continuum equations of motion it was formulated to solve. For well-tuned shallow dynamical cores, the steady-state can

345 be maintained for several weeks (Jablonowski and Williamson, 2006; Lauritzen et al., 2010). Ullrich et al. (2014) only derives the deep-atmosphere steady-state when $g = \hat{r}^{-2}g_0$. However, the functional form of g only determines the functional form for temperature for the family of steady-states used in this test case (Staniforth and Wood, 2013). As a result, by omitting the term $\hat{r}^2$ in Equation 13 in Ullrich et al. (2014), we derive a steady-state for the deep-atmosphere equations with constant gravity.

We performed steady-state simulations in NHS HOMME and NHD HOMME using their respective steady-states. We used a

350 planetary radius of $a/10$. Skamarock et al. (2021) found that on planets of radius $a/20$, the temperature profile in Ullrich et al.

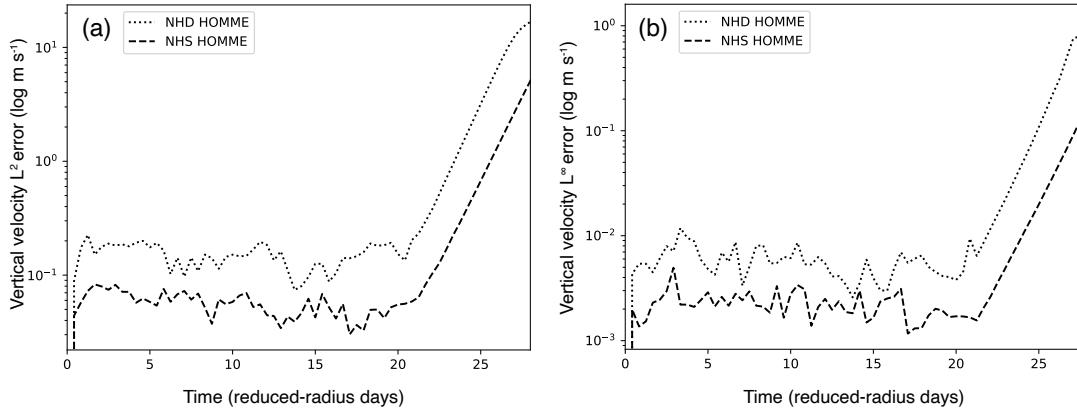


Figure 1. Time series of steady-state maintenance error metrics from steady-state simulations with planetary radius $a/20$. Panel (a) shows evolution of the vertical velocity error in the L^2 norm, and panel (b) shows the vertical velocity error in the L^∞ norm.

(2014) causes the base state to be statically unstable near the equator and derives a modified temperature profile that removes this instability. For the steady-state experiments, we use the default meridional profile for temperature. Our simulations use the 30 vertical levels utilized in the Community Atmosphere Model version 5 (CAM5) from the National Center for Atmospheric Research (NCAR) (Neale et al., 2010). Specific vertical coordinate coefficients for the 30 levels can be found in Reed and Jablonowski (2012). The height of the top interface level is approximately 38 km, with a pressure of approximately 2 hPa. We use the initialization procedure described in Appendix C to ensure that NHS and NHD HOMME were initialized to be consistent with their respective mass coordinate definitions.

Figure 1 compares the time evolution of the L^∞ and L^2 errors for vertical velocity w in NHS HOMME and NHD HOMME. The analytic reference solution is $w \equiv 0$. The L^∞ error is approximated as the point-wise maximum over all grid point values. The L^2 error is calculated as

$$\left(\iint w(\lambda, \varphi, s)^2 dA ds \right)^{1/2},$$

using the discrete Gauss-Lobatto-Legendre quadrature to integrate over the unit sphere. The exponential growth in the vertical wind after 20 simulation days is due to the breakdown of the base state. Note that the vertical scales for the two panels are different. In both metrics, the error in the NHD simulation is approximately an order of magnitude larger than the error in NHS HOMME over the first 20 days. The breakdown of the base state occurs at nearly identical times in both models, after approximately 20 days. The additional error in NHD HOMME is concentrated in zonally symmetric, fast-moving waves that do not amplify over time. The time-mean zonal mean vertical velocity does not show any coherent meridional or vertical structure in the error [not shown]. Vertical velocity was selected for this comparison because it was the most sensitive to errors in initialization. However, similar results can be derived for meridional velocity. The ability of NHD HOMME to maintain the deep-atmosphere steady-state for 20 days validates that our implementation of the added metric and Coriolis terms is correct.



Several factors may contribute to the increased discretization error in NHD HOMME compared to NHS HOMME. This error is sensitive to the numerical quadrature scheme used in the initialization routine. The mismatch between discrete and continuous hydrostatic balance becomes larger on smaller planets in NHD HOMME, while it remains constant in NHS HOMME. In addition, isobaric surfaces and surfaces of constant π coincide in NHS HOMME, but diverge in NHD HOMME. Consequently, the NHD HOMME results are sensitive to discretization errors in the pressure gradient force. Conversely, the pressure gradient force is zero to machine precision in NHS HOMME. Since NHD HOMME is designed for simulating earth's atmosphere, these initialization errors are negligible in practical simulation. Further improvements to initialization are beyond the scope of the current work.

3.2 Baroclinic wave

Idealized simulations of baroclinic waves are commonly used to assess the performance of atmospheric dynamical cores. When a small perturbation is added to a baroclinically unstable base state, the instability amplifies into a synoptic-scale baroclinic wave. Several baroclinic wave test cases have been developed for dynamical cores (Jablonowski and Williamson, 2006; Polvani et al., 2004; Ullrich et al., 2014; Hughes and Jablonowski, 2023). However, test cases based on the steady-state from Ullrich et al. (2014) are presently the only baroclinic wave tests that have an analytic specification for the NHD equations. Several previous deep-atmosphere dynamical cores (Skamarock et al., 2021; Borchert et al., 2019) have published results from this test case. The waves evolve almost identically under the NHS equations and the NHD equations on a planet the size of earth. We found that NHD HOMME can reproduce this baroclinic wave. However, since the simulation is nearly identical to NHS HOMME, we do not show these results. Instead, we examine baroclinic wave simulations on a planet with radius $a/20$.

Our simulations were configured as follows. All simulations are performed on an ne30 grid with the same 30 vertical levels as in the steady-state case. We use the initialization procedure from the steady-state case, but reintroduce the vertical variation of gravity in the NHD(g) HOMME simulations. We used the five-stage 2nd-order scheme derived in Steyer et al. (2019) with a 15 second time step. Vertical remapping was performed every 3 dynamics time steps. Surface pressure was diagnosed as $p_{\text{surf}} = p_N - \Delta\pi_N/2$. Of the two perturbation types described in Ullrich et al. (2014), we use the one derived from a streamfunction. This choice allows us to directly compare our results against simulations in the literature.

We found that the NHD(g) HOMME simulations are comparable to the results in the literature. Figure 3 shows the evolution of surface pressure using identical parameters to the original test case description. The columns depict the evolution of the wave in NHS HOMME, NHD HOMME, and NHD(g) HOMME, respectively. The first and third columns show close agreement with Skamarock et al. (2021) and Borchert et al. (2019). These previous studies concluded that the leading low pressure system is well developed after ten simulation days and is comparable between the ICON and MPAS dynamical cores. However, Skamarock et al. (2021) found that the trailing eddies were sensitive to changes in diffusion constants. This sensitivity makes it unlikely that different dynamical cores would produce the same structure. As a result, the ICON, MPAS, and NHD(g) HOMME simulations all differ in the small-scale features to the west of the leading edge of the wave. The second column in Fig. 3 shows substantial changes to the structure of the wave when the strength of gravity is constant. The wave amplifies at a slower rate: while the variable gravity simulation has a minimum surface pressure of around 900 hPa, the minimum surface pressure in the



405 constant gravity simulation is around 950 hPa. On small planets, the magnitude of the vertical gradient of gravity is greatly increased. However, the structure of the wave is impacted by a combination of the change of equation set as well as the change in the structure of the base state.

