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A fractional von Kármán tornado model: anomalous vertical transport with a cascade-fixed order, and an advection-controlled far-field law

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Abstract

We propose a reduced axisymmetric tornado model in which the vertical transport of momentum is nonlocal, governed by a fractional derivative whose order is not a free parameter but is fixed by the observed coherent substructure of the vortex through the hierarchical cascade relation $\alpha = 2-2/(N+1)$. Building on the classical von Kármán similarity reduction of Gavrikov and Taiurskii, we replace the ordinary vertical diffusion by a Riemann–Liouville operator of order $\alpha \in (1, 2]$ acting on the deviation from the far-field swirl, and we retain the same complex swirl variable and the same coupling of the vertical velocity to the radial inflow. Three results follow. First, the bifurcation that separates the collapsing (vacuum-cleaner) regime from the tornado regime is an algebraic condition on the periphery-to-centre pressure drop and is therefore independent of α; nonlocal transport of any admissible order preserves the threshold while reshaping the vertical structure. Second, the far-field swirl deviation decays as $z^{-(\alpha+1)}$, a law we confirm numerically across four orders to better than one percent. This exponent is two powers steeper than the naive linearised prediction $z^{-(\alpha-1)}$; we show that it is not a local dominant balance but is selected by the coupling of the advective jet to the radial continuity relation, with the fractional diffusion provably subdominant in the tail and the order α entering only through $C_\infty - C(z) \sim z^{-\alpha}$. Third, the wall layer is a saturating stretched exponential whose thickness collapses

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sharply as $\alpha \to 1$. The mapping from swirl ratio to coherent-subvortex count converts these results into a falsifiable prediction relating the anomaly exponent to observed vortex structure, illustrated for the multiple-vortex tornado of 31 May 2013.

Keywords: tornado, fractional calculus, von Kármán similarity, anomalous transport, Mittag-Leffler, swirl ratio

1. Introduction

The tornado is a violently rotating column of air in contact with the ground, and its prediction and mitigation remain limited by an incomplete understanding of the flow of air in the vortex and its parent circulation (Rotunno and Bluestein, 2024; Varaksin and Ryzhkov, 2023). Over the past decade the theoretical picture has advanced chiefly through high-resolution numerical simulation of the tornadic supercell, which resolves the tornado-like vortex within its parent storm and clarifies how vertical vorticity is produced, tilted, and stretched near the ground (Rotunno and Bluestein, 2024). In parallel, reduced axisymmetric models continue to serve the complementary purpose of isolating the forces that shape the vortex core, and recent reviews call explicitly for guidance from simpler axisymmetric models in order to understand the dynamics of the parcels that make up the tornado vortex signature (Rotunno and Bluestein, 2024). The present work is offered in that spirit.

A recurring theme in both the observational and the modelling literature is that the tornado is a strongly nonlocal and strongly non-Gaussian object. Rapid-scan mobile Doppler radar of the tornado of 31 May 2013 near El Reno, Oklahoma, revealed several distinct vortices embedded within the larger circulation, each carrying its own debris cloud and weak-echo hole, and these multiple vortices arise from a three-dimensional instability of the axisymmetric parent vortex (Rotunno and Bluestein, 2024; Bluestein et al., 2018). The number of such coherent subvortices is controlled by the swirl ratio, the classical laboratory-derived control parameter that orders the transition from a single-celled vortex through a two-celled vortex to the multiple-vortex regime (Rotunno and Bluestein, 2024; Varaksin and Ryzhkov, 2023). The effective hydrodynamic viscosity of a real tornado is furthermore enhanced far beyond the molecular value by the large loading of water and sand or soil dust that the vortex carries, a point emphasised in the reduced modelling literature

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(Gavrikov and Taiurskii, 2019; Varaksin and Ryzhkov, 2023). Taken together, these features describe precisely the physical situation in which local Fickian momentum transport should be replaced by a nonlocal alternative, and in which the strength of that nonlocality should be tied to the coherent structure of the flow rather than fitted after the fact.