Due to the nature of baroclinic instability, the meridional gradient of the analytic temperature profile heavily influences the growth rate of the resulting wave. Borchert et al. (2019) observed grid-scale noise at the equator in simulations of the small-planet baroclinic wave performed in ICON. The same noise was observed in MPAS simulations, and Skamarock et al. (2021) consequently identified that the equatorial base state for the NHD equations exhibits a region of static instability that is not present in the NHS steady-state. They identified that this static instability could be eliminated by redefining the T_0 parameter as $T_0 = \alpha T_E + (1 - \alpha) T_P$, with $\alpha = 0.8$, see Ullrich et al. (2014) for the full formulation of the initial conditions. The original steady-state is recovered by setting $\alpha = 0.5$. Figure 2 depicts the zonal wind (filled contours) and potential temperature (black contours) for the base states derived using the NHS, NHD (constant gravity), and NHD (variable gravity) equations. The first row shows the base states when $\alpha = 0.5$, and the second shows the modified base state used by Skamarock et al. (2021). A thick black contour indicates a region of static instability. Figure 2(c) highlights that the potential temperature and zonal wind structure of the steady-state for the NHD equations with variable gravity differ markedly from the other two equation sets when $\alpha = 0.5$. However, when $\alpha = 0.8$, the magnitude of the temperature gradient is reduced, and the static instability at the equator is eliminated. Furthermore, the vertical extent of the zonal jet and the altitude at which the jet is maximized become comparable to the NHS HOMME base state. Comparing panels (a), (c), and (f) in Fig. 2 shows that $\alpha = 0.8$ results in a steady-state solution to the NHD (variable gravity) equations that has a more comparable temperature gradient in the mid-latitudes to the NHS steady-state. Conversely, Fig. 2(e) demonstrates that increasing alpha results in a steady-state solution to the NHD (constant gravity) equations with a weaker temperature gradient than the NHS steady-state.

425 Figure 4 depicts surface pressure after ten days of simulation in NHD HOMME and NHD(g) HOMME using base states with varying values of α . Panel (a) shows NHD(g) HOMME with $\alpha = 0.8$, panel (b) shows NHD HOMME with $\alpha = 0.8$, and panel (c) shows NHD HOMME with $\alpha = 0.4$. Figure 4(a) displays qualitative agreement with the lower-right panel of Figure 9 in Skamarock et al. (2021). When $\alpha = 0.8$ and gravity is constant, the surface pressure in Figure 4(b) exhibits slower amplification of the wave. This is consistent with the weaker mid-latitudinal temperature gradient in the base state. Since the rate of amplification in NHD HOMME with $\alpha = 0.5$ is slower than NHS HOMME (Figure 3(h)), we should instead decrease alpha to $\alpha = 0.4$. This results in a base state with a stronger meridional temperature gradient. However, the atmosphere is also statically unstable at the equator [not shown]. Therefore, the variation in the rate of amplification between NHS HOMME, NHD HOMME, and NHD(g) HOMME reflects the changes to the base state induced by changing equation set. Comparing Fig. 3(g), Fig. 4(a), and Fig. 4(c), we can see that qualitatively similar baroclinic waves can be achieved in each equation set if the base states are chosen to have comparable available potential energy in the mid-latitudes.

3.3 Energy conservation

Section 2 and Appendix A demonstrate that the mathematical description of NHD HOMME is energy consistent. We now validate that the software implementation of NHD HOMME satisfies energy consistency to machine precision using a modified

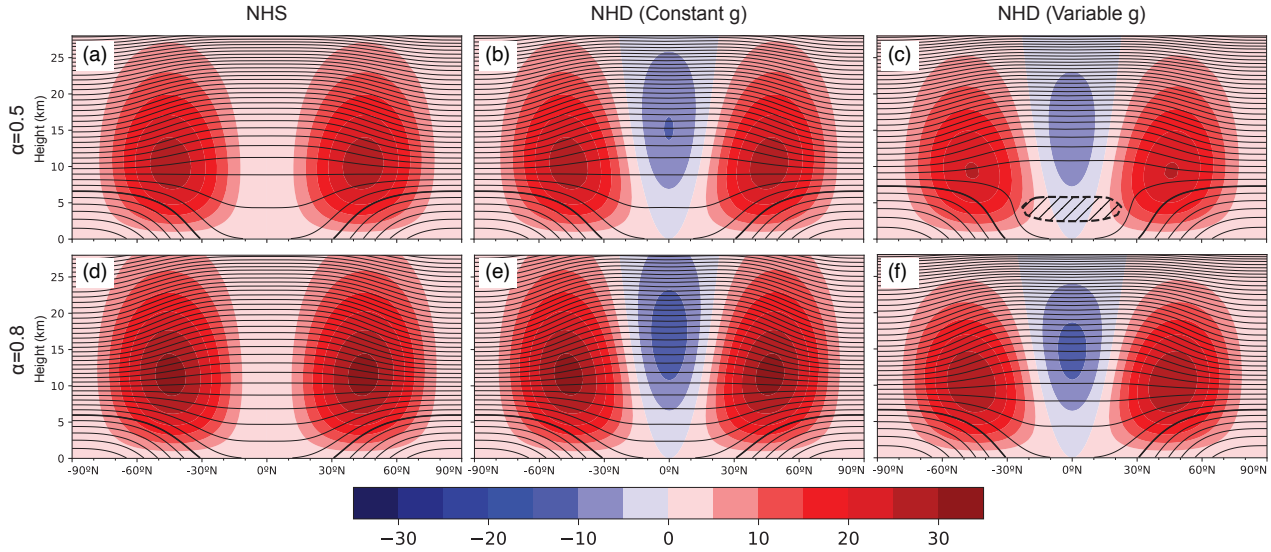


Figure 2. Latitude-height cross sections of zonally-symmetric analytic steady-states for different parameter choices on a 20x smaller planet. Zonal wind (m s^{-1}) is shown in colored contours. Potential temperature is shown in thin black contours, with a contour interval of 10 K, where the thick solid contour indicates $\theta = 300$ K. Regions where the atmosphere is statically unstable are hatched. The first row shows the steady-states with the parameters described in Ullrich et al. (2014), and the second row uses the modified temperature described in Skamarock et al. (2021).

version of the approach from Taylor et al. (2020). In the model code, energy consistency means that when all forms of diffusion
 440 and vertical remapping are disabled, the loss of total energy E incurred over the temporal integration decreases proportionally
 to Δt^k , where k is the order of accuracy of the time stepping method. This experiment is based on the baroclinic wave test with
 $\alpha = 0.5$, on a small planet with $X = 20$. We use the same 30 levels as in the previous baroclinic wave simulations, but we use
 a "ne8" (nominal 4° grid) as in Taylor et al. (2020). On coarse grids, the atmospheric base state breaks down quickly, and the
 details of the baroclinic wave are not resolved. The test is performed in two phases. In the first phase, the baroclinic wave is
 445 simulated for a period of eight days with vertical remapping and hyperviscosity settings enabled. During the second phase, the
 model is re-initialized from the atmospheric state at the end of the first stage, but vertical remapping and all forms of external
 diffusion are disabled. We perform simulations with a duration of two hours using time steps of $\Delta t = 60$ s, 6 s, 0.6 s, and 0.06 s
 and measure the discrete total energy change over the duration of the integration. Relative energy change was calculated as

$$\frac{E(t_{\text{end}}) - E(t_{\text{start}})}{E(t_{\text{start}})}.$$

450 Discrete energies are calculated using the definitions of internal, kinetic, and potential energy above, where interface quantities
 are averaged to model levels when necessary. Calculations were done on the native GLL grid, using GLL quadrature for hori-
 zontal integrals and mass-weighted sums for vertical integrals. We observe comparable energy conservation behavior between