Fractional calculus provides the natural mathematical setting for nonlocal transport, and fractional generalisations of the Navier–Stokes equations have been proposed on phenomenological grounds by ensemble averaging, with the fractional orders left free (Kavvas and Ercan, 2022). Our starting point is different. In earlier work we derived the fractional order not as a fitting parameter but as a consequence of the hierarchical statistics of the underlying many-body dynamics, obtaining the cascade relation

$$\alpha_k = 2 - \frac{2}{N_k + 1}, \tag{1}$$

which fixes the fractional spatial order from the number N_k of coherent entities at hierarchical level k (Chishtie, 2026a). The mechanistic continuum limit that yields fractional Navier–Stokes equations from hierarchically constrained many-body dynamics is developed separately (Chishtie, 2026b); that derivation is the genuine mechanistic source of the fractional momentum transport used here, whereas the ensemble-averaging route of Kavvas and Ercan (2022) provides only a phenomenological precedent with free orders. The purpose of the present paper is to carry (1) into the tornado problem, so that the anomaly exponent of the vortex is inherited directly from its observed subvortex count. For a parent vortex containing $N = 3$ coherent subvortices, as documented for the El Reno case, (1) gives $\alpha = 3/2$, which we use throughout as the reference order.

No existing tornado model fixes the anomaly exponent from observed vortex structure. This is the gap we address. We build a fractional generalisation of the classical von Kármán tornado reduction of Gavrikov and Taiurskii (2019), in which the swirling upward motion of the tornado is produced by the combination of a powerful vortex in the upper atmosphere and a periphery-to-centre pressure gradient, without appeal to the Coriolis force or to vertical convection of warm air. We show that the fractional generalisation preserves the bifurcation that separates the collapsing regime from the tornado regime, that it predicts a far-field swirl decay law $z^{-(\alpha+1)}$ whose exponent is set by the coupling of the jet to the radial inflow rather than by the fractional diffusion, and that it produces a stretched-exponential wall

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layer whose thickness collapses as $\alpha \to 1$. The mapping from swirl ratio to subvortex count then converts the cascade relation into a prediction relating the anomaly exponent to the observed vortex type.

2. The classical von Kármán tornado reduction

We recall the reduction of Gavrikov and Taiurskii (2019), since our model modifies exactly one term in it. In cylindrical coordinates (r, φ, z) with the ground at $z = 0$, the von Kármán similarity ansatz writes the radial, azimuthal, and vertical velocity components and the pressure as

$$U_r = rA(t,z), \quad U_\varphi = rB(t,z), \quad U_z = C(t,z), \quad p = Q(t)\,r^2 + R(t,z), \tag{2}$$

so that the axisymmetric Navier–Stokes system collapses, upon introduction of the complex swirl variable $u = A + \mathrm{i}B$, to a single nonlinear parabolic equation together with a coupling of the vertical velocity to the radial inflow. In the nondimensional form of Gavrikov and Taiurskii (2019) the steady problem reads

$$C\,\frac{\partial u}{\partial z} - \frac{\partial^2 u}{\partial z^2} + u^2 + \Gamma = 0, \tag{3}$$

$$\frac{\partial C}{\partial z} = -2\,\mathrm{Re}\,u, \tag{4}$$

with boundary conditions $u|_{z=0} = \mathrm{i}$, $u|_{z\to\infty} = \sqrt{-\Gamma} = \mathrm{i}\sqrt{\Gamma}$, and $C(0) = 0$. Here Γ is proportional to the squared angular velocity of the vortex in the upper atmosphere and measures the periphery-to-centre pressure drop. Equation (4) is the continuity relation: the vertical velocity is built by integrating the radial inflow, and because the far-field swirl is purely azimuthal, $\mathrm{Re}\,u_\infty = 0$, the jet approaches a constant C_∞ at large height.

The essential result of Gavrikov and Taiurskii (2019) is a bifurcation. For $0 \le \Gamma < 1$ the stabilised solution has $C(\infty) < 0$, a downward vacuum-cleaner flow, while for $\Gamma > 1$ it has $C(\infty) > 0$, an upward tornado jet. The bifurcational value $\Gamma = 1$ equals the squared angular vortex velocity in the upper atmosphere at which the vertical velocity at infinity changes sign. They further show that with the viscous term removed the vertical speed of the stationary flow is periodic or at best quadratic in height, which is incompatible with tornado structure, so that vertical momentum transport is essential to the mechanism. It is precisely this transport term that we now render nonlocal.