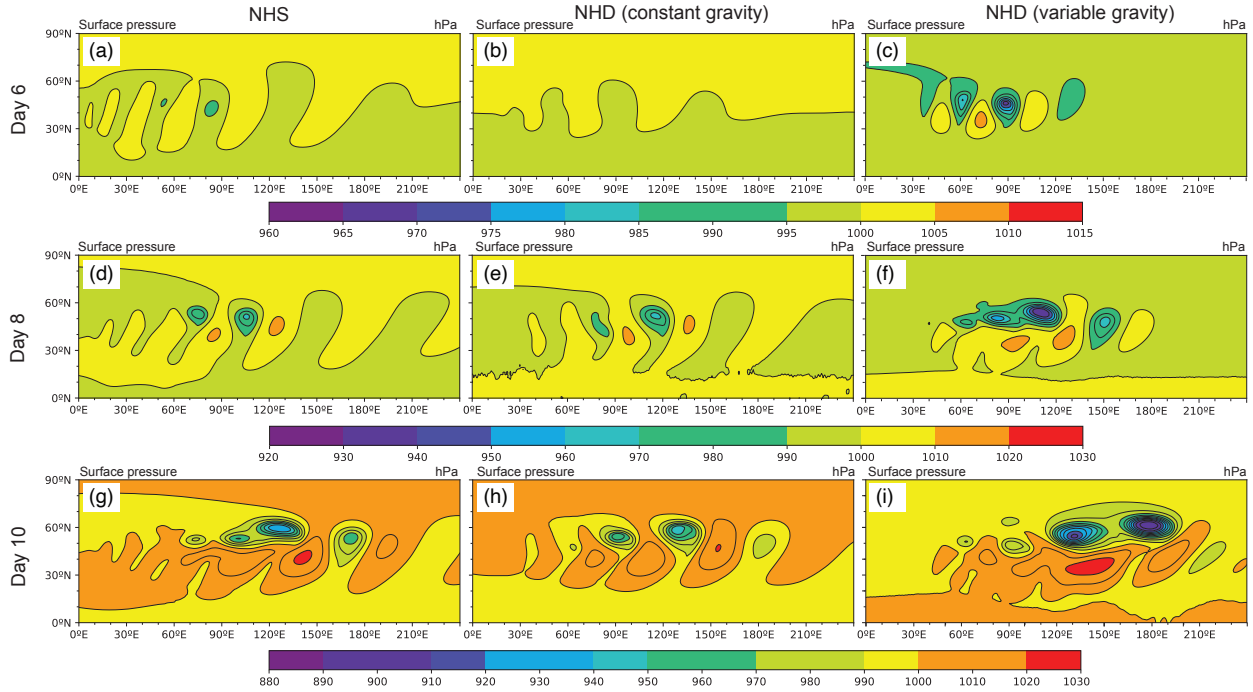


Figure 3. Latitude-longitude cross sections of the evolution of surface pressure for the Ullrich et al. (2014) baroclinic wave on a 20x smaller planet with $\alpha = 0.5$. The first column shows the evolution in NHS HOMME, the second column shows NHD HOMME with constant gravity, and the third column shows NHD HOMME with variable gravity.

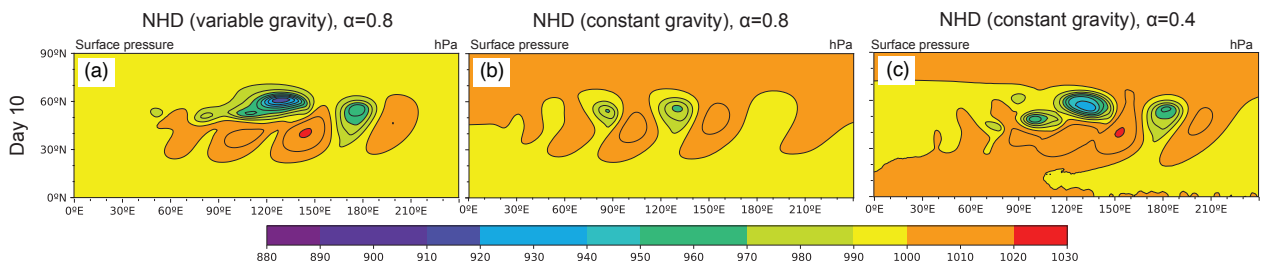


Figure 4. Latitude-longitude cross sections of surface pressure at day 10 for variants of the Ullrich et al. (2014) baroclinic wave on a 20x smaller planet with modified temperature profiles. Panel (a) shows the simulation in NHD(g) HOMME for $\alpha = 0.8$, panel (b) shows NHD HOMME for $\alpha = 0.8$, and panel (c) shows NHD HOMME for $\alpha = 0.4$.

NHS HOMME and NHD HOMME. Figure 5 shows the rate of convergence in the total energy change for NHS HOMME (dotted) and NHD HOMME (dashed).

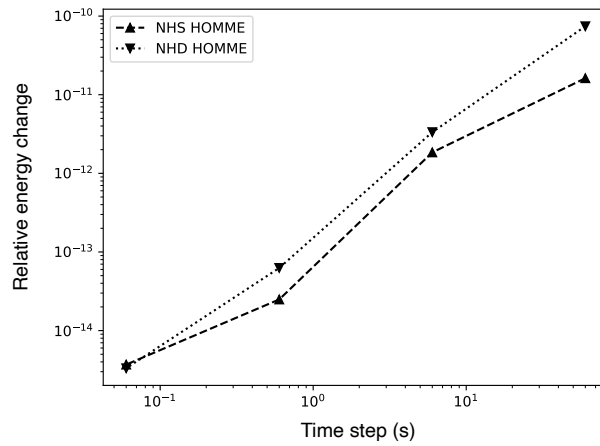


Figure 5. A log-log plot of relative energy change vs timestep for two-hour simulations of a baroclinic instability simulation on an ne8 grid when vertical remapping and diffusion are disabled. The dotted line shows NHD HOMME simulations, and the dashed line shows NHS HOMME.

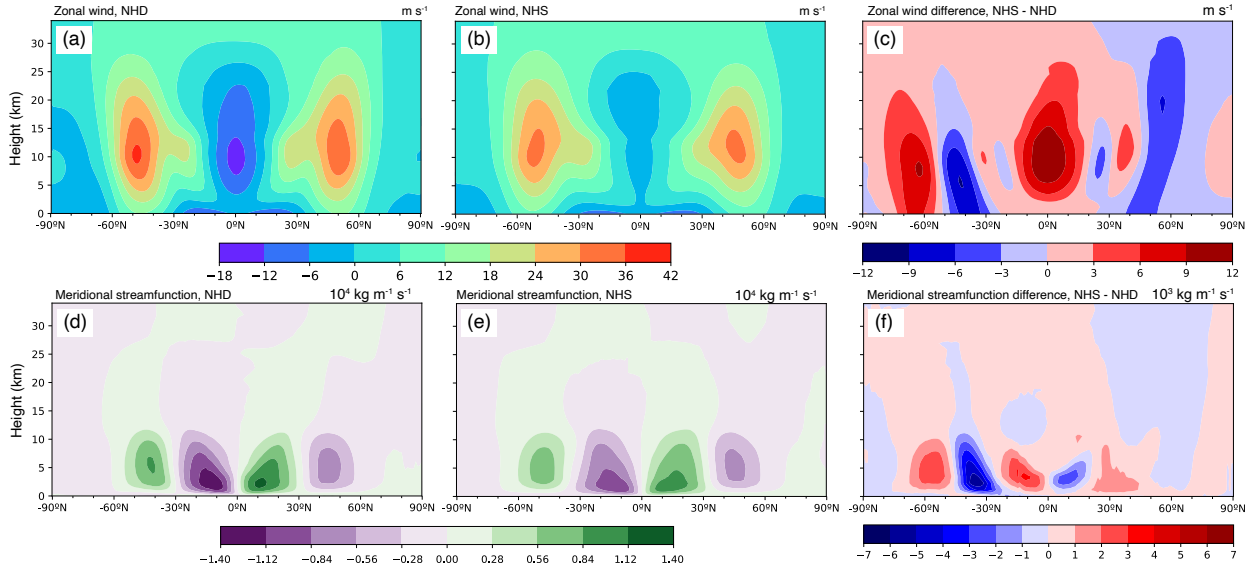


Figure 6. Latitude-height cross sections of (top row) time-mean zonal-mean zonal velocity and meridional mass streamfunction (bottom row) for small-planet Held-Suarez simulations in the (a, d) NHD HOMME and (b, e) NHS HOMME models. Panels (c) and (f) show the difference between the NHD HOMME and NHS HOMME simulations for zonal wind and meridional streamfunction, respectively.



455 3.4 Held Suarez

We also assess the behavior of NHD HOMME in decadal simulations using a Held-Suarez (Held and Suarez, 1994) simulation on a small planet. This highly idealized Held-Suarez forcing scheme produces an atmospheric circulation that equilibrates to a stable climatology after several years of spin-up. Held-Suarez results from existing deep-atmosphere dynamical cores on small planets can be found in the literature (Yessad and Wedi, 2011). This spin-up period is discarded from the analysis. The test case is typically specified using a terrain-following pressure coordinate, $\sigma = p/p_s$ where p_s denotes the surface pressure. The idealized forcing comes in two parts. Firstly, the temperature in the atmosphere is nudged via a Newtonian relaxation towards a physically plausible analytic reference profile with a time-invariant relaxation timescale. Secondly, the horizontal wind components below 700 hPa are slowed via Rayleigh friction with a time-invariant relaxation timescale to mimic the dissipative effects in the boundary layer. The relaxation timescales depend on latitude and σ . Due to the chaotic evolution of the atmosphere, the climatology of Held-Suarez tests is not sensitive to how the atmosphere is initialized. In our simulations, we use the baroclinic wave from the previous section. Wedi and Smolarkiewicz (2009) adapted the Held-Suarez forcing to be suitable for small planets by decreasing the relaxation timescales proportional to the factor by which the planetary radius is scaled. We simulate on a planet with a radius of $a_{\text{earth}}/20$; thus, the relaxation timescales are 20 times smaller.

The Held-Suarez test case was originally formulated for hydrostatic models, and generalizing it to non-hydrostatic models can be done in several non-equivalent ways (see, e.g., (Wedi and Smolarkiewicz, 2009; Smolarkiewicz et al., 1998)). In our simulations, $\sigma = p/p_s$, where p, p_s are thermodynamic pressures. Calculating the forcing profiles from pressure ensures that the NHS HOMME and NHD HOMME simulations are not impacted by the redefinition of the mass coordinate. Yessad and Wedi (2011) does not define how they construct σ , which diminishes our ability to compare it to their results.

We performed simulations for 2000 reduced-radius days in NHS HOMME and NHD HOMME on a "ne16" (nominal 2°) grid with the same 30 vertical levels as the baroclinic wave tests. The reduction of the planetary radius results in a nominal grid spacing of 12 km. The first 720 days were discarded as spin-up. Figure 6(a) and (b) show the zonal-mean time-mean zonal wind from NHD HOMME and NHS HOMME, respectively. The Held-Suarez forcing heats the equator, inducing a direct Hadley-like circulation. In NHD HOMME, the presence of the NCTs causes a westward deflection of rising air. This results in the region of westward mean flow at the equator in NHD HOMME. Figure 1 in Yessad and Wedi (2011) shows the results of reduced-radius Held-Suarez simulations from several deep-atmosphere configurations of the Integrated Forecast System's (IFS) dynamical core. All simulations depict a similarly pronounced westward mean flow that we observe in NHD HOMME. Figure 6(c) shows the difference in zonal mean climatology between NHD HOMME and NHS HOMME. The differences exhibited in the mid-latitudes are likely sampling errors and, therefore, not systematic. The different dynamical core configurations in Yessad and Wedi (2011) produce small changes in the position and structure of the mid-latitude jets. Therefore, it is harder to attribute mid-latitudinal differences between NHS and NHD HOMME to the use of the NHD equations. Figure 6(f) shows the difference between the vertical mass streamfunction in NHS and NHD HOMME. The sign of the change to the equatorial Hadley circulation is opposite to the sign of the streamfunction. Therefore, the re-introduction of the NCTs weakens the Hadley

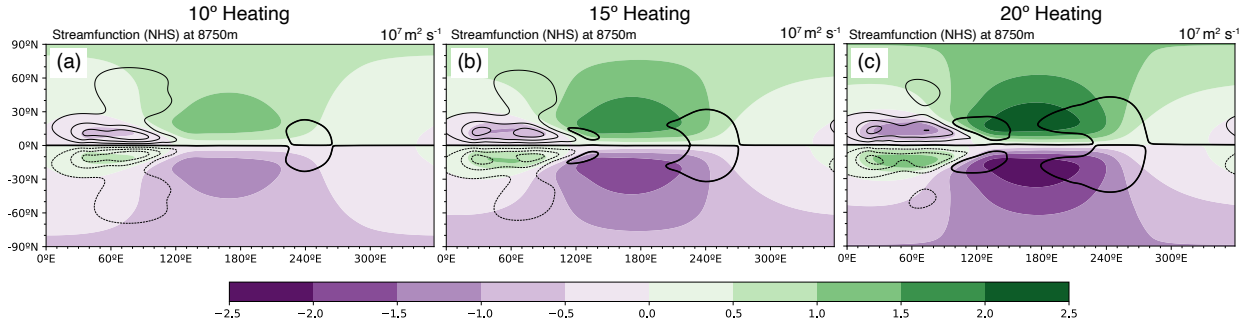


Figure 7. NHS HOMME horizontal streamfunction (filled contours) and difference in horizontal streamfunction NHS - NHD HOMME (solid contours) along surfaces of constant height $z = 8750$ m for the Matsuno-gill forcing. Both quantities are calculated at day 6. Solid black contours indicate positive values, and dashed lines indicate negative values, and the thick solid contour is zero. The contour interval for streamfunction difference is $5 \cdot 10^5 \text{ m}^2 \text{ s}^{-1}$. Panels (a), (b), and (c) show simulations with total meridional heating profiles with 10° , 15° , and 20° , respectively.

circulation on small planets. Due to the high variability of the circulation in the mid-latitudes, the changes to the streamfunction outside of the equator cannot be directly attributed to the change in equation set.

490 3.5 Tropical Heating

We showed that our implementation of the NHD equation set can reproduce the effects of the NCTs on the equatorial response to idealized cumulus-like heating patterns on earth-sized planets. Hayashi and Itoh (2012) and Ong and Roundy (2019) demonstrated that the restoration of the NCTs systematically changes the dynamical response to sinusoidal synoptic diabatic forcings in linearized numerical models. These forcings resemble the analytic work done in Gill (1980), but they differ in their choice of functional forms for the forcings. Similarly, the so-called "Matsuno-Gill" forcing that has been used to validate dynamical cores (Tomita and Satoh, 2004) produces qualitatively similar circulations to linearized studies but uses yet another functional form for the heating pattern. In this section, we use the functional forms from Tomita and Satoh (2004) to compare simulations in NHS HOMME and NHD HOMME and use the dynamical explanation of the effects of the NCTs discovered in Hayashi and Itoh (2012) to interpret the resulting circulation changes.