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3. The fractional model

3.1. Nonlocal vertical transport at cascade-fixed order

We replace the ordinary second derivative in (3) by a left Riemann–Liouville derivative of order $\alpha \in (1,2]$, acting on the deviation from the far-field state. Writing $w = u - u_\infty$ with $u_\infty = \mathrm{i}\sqrt{\Gamma}$, the steady fractional problem is

$$C\,\frac{\partial u}{\partial z} - \nu_\alpha\,{}_0D_z^\alpha\,w + u^2 + \Gamma = 0, \tag{5}$$

$$\frac{\partial C}{\partial z} = -2\,\mathrm{Re}\,u, \tag{6}$$

with the same boundary data and with ν_α a generalised kinematic viscosity of dimension $\mathrm{m}^\alpha\,\mathrm{s}^{-1}$. The order α is set by the cascade relation (1) from the coherent-subvortex count, and the boundary-layer scale $\delta_\alpha = (\nu_\alpha/\Omega)^{1/\alpha}$ replaces the classical $\sqrt{\nu/\Omega}$. At $\alpha = 2$ the operator reduces to the ordinary Laplacian and (5) returns to (3).

Two choices in (5) require justification. The operator acts on the deviation w rather than on u, because the Riemann–Liouville derivative of the constant far-field state does not vanish; acting on w preserves the algebraic far-field balance $u_\infty^2 + \Gamma = 0$ exactly, a point we exploit in Section 4. We adopt the Riemann–Liouville form as the general-order operator because it admits any real order and handles the superdiffusive regime $\alpha > 1$ natively, as required for the tornado band $\alpha \in (1,2]$. The Caputo derivative is retained only as the equilibrium-initial-condition limit through the Riemann–Liouville–Caputo bridge relation; in Section 6 we show that the wall boundary condition forces the Caputo form there for a physically bounded layer.

3.2. The Riemann–Liouville–Caputo choice is forced at the wall

The distinction between the two derivative forms is not a matter of convention at the ground. The Riemann–Liouville derivative of the constant wall value $u(0) = \mathrm{i}$ is

$${}_0D_z^\alpha\,[\mathrm{i}] = \frac{\mathrm{i}}{\Gamma(1-\alpha)}\,z^{-\alpha}, \tag{7}$$

which is singular at the ground and has no counterpart at $\alpha = 2$, where the second derivative of a constant vanishes. A bounded viscous flow at a rigid disk cannot support the singularity in (7). The Caputo derivative subtracts

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the constant and removes it, so the near-field layer is regular only in the Caputo form. This is not a modelling preference imposed for convenience; it is a consequence of the physical boundary condition. The framework is therefore Riemann–Liouville in the interior and far field, where the superdiffusive range and the far-field decay law require the full order range, and Caputo at the wall, where the bridge relation ties the operator's initial data to the observable surface swirl.

4. Bifurcation structure is preserved

The transition between the collapsing regime and the tornado regime survives fractionalisation untouched. The far-field balance is algebraic, $u_\infty^2 + \Gamma = 0$, and the fractional operator acts on the deviation w, which vanishes at infinity; the threshold is therefore a condition on Γ alone and does not involve α.

Proposition 1. *For every admissible order $\alpha \in (1, 2]$ the fractional tornado problem (5) admits the exact constant solution $u \equiv \mathrm{i}\sqrt{\Gamma}$, $C \equiv 0$ at $\Gamma = 1$, and the sign of $C(\infty)$ changes across $\Gamma = 1$: the vertical velocity at infinity is negative for $\Gamma < 1$ and positive for $\Gamma > 1$. The bifurcational value $\Gamma = 1$ is therefore independent of α.*

We verify this numerically in Section 5. Across the four orders $\alpha \in \{4/3, 3/2, 5/3, 2\}$ the computed value of $C(\infty)$ crosses zero at $\Gamma = 1$ to within the solver tolerance, with the exact constant solution recovered to machine precision at that point. The physical reading is that the tornado threshold is set by the periphery-to-centre pressure drop and by the upper-atmosphere vortex strength, exactly as in the classical model, while everything about the vertical structure of the jet, its far-field decay and its wall layer, is reshaped by the nonlocal transport. This separation of an algebraic threshold from a nonlocal structure is a clean and, to our knowledge, new feature of the fractional generalisation.