500 We performed "Matsuno-Gill" simulations using the test specification given in Tomita and Satoh (2004). The specific functional form for diabatic heating is given by Equation 72 in Tomita and Satoh (2004). The initial state in both deep and shallow models was assumed to be at rest and in hydrostatic balance. The temperature profile was derived by assuming a constant Brunt-Väisälä frequency $N = 0.01 \text{ s}^{-1}$, with a surface temperature of 300 K. We used a quasi-uniform 1° grid with 20 vertical levels. The level coefficients were derived from the initial state so that the vertical grid spacing was constant with $\Delta z = 500$ m. Consequently, the model top was located at 10 km. To examine the dependence of the effects of the NCTs on the meridional gradient of diabatic heating, we performed simulations with variable meridional heating extents (that is, $2\Delta\phi$ in the notation of Tomita and Satoh (2004)). We used meridional forcings with 10° , 15° , and 20° widths. All simulations are performed with a

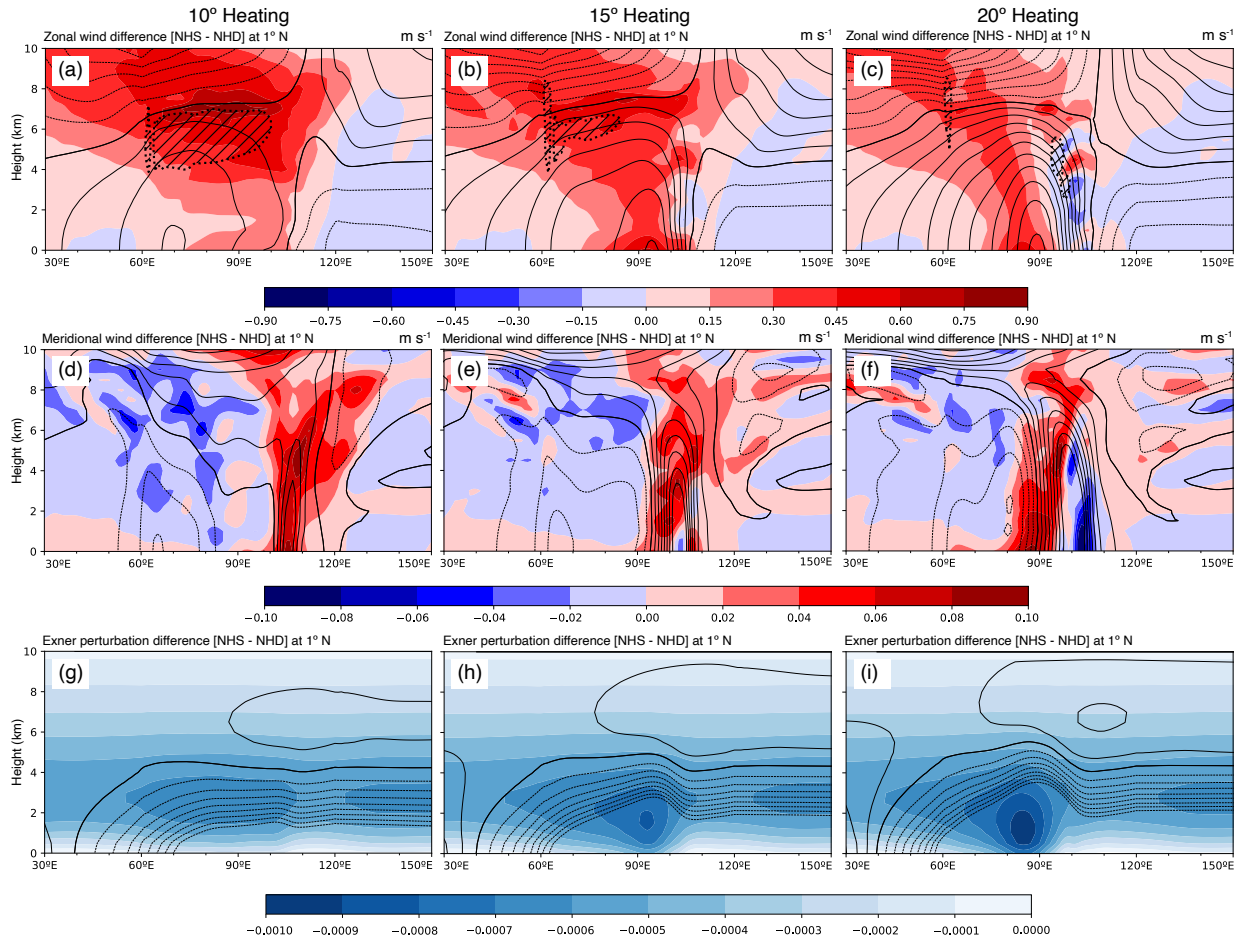


Figure 8. Longitude-height cross sections of NHS HOMME atmospheric state (solid contours) and difference NHS - NHD HOMME (filled contours) at day 6 using the Matsuno-gill forcing. The first, second, and third columns show results from simulations with total meridional heating extents of 10° , 15° , and 20° . The top row shows zonal wind, the middle row meridional wind, and the bottom row the perturbations of the Exner function. Solid black contours indicate positive values, dashed contours indicate negative values, and the thick solid contour is zero. The hatched region in the top row shows regions where $w > 0.015 \text{ m s}^{-1}$. Contour interval for the top, middle, and bottom rows are 3.0 m s^{-1} , 0.5 m s^{-1} , and 0.0005 , respectively.

60° zonal heating extent. The 20° simulation matches Tomita and Satoh (2004) exactly. Although Hayashi and Itoh (2012) use a different model top and functional form for heating, the meridional gradient of heating in our 10° simulations is comparable to their “RE” experiment and the gradient in the 20° simulations matches their “SQ” experiment.

The horizontal structure of the Matsuno-Gill flow confirms that the NCTs produce a non-negligible change in circulation. Figure 7 shows the horizontal streamfunction and the difference between NHD and NHS streamfunctions along constant-height

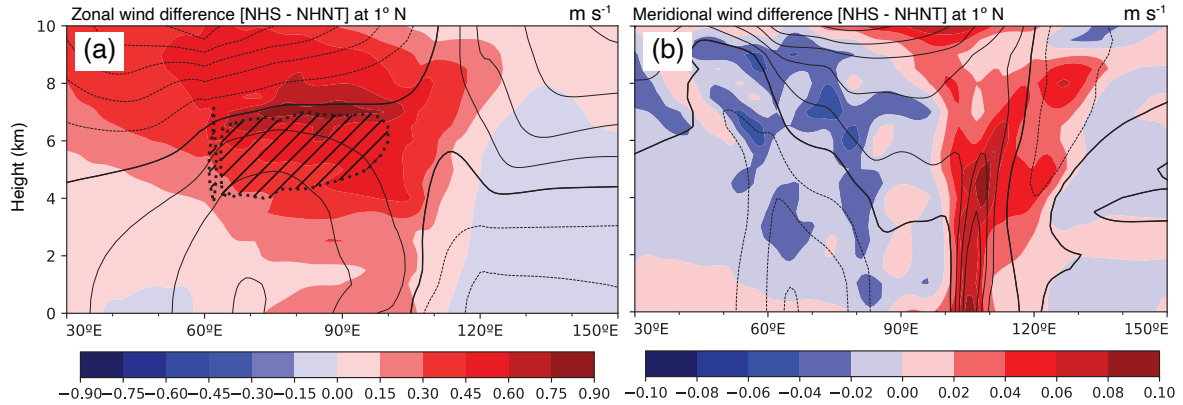


Figure 9. Longitude-height cross sections of NHS HOMME atmospheric state (solid contours) and difference NHS - NHNT HOMME (filled contours) at day 6 using the Matsuno-gill forcing. The panels show the (a) zonal wind and (b) meridional wind information. Solid black contours indicate positive values, dashed contours indicate negative values, and the thick solid contour is zero. The hatched region in the top row shows regions where $w > 0.015 \text{ m s}^{-1}$. Contour interval for panels (a) and (b) are 3.0 m s^{-1} , and 0.5 m s^{-1} respectively. The region contained in the thick dotted contour has a vertical velocity exceeding 0.015 m s^{-1} .

surfaces with $z = 8750 \text{ m}$ for each meridional forcing extent. Although studies in the literature differ on the specific form of the forcing, these forcings all produce a Kelvin wave response to the east of the forcing and a Rossby wave to the west of the forcing. Thuburn et al. (2002) found that restoring the NCTs changes the theoretical phase speed of equatorial Kelvin waves by several percent. In our simulations, the leading edge of the Kelvin wave begins to overlap with the Rossby response after eight days. Consequently, we were not able to discern a quantitative difference in phase speed between NHD HOMME and NHS HOMME. The changes induced by the NCTs were much more significant to the west of the forcing. Figures 7(a)-(c) depict that the strength of the circulation to the west of the forcing increases as the meridional extent of the forcing is increased. The black contours showing the difference in streamfunction demonstrate that the NCTs reduce the strength of the circulation, with the maximum change occurring at 60° E . The absolute strength of the change induced by the NCTs is nearly the same magnitude in the 10° and 20° cases. Consequently, the impacts of the NCTs relative to the strength of the underlying circulation increase as the meridional extent of the heating decreases. This is consistent with Hayashi and Itoh (2012), who found that the impact of the NCTs is proportional to the meridional gradient of vertical velocity, which is proportional to the meridional gradient of diabatic heating. We can further apply their framework to explain the vertical structure of the changes induced by the NCTs.