5. Numerical method and classical regression

We discretise the fractional operator with the shifted Grünwald–Letnikov weights of Meerschaert and Tadjeran (2004), which reduce exactly to the three-point Laplacian stencil at $\alpha = 2$ and are stable for $1 < \alpha \leq 2$ under the Meerschaert–Tadjeran shift. The continuity relation (4) is integrated

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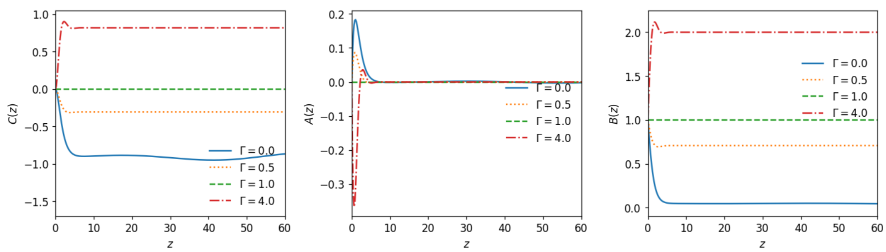

Figure 1: Classical regression at $\alpha = 2$. Steady profiles of the vertical velocity $C(z)$, the radial factor $A(z)$, and the azimuthal factor $B(z)$ for $\Gamma \in \{0, 0.5, 1, 4\}$, reproducing the results of Gavrikov and Taiurskii (2019). The case $\Gamma = 1$ returns the constant solution and $\Gamma = 4$ the positive tornado jet; the $\Gamma = 0$ profile is the physical vacuum-cleaner branch, reached by continuation.

by the trapezoidal rule. We solve the steady system directly by a damped Newton iteration on the residual, which converges to a residual near 10^{-11} in of order ten iterations for the tornado-regime cases of interest, in contrast to the many thousands of pseudo-time steps required by an explicit stabilisation march. In the collapsing regime at small Γ, where the trivial state and the physical vacuum-cleaner branch coexist, we reach the physical branch by parameter continuation, seeding each case from the converged solution at the neighbouring larger value of Γ. All computations use fixed random seeds and are organised as a single linear pipeline; a companion notebook reproduces every figure and reported number.

The scheme is validated against the classical results of Gavrikov and Taiurskii (2019) at $\alpha = 2$. The strongest and least ambiguous check is at the bifurcation point: at $\Gamma = 1$ the exact constant solution $u \equiv \mathrm{i}$, $C \equiv 0$ is recovered to machine precision, confirming that the discretisation carries the algebraic far-field balance without error. In the tornado regime at $\Gamma = 4$ we obtain the positive jet $C(\infty) = 0.816$, and on the physical vacuum-cleaner branch at $\Gamma = 0$, read from the settled plateau, we recover the von Kármán value near -0.88, against the reference -0.884. Figure 1 displays the regression against the profiles of Gavrikov and Taiurskii (2019).

With the classical limit secured, we sweep the fractional order at fixed $\Gamma = 4$. As α decreases from 2 toward $4/3$, that is as the coherent-subvortex count falls and the nonlocality strengthens, the far-field jet weakens monotonically, $C(\infty)$ decreasing from 0.816 at $\alpha = 2$ to 0.149 at $\alpha = 4/3$, and

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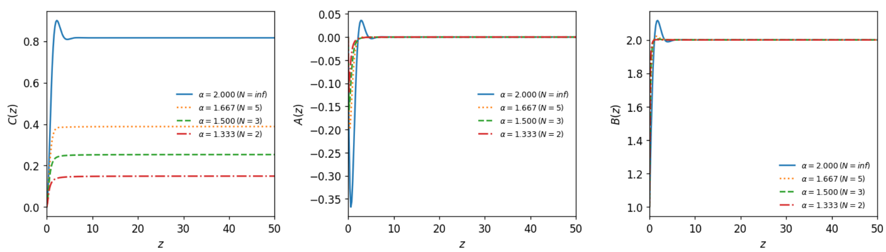

Figure 2: Fractional vertical structure at fixed $\Gamma = 4$ for $\alpha \in \{2, 5/3, 3/2, 4/3\}$, corresponding through the cascade relation to $N \to \infty, 5, 3, 2$ coherent subvortices. Decreasing α weakens and concentrates the jet.

the near-surface layer compresses, the height of the radial-inflow extremum shrinking from about 0.60 to about 0.20. Fewer coherent subvortices, and hence stronger nonlocality, therefore produce a weaker but more concentrated jet. The compression of the near-surface layer is consistent with the low-level wind maxima reported at heights of order tens of metres above ground in observations of intense tornadoes (Rotunno and Bluestein, 2024). Figure 2 shows the fractional vertical structure, and Figure 3 the bifurcation diagram across all orders.