We find that the vertical structure of the changes induced by the NCTs is consistent with the tilted planetary vorticity framework of Hayashi and Itoh (2012). They found that the restoration of the NCTs causes tilting of the planetary vorticity vector when there is a meridional gradient of vertical velocity. Under the reasonable assumption that the NCTs do not induce divergent motion, the zonal gradient of the streamfunction decreases to account for this tilt (see, e.g., their Equation 28). The filled contours in Fig. 8 show cross-sections of the difference (NHS-NHD) between zonal wind (top row), meridional wind (middle row), and Exner perturbation (bottom row) for each heating profile after 6 days of simulation. These difference plots



can be compared to Figs. 2-3 in Hayashi and Itoh (2012), which show what they term the "contribution" of the NCTs. Their "contribution" plots subtract the results with the NCTs omitted from those with the NCTs present, so the sign of the difference must be flipped before comparing to our Fig. 8. These figures show that the contribution has an equivalent barotropic structure. In particular, the winds remain approximately parallel over the vertical extent of the atmosphere. The Exner perturbation is calculated as in Hayashi and Itoh (2012) by taking the difference between the Exner function at day 6 from its initial reference value, then subtracting the difference of these values in NHD HOMME from those in NHS HOMME. As in Fig. 3 in Hayashi and Itoh (2012), the maximum amplitude of the zonal wind difference is approximately collocated with the maximum value of vertical velocity (shown in hatching in the first row) for each heating profile. The maximum vertical velocity has a magnitude of $0.02 \pm 0.003 \text{ m s}^{-1}$ across all of the heating profiles. The maximum zonal wind difference increases as the meridional heating extent decreases. Therefore, the change in horizontal wind is correlated with the meridional variation in heating, rather than the strength of the upwelling. Unlike the previous validation experiments in this paper that were performed on small planets, these results imply that the NCTs are non-negligible even for simulations on earth-sized planets with 1° gridpoint spacing.

The difference between NHS and NHD HOMME simulations confirm that discarding the NCTs induces measurable, systematic biases in equatorial synoptic heating. However, we found that the differences between NHS and NHD HOMME can be rectified by reintroducing only the NCTs, resulting in the NHNT equation set. Figure 9 shows the same difference quantities as in Fig. 8 (a) and (d), but calculated using simulations that use the NHNT equation set. The filled contour patterns are nearly identical to those calculated using the NHD equations. Therefore, the NHNT equations would be the most computationally efficient way to rectify the tropospheric biases introduced by the traditional approximation.

4 Conclusions

In this work, we introduced a method for restoring spherical geometry to the HOMME dynamical core by removing the SA and T approximations. The mimetic discretizations used to show energy consistency of NHS HOMME were adapted to spherical geometry. Consequently, showing that NHD HOMME is energy consistent follows from modified versions of the arguments used for NHS HOMME. These modifications include the redefinition of discrete differential operators, the careful choice of discrete vertical averaging, and a modified artificial upper boundary condition. Our approach was designed to be readily integrated into the Fortran and C++ software implementations of NHS HOMME.

We validated the correctness of our software implementation using several idealized test cases and showed agreement with existing deep-atmosphere dynamical cores in the literature. The steady-state maintenance tests demonstrated that NHS HOMME and NHD HOMME approximate their respective equations of motion with comparable fidelity. Baroclinic wave simulations on small planets exhibited agreement with existing simulations in the literature when vertical variation of gravity was included. On small planets, using a constant value of gravity substantially changes the structure of the atmospheric base state. The resulting baroclinic wave simulations differ substantially from both shallow-atmosphere and variable-gravity deep-atmosphere dynamical cores. Skamarock et al. (2021) observed that differences between baroclinic waves in shallow and deep-atmosphere dynamical cores on small planets are closely related to the structure of the base state. Our results support their



565 contention that differences in wave structure and amplification are not primarily due to dynamical biases induced by the SA and T approximations. Next, Held-Suarez simulations on a small planet demonstrated the easterly bias induced by omitting the NCTs. The effects of the NCTs are pronounced on smaller planets, but the magnitude of the bias in NHD HOMME simulations closely matches results from the literature. Finally, we demonstrated that simulations of the Matsuno-Gill heating pattern on an earth-sized planet show systematic changes in synoptic flow. Simulations showed that the magnitude of the zonal wind bias
570 can match 10% of the base value. The errors are systematic and align closely with the predictions from linear studies.

Determining which of the three equation sets introduced in this paper (NHD HOMME, NHD(g) HOMME, and NHNT HOMME) should be used in operational E3SM will require further investigation. The results in this work show that the NHNT equations can rectify the well-studied equatorial tropospheric biases introduced by the traditional approximation at very little additional computational expense compared to NHS HOMME. The additional code required to implement NHD(g) HOMME
575 likely cannot be justified due to the lack of evidence that neglecting the vertical variation of gravity introduces dynamically significant biases in tropospheric simulation. However, this assessment might change if applications with very high model tops above the mesosphere are considered. If the NHNT and NHD HOMME models produce near-identical results using intermediate-complexity simulation frameworks such as Atmospheric Model Intercomparison Project (AMIP) simulations, this would support the adoption of the NHNT equations. However, careful examination of the impacts of the NCTs on equatorial
580 climatology is beyond the scope of the present study and is a subject for future research.

Code and data availability. Model data, the E3SM branch used to generate data for this paper, and code that reproduces the plots in this paper is available at <https://doi.org/10.5281/zenodo.21382684> (Owen et al., 2026).

Appendix A: Energy conservation

This appendix derives the constraints on how discrete quantities must be averaged to ensure energy conservation under perfect
585 temporal integration. In this section, a quantity $\bar{\xi}$ represents either the interpolation of an interface quantity to a model level or the interpolation of a model level quantity to an interface level. The work in this appendix is sufficient to guarantee energy conservation using the summation-by-parts and averaging identities that are used to show conservation of energy in Taylor et al. (2020). In the discrete model, the standard Lorenz staggering is used for the prognostic equations in NHS HOMME. The NHD HOMME quantity $\hat{r}^2$ can be derived directly from ϕ , and so is located on model interfaces. The quantity $\hat{r}_{\text{avg}}^2 =$
590 $(\hat{r}_{k+\frac{1}{2}}^2 + \hat{r}_{k+\frac{1}{2}} \hat{r}_{k-\frac{1}{2}} + \hat{r}_{k-\frac{1}{2}}^2)$, which is the vertical average of $\hat{r}^2$ between two interfaces, is required for the consistency of the vertical coordinate. $\hat{r}_{\text{avg}}^2$ only appears in the equation of state, and energy conservation holds even though this quantity does not follow the standard averaging for interface quantities in the HOMME vertical discretization.