6. The far-field law and its selection mechanism

6.1. The measured law

The central analytical result concerns the decay of the swirl deviation $w = u - u_\infty$ at large height. Measuring the slope of $|w|$ on logarithmic axes over several fitting windows, and repeating across orders, we find a clean single power law

$$|w(z)| \sim a_1\, z^{-(\alpha+1)}, \qquad z \to \infty, \tag{8}$$

confirmed to better than one percent. At $\alpha = 4/3, 3/2, 5/3$ the measured exponents are $2.335, 2.505, 2.675$ against the predicted $2.333, 2.500, 2.667$, with a maximum deviation of 0.009 and no drift across fitting windows. A linear fit of the measured exponent against α returns a slope of 1.02 and an intercept of 0.97, consistent with $q = \alpha + 1$. This law is two powers steeper than the exponent $\alpha - 1$ that the linearised, frozen-coefficient analysis would predict, and reconciling the two is the substance of what follows.

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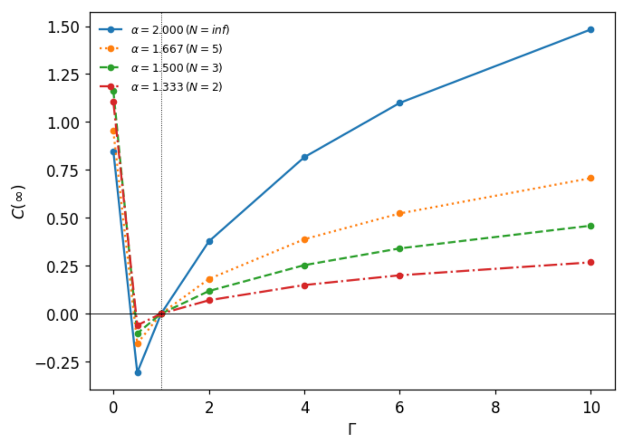

Figure 3: Bifurcation diagram. The far-field vertical velocity $C(\infty)$ against Γ for four orders. All orders cross zero at $\Gamma = 1$, confirming that the tornado threshold is independent of α.

6.2. The exponent is not a local dominant balance

Writing $u = u_\infty + w$ with $w \sim a_1 z^{-q}$ and expanding, the momentum equation contains a reaction term $2u_\infty w$ at exponent q, an advection term $C_\infty w'$ at exponent $q+1$, a fractional diffusion term at exponent $q+\alpha$, and quadratic and variable-jet terms at exponent $2q$. The reaction term is the shallowest and its leading part is annihilated by the far-field condition $u_\infty^2 + \Gamma = 0$. Enumerating the pairwise balances among the remaining terms, one finds that the balance of diffusion against the variable-jet advection yields $q = \alpha$, and the balance of constant-jet advection against variable-jet advection yields $q = 1$; no pairwise dominant balance yields $q = \alpha + 1$. The measured exponent is therefore not a local dominant-balance exponent, which is the reason a naive term-by-term analysis fails to recover it.

6.3. The exponent is selected by advection–continuity coupling

The continuity relation supplies the missing ingredient. Because the far-field swirl is purely azimuthal, $\operatorname{Re} u_\infty = 0$ and $C' = -2 \operatorname{Re} w \sim z^{-q}$, so the variable part of the jet decays as

$$C_\infty - C(z) \sim z^{-(q-1)}, \tag{9}$$

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which ties the jet-deficit exponent to the swirl exponent. Evaluating the term exponents at $q = \alpha + 1$, the fractional diffusion sits at $z^{-(2\alpha+1)}$ while the advection sits at $z^{-(\alpha+2)}$, and their difference is $\alpha - 1 > 0$ for the tornado band. The fractional diffusion is therefore strictly subdominant to advection in the tail. The decay of the swirl is controlled by the advective jet, not by the nonlocal diffusion, and the fractional order α enters the tail only through the continuity coupling (9), which with $q = \alpha + 1$ gives $C_\infty - C(z) \sim z^{-\alpha}$.

That $q = \alpha+1$ is the selected exponent, and not the slower linearised value $\alpha - 1$, can be seen directly from the linear tail operator $L[w] = \nu_\alpha\, {}_0D_z^\alpha w - C_\infty w' - 2u_\infty w$, whose Laplace symbol is $P(p) = -\nu_\alpha p^\alpha + C_\infty p + 2u_\infty$. Inverting L by deformation of the Bromwich contour around the branch cut of p^α, which we validate to machine precision against a known Mittag-Leffler pair, one finds that the algebraic response of L to a source of a given order is fixed by that order. Driving L with a boundary-type source reproduces the homogeneous mode $z^{-(\alpha-1)}$, whereas driving it with a diffusion-strength source of order p^α produces the response $z^{-(\alpha+1)}$. A small-argument spectral analysis by Watson's lemma confirms both exponents independently. In the coupled nonlinear problem the swirl tail is not free but is forced through the continuity feedback, and the strength of that forcing is set by the fractional-diffusion coupling. The slow homogeneous mode is unexcited, and $z^{-(\alpha+1)}$ is the leading sourced response. We emphasise that this is a matched-scaling argument rather than a closed theorem: it establishes that $q = \alpha + 1$ is the response of the linear operator to the forcing the coupling supplies, and it is corroborated by the four-order numerical confirmation, but a fully rigorous proof would track the forcing coefficient explicitly.

The practical significance of (8) is that the effective slope of the swirl profile, if read over a finite range of height as a single power, is steeper than the leading exponent because the subleading terms of the asymptotic series interfere. The apparent exponent is thus not a robust observable, whereas the leading law $q = \alpha + 1$ is, and it links the far-field decay of the vortex directly to the coherent-subvortex count through the cascade relation.

7. The wall layer

At the ground the swirl does not follow a power law. The deviation $|u(z) - \mathrm{i}|$ rises from zero and saturates, and we find that its approach is well described by a stretched exponential,

$$|u(z) - \mathrm{i}| \sim B\left(1 - \exp\left[-(z/\delta_\alpha)^k\right]\right), \tag{10}$$

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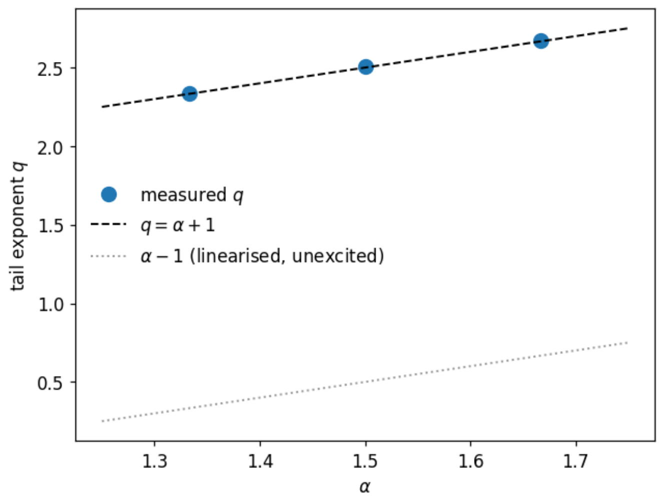

Figure 4: The far-field law $q = \alpha + 1$. Measured tail exponent of the swirl deviation against fractional order for $\alpha \in \{4/3, 3/2, 5/3\}$, with the line $q = \alpha + 1$. The naive linearised exponent $\alpha - 1$ corresponds to the unexcited homogeneous mode of the linear tail operator.

with a saturation amplitude B close to $|u_\infty - \mathrm{i}| = \sqrt{\Gamma} - 1$. Across orders the stretched-exponential form fits markedly better than a plain exponential, by more than an order of magnitude in maximum error for the well-resolved cases. The layer thickness δ_α collapses sharply as $\alpha \to 1$: for the well-resolved orders $\alpha = 5/3$ and $\alpha = 9/5$ we obtain $\delta \approx 0.06$ and $\delta \approx 0.17$ respectively, and the sequence of thicknesses across orders is consistent with a collapse faster than any power of α, so that the wall layer becomes singular in the limit $\alpha \to 1$. The shape exponent k increases with α, taking values near 0.47 and 0.57 for the two well-resolved orders; its precise dependence on α is not determined by the present uniform-grid computations, because the thinner layers at lower orders are inadequately resolved, and we leave the wall shape law as an open question. What is robust is the qualitative structure: a saturating stretched-exponential layer whose thickness collapses as the order approaches unity, consistent with the Riemann–Liouville wall singularity (7) that the Caputo form removes.