The following are the NHD equations in discretized form, where $[\xi]$ denotes a quantity for which the correct averaging of ξ must be determined from energy conservation.

$$595 \quad \frac{\partial \mathbf{u}}{\partial t} + \left(\hat{r}^{-1} \nabla_s \times \mathbf{u} + f_s \right) \hat{k} \times \mathbf{u} + \frac{1}{2\hat{r}} \nabla_s (\mathbf{u}^2 + \overline{w^2}) - \hat{r}^{-1} \overline{w \nabla_s(w)} + \quad (A1)$$

$$\frac{\overline{\mathbf{u}w}}{\bar{r}} + f_c \begin{pmatrix} \overline{w} \\ 0 \end{pmatrix} + c_p \theta_v \hat{r}^{-1} \nabla_s (\Pi) + \hat{r}^{-1} \overline{\mu \nabla_s(\phi)} = 0 \quad (A2)$$

$$\frac{\partial w}{\partial t} + [\mathbf{u}/\hat{r}] \cdot \nabla_s(w) - [\mathbf{u}^2/r] - f_c[u] + g - \mu g = 0 \quad (A3)$$

$$\frac{\partial \phi}{\partial t} + [\mathbf{u}/\hat{r}] \cdot \nabla_s(\phi) - gw = 0 \quad (A4)$$

$$\frac{\partial \Theta}{\partial t} + \theta_v \nabla_s \cdot \left(\hat{r}^{-1} \frac{\partial \pi}{\partial s} \mathbf{u} \right) + \hat{r}^{-1} \frac{\partial \pi}{\partial s} \mathbf{u} \cdot \nabla_s(\theta_v) = 0 \quad (A5)$$

$$600 \quad \frac{\partial}{\partial t} \left(\frac{\partial \pi}{\partial s} \right) + \nabla_s \cdot \left(\hat{r}^{-1} \frac{\partial \pi}{\partial s} \mathbf{u} \right) = 0 \quad (A6)$$

$$\frac{\partial}{\partial t} \left(q \frac{\partial \pi}{\partial s} \right) + \nabla_s \cdot \left(\hat{r}^{-1} q \frac{\partial \pi}{\partial s} \mathbf{u} \right) = 0. \quad (A7)$$

with

$$\frac{\partial \phi}{\partial s} = -R\Theta \frac{\Pi}{\hat{r}_{\text{avg}}^2 p}, \quad \mu = \left[\hat{r}^2 \frac{\partial p}{\partial s} \right] \left(\frac{\partial \pi}{\partial s} \right)^{-1}.$$

The resulting energy tendencies are

$$\begin{aligned} 605 \quad \frac{\partial P}{\partial t} &= -\phi \nabla_s \cdot \left(\hat{r}^{-1} \frac{\partial \pi}{\partial s} \mathbf{u} \right) - \frac{\partial \pi}{\partial s} [\mathbf{u}/\hat{r}] \cdot \nabla_s(\phi) + \frac{\partial \pi}{\partial s} gw \\ \frac{\partial I}{\partial t} &= -c_p \Pi \nabla_s \cdot \left(\hat{r}^{-1} \Theta \mathbf{u} \right) - \left[\hat{r}^2 \frac{\partial p}{\partial s} \right] (-[\mathbf{u}/\hat{r}] \cdot \nabla_s(\phi) + gw) \\ \frac{\partial K}{\partial t} &= \frac{\partial \pi}{\partial s} \mathbf{u} \cdot \left(-\left(\hat{r}^{-1} \nabla_s \times (\mathbf{u}) + f_s \right) \hat{k} \times \mathbf{u} - \frac{1}{2\hat{r}} \nabla_s (\mathbf{u}^2 + \overline{w^2}) + \hat{r}_m^{-1} \overline{w \nabla_s w} \right. \\ &\quad \left. - \frac{\overline{\mathbf{u}w}}{\bar{r}} - f_c \begin{pmatrix} \overline{w} \\ 0 \end{pmatrix} - c_p \theta_v \hat{r}^{-1} \nabla_s \Pi - \hat{r}_m^{-1} \overline{\mu \nabla_s \phi} \right) \\ &\quad + \frac{\partial \pi}{\partial s} w (-[\mathbf{u}/\hat{r}] \cdot \nabla_s w + [\mathbf{u}^2/r] + f_c[\mathbf{u}] - g + \mu g) \\ 610 \quad &- \frac{1}{2} (\mathbf{u}^2 + \overline{w^2}) \nabla_s \cdot \left(\hat{r}^{-1} \frac{\partial \pi}{\partial s} \mathbf{u} \right). \end{aligned}$$

Like in the continuum case, these tendencies cancel only when these equations are integrated over the full model domain. In the discrete case, this turns into a double summation, where the first sum is over horizontal grid points and the second is over model levels. The details of discrete energy conservation do not depend on the details of these summations, so we omit them for brevity.



615 Terms that are not modified for the deep atmosphere are assumed to cancel as in Taylor et al. (2020). We now go through pairs of terms that must be shown to cancel in NHD HOMME. The pair

$$-\phi \nabla_s \cdot \left(\bar{\hat{r}}^{-1} \frac{\partial \pi}{\partial s} \mathbf{u} \right) - \frac{\partial \pi}{\partial s} ([\mathbf{u}/\hat{r}] \cdot \nabla_s(\phi)) = 0$$

cancel if we define $[\mathbf{u}/\hat{r}] = \frac{\partial \pi}{\partial s} \bar{\hat{r}}^{-1} / \frac{\partial \pi}{\partial s}$ via horizontal and vertical summation by parts. Note that interpolation is from model levels to interfaces. Next, the pairs

$$\begin{aligned} 620 \quad & -\frac{\partial \pi}{\partial s} \mathbf{u} \frac{1}{2\hat{r}} \nabla_s \mathbf{u}^2 - \frac{1}{2} \mathbf{u}^2 \nabla_s \cdot \left(\bar{\hat{r}}^{-1} \mathbf{u} \frac{\partial \pi}{\partial s} \right) = 0 \\ & -\frac{\partial \pi}{\partial s} \mathbf{u} \frac{1}{2\hat{r}} \nabla_s w^2 - \frac{1}{2} w^2 \nabla_s \cdot \left(\bar{\hat{r}}^{-1} \mathbf{u} \frac{\partial \pi}{\partial s} \right) = 0 \end{aligned}$$

cancel via integration by parts in the horizontal with no additional work. Thirdly, the terms

$$\begin{aligned} & -f_c \bar{w} \frac{\partial \pi}{\partial s} \mathbf{u} + f_c [\mathbf{u}] w \frac{\partial \pi}{\partial s} = 0 \\ & -\frac{\partial \pi}{\partial s} \mathbf{u}^2 \bar{\hat{r}}^{-1} \bar{w} + \frac{\partial \pi}{\partial s} w [\mathbf{u}^2/r] = 0 \end{aligned}$$

625 cancel if $[u] = \frac{\partial_s \pi \mathbf{u}}{\partial_s \pi}$ in the first equation and $[\mathbf{u}^2/r] = \frac{\partial_s \pi \mathbf{u}^2 \bar{\hat{r}}^{-1}}{\partial_s \pi}$ in the second via summation-by-parts in the vertical. Next,

$$\frac{1}{\hat{r}_m} \bar{w} \nabla_s w \frac{\partial \pi}{\partial s} \mathbf{u} - \frac{\partial \pi}{\partial s} [\mathbf{u}/\hat{r}] w \nabla_s w = 0$$

cancel if $[\mathbf{u}/\hat{r}] = (\partial_s \pi \mathbf{u} / \hat{r}_m) / \partial_s \pi$ using summation by parts in the vertical. The terms

$$-\frac{\partial \pi}{\partial s} \mathbf{u} c_p \theta_v \frac{1}{\hat{r}_m} \nabla_s \Pi - c_p \Pi \nabla_s \cdot \left(\frac{1}{\hat{r}_m} \mathbf{u} \Theta \right) = 0$$

630 from kinetic and internal energy, respectively, cancel using integration by parts in the horizontal. The terms

$$\mu \frac{\partial \pi}{\partial s} [\mathbf{u}/\hat{r}] \nabla_s \phi - \frac{\partial \pi}{\partial s} \mathbf{u} \frac{1}{\hat{r}_m} \mu \nabla_s \phi = 0$$

from internal and kinetic energy cancel via summation by parts in the vertical. We choose to define $\hat{r}^2 \partial_s p = \mu \bar{\partial_s \pi}$ so that the terms

$$\frac{\partial \pi}{\partial s} w \mu g - \hat{r}^2 g w \frac{\partial p}{\partial s} = 0$$

635 cancel pointwise. Finally, the following terms appear in internal and kinetic energy, respectively, and cancel pointwise:

$$\frac{\partial \pi}{\partial s} (gw) - \frac{\partial \pi}{\partial s} (gw) = 0.$$



Appendix B: IMEX

The process of solving for ϕ_j^{guess} , which is the state of ϕ at the j th internal stage in the chosen time-stepping methods can be reduced to solving the equation

$$640 \quad G_j(\phi_j^{\text{guess}}) = \phi_j^{\text{guess}} - E_j^\phi - g\Delta t \hat{A}_{j,j} E_j^w + (g\Delta t \hat{A}_{j,j})^2 (1 - \mu_j). \quad (\text{B1})$$

Here E_j^ϕ and E_j^w are the explicit internal stages for ϕ and w , respectively, which are defined in Steyer et al. (2019). $\hat{A}_{j,j}$ are the coefficients from the time stepping method. Because the vertical variation of g is quite small we treat it as a constant. After rearranging to cancel out w_j^{guess} , μ_j can be written solely as a function of ϕ_j^{guess} . Once the vertical dimension is discretized, $\phi_{j,k+\frac{1}{2}}^{\text{guess}}$ becomes a vector that has the dimension of the number of interface levels. The finite-difference expression for μ in
 645 the discretized equations means that, excluding the boundaries, $\mu_{k+\frac{1}{2}}$ depends only on $\phi_{k-\frac{1}{2}}$, $\mu_{k+\frac{3}{2}}$, and $\phi_{k+\frac{1}{2}}$. Therefore, the only quantity required to calculate the Jacobian of Eq. (B1) is $\partial_{\phi_{k+l+\frac{1}{2}}} \mu_{k+\frac{1}{2}}$, $l \in \{-1, 0, 1\}$. This results in a block-tridiagonal structure that can be efficiently solved with computational cost that grows linearly in the number of vertical levels. We make the following definitions

$$\begin{aligned} J_{\text{lower}} &= \frac{\partial \mu_{i+1/2}}{\partial \phi_{i-1/2}} \\ 650 \quad J_{\text{upper}} &= \frac{\partial \mu_{i+1/2}}{\partial \phi_{i+3/2}} \\ J_{\text{diag}} &= \frac{\partial \mu_{i+1/2}}{\partial \phi_{i+1/2}} \end{aligned}$$

and use the equation of state to calculate the entries of the Jacobian. After defining

$$\begin{aligned} a_k &= \frac{(\Delta t g)^2}{(1 - \kappa)} \\ b_k &= \frac{a}{\left(\frac{\partial \pi}{\partial s}\right)_{\text{int},k}} \\ 655 \quad c_k &= \frac{p_k}{\Delta \phi_k} \\ q_k &= \frac{(p_k - p_{k+1}) \partial_{\phi_{k+1/2}} [\hat{r}_{k+1/2}^2]}{(\pi_{k+1/2})} \\ d_k &= (\phi_{k-3/2} - \phi_{k-1/2}) \partial_{\phi_{k+1/2}} [\hat{r}_{\text{avg},k}^2] \\ f_k &= (\phi_{k-1/2} - \phi_{k+1/2}) \partial_{\phi_{k+1/2}} [\hat{r}_{\text{avg},k}^2], \end{aligned}$$



660 we can write the Jacobian explicitly. The Jacobian reads as at the top interface as

$$q_1 = \Delta t^2 \left(\frac{4p_k(ag + \phi_{1/2})}{a^2 \left(\frac{\partial \pi}{\partial s} \right)_k} \right)$$

$$J_{\text{upper}} = 2b_1 c_1 \hat{r}_{1/2}^2 \left(\frac{d_1 + \hat{r}_{\text{avg},1}^2}{\hat{r}_{\text{avg},1}^2} \right)$$

$$J_{\text{diag}} = 1 - q_1 - 2b_1 c_1 \hat{r}_{1/2}^2 \left(\frac{f_1 + \hat{r}_{\text{avg},1}^2}{\hat{r}_{\text{avg},1}^2} \right)$$

The Jacobian at the bottom interface read as follows:

665

$$J_{\text{lower}} = b_k c_{k-1} \hat{r}_{k-1/2}^2 \left(\frac{f_k + \hat{r}_{\text{avg},k+1}^2}{\hat{r}_{\text{avg},k+1}^2} \right)$$

$$J_{\text{diag},k \neq 1} = 1 - b_k c_{k-1} \hat{r}_{k-1/2}^2 \left(\frac{d_k + \hat{r}_{\text{avg},k-1}^2}{\hat{r}_{\text{avg},k-1}^2} \right) - b_k \hat{r}_{k-1/2}^2 \left(\frac{f_k + \hat{r}_{\text{avg},k}^2}{\hat{r}_{\text{avg},k}^2} \right).$$

On all internal interfaces, it reads as follows:

$$J_{\text{lower}} = b_{k+1} c_k \hat{r}_{k+1/2}^2 \left(\frac{f_{k-1} + \hat{r}_{\text{avg},k-1}^2}{\hat{r}_{\text{avg},k-1}^2} \right)$$

$$J_{\text{upper}} = b_k c_k \hat{r}_{k-1/2}^2 \left(\frac{d_{k+1} + \hat{r}_{\text{avg},k}^2}{\hat{r}_{\text{avg},k}^2} \right)$$

670

$$J_{\text{diag}} = 1 - q_k - b_k c_{k-1} \hat{r}_{k-1/2}^2 \left(\frac{d_k + \hat{r}_{\text{avg},k-1}^2}{\hat{r}_{\text{avg},k-1}^2} \right) - b_k c_k \hat{r}_{k-1/2}^2 \left(\frac{f_k + \hat{r}_{\text{avg},k}^2}{\hat{r}_{\text{avg},k}^2} \right).$$

It can be seen that if $\hat{r}^2 = 1$ and the necessary derivatives are canceled, the Jacobian for NHS HOMME is recovered.

Appendix C: Initialization

This appendix contains one method for initializing a dynamical core that uses the deep-atmosphere mass coordinate described in Wood and Staniforth (2003). There are small differences between the deep and shallow mass coordinates on planets the size
675 of earth. In practice, these differences can be ignored, and the simulation results will be nearly identical. On small planets, the difference between the two versions of the mass coordinate is non-negligible. For example, an analytic steady state will break down after less than a day. The method presented in this appendix is designed for idealized test cases, and requires an analytic specification of $\rho(z), p(z), T(z)$. Here, ρ should be the density used in the equation of state of a dynamical core, so it may or may not include moisture. In HOMME, this density includes moisture. For initial conditions that specify $z(p)$, numerical root
680 finding or binary search can be used to invert this function.

In this initialization procedure, we assume we have $A_{\text{int},k}, B_{\text{int},k}, k = 1, \dots, N_{\text{lev}}+1$. By construction, $A_{\text{mid},k} = \frac{1}{2} (A_{\text{int},k} + A_{\text{int},k+1})$, for $k = 1, \dots, N_{\text{lev}}$. The initialization proceeds as follows:

1. Calculate z_{top} , at which $\hat{r}^{-2} p_{\text{top}} = \pi_{\text{top}}$, which ensures that the upper boundary condition is satisfied.



2. Calculate the total amount of mass in the atmosphere,

685

$$\pi_{\text{surf}} = \int_{z_{\text{surf}}}^{z_{\text{top}}} \hat{r}^2 \rho(z) dz.$$

3. Calculate $z_{\text{int},k}$, $z_{\text{mid},k}$ that satisfy

$$A_{\text{int},k} \pi_0 + B_{\text{int},k} \pi_{\text{surf}} = \int_{z_{\text{surf}}}^{z_{\text{int},k}} \hat{r}^2 \rho(z') dz',$$

where the analogous equation holds for midpoint levels.

4. Initialize midpoint prognostic variables using $z_{\text{mid},k}$ and interface prognostic variables using $z_{\text{int},k}$.

690

In the simulations presented in Section 3, we use the following numerical implementation of the above strategy. In step 1, we use Newton's method to find the height of the top interface. In step 2, we use numerical quadrature, dividing the atmosphere into N_z segments and using second order quadrature within each segment to calculate the total mass in the atmosphere. In the third step, we use Newton's method and second order numerical quadrature to solve for $z_{\text{int},k} - z_{\text{int},k+1}$ and $z_{\text{mid},k} - z_{\text{int},k+1}$. We iterate from the surface to the top of the atmosphere to calculate interface and midpoint heights.

695

Author contributions. MAT and OG designed the specifications for the mimetic discretization. OH and OG developed the software implementation. OH and CJ validated the model and revised the discretization. OH and CJ prepared the manuscript for publication.

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