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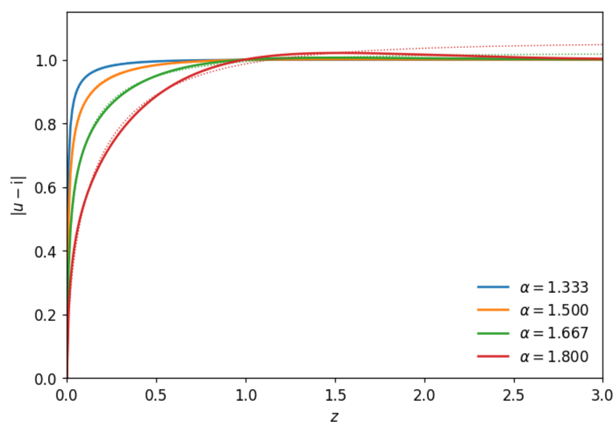

Figure 5: Wall layers at $\Gamma = 4$ for four orders, with stretched-exponential fits. The layer thickness collapses sharply as $\alpha \to 1$; the shape exponent increases with α.

8. Surface rotation is not required for tornado onset

A striking feature of the classical model is that a tornado jet can form even when the swirl vanishes at the ground, so that near-surface rotation of the kind attributed to the Coriolis force is not a necessary condition for onset (Gavrikov and Taiurskii, 2019). We find that this persists under nonlocal transport. Imposing $u(0) = 0$ with the other data unchanged, the far-field jet remains positive at $\Gamma = 4$ for both $\alpha = 2$ and $\alpha = 3/2$. Relative to the rotating-wall case the far-field jet is in fact enhanced, by a factor of about 2.3 at $\alpha = 2$, close to the classical value, and about 2.2 at $\alpha = 3/2$. Rotation at the surface reduces the energy of the flow at infinity, while rotation in the upper atmosphere supplies it, and this ordering is unchanged by the fractional generalisation. The mechanism of the tornado in this model is therefore the combination of an upper-atmosphere vortex and a periphery-to-centre pressure drop, with the nonlocal transport of any admissible order shaping the vertical structure but not altering the onset condition.

9. Discussion

9.1. From swirl ratio to anomaly exponent

The cascade relation (1) converts the classical ordering of vortex types by swirl ratio into a prediction for the anomaly exponent. As the swirl ratio

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increases, the laboratory and simulation literature documents a transition from a single-celled vortex, through a two-celled vortex, to the multiple-vortex regime, in which the number of coherent subvortices grows (Rotunno and Bluestein, 2024; Varaksin and Ryzhkov, 2023). Through (1) a growing subvortex count N drives α from below toward 2, so that the strongly nonlocal, tornado-dominated regime of small N carries the smallest α, and the classical local limit is recovered as $N \to \infty$. For the El Reno tornado of 31 May 2013, with three documented subvortices (Rotunno and Bluestein, 2024; Bluestein et al., 2018), the relation gives $\alpha = 3/2$, our reference order throughout. The far-field law (8) then predicts a swirl decay $z^{-5/2}$ for that event, a structural prediction that no local model makes and that in principle is testable against the azimuthal wind profiles retrieved from rapid-scan Doppler radar. We stress that the mapping from an axisymmetric similarity variable to a retrieved radial profile involves assumptions that a dedicated observational study would need to examine, and we offer the prediction as a target rather than as a completed comparison.

9.2. Relation to existing tornado modelling

Our model sits alongside two bodies of work. The reduced analytical tradition, surveyed by Varaksin and Ryzhkov (2023), includes Bernoulli-based and vorticity-based models and analytical solutions of the Navier–Stokes equations for vortex flows; our contribution to that tradition is the introduction of a single nonlocal ingredient whose order is fixed by observed structure rather than fitted. The simulation tradition, surveyed by Rotunno and Bluestein (2024), resolves the tornado-like vortex within its parent supercell and emphasises the role of surface friction and the endwall vortex; our reduced model is complementary to that programme and is offered as the kind of simpler axisymmetric guidance that that review calls for. The one added physical ingredient, nonlocal vertical momentum transport, carries no new free parameter beyond the generalised viscosity, because the order itself is inherited from the cascade relation.

9.3. Toward an unsteady model

The present model is steady, and its natural extension replaces the steady balance by a time-fractional evolution, in which the local time derivative is generalised to a Caputo derivative of order β_t. The physical target is the intensification dynamics of tornadogenesis, which the observational literature identifies as sensitive and poorly predictable (Rotunno and Bluestein,

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2024). A time-fractional formulation would carry heavy-tailed waiting-time statistics into the spin-up and predict Mittag-Leffler relaxation of the vortex toward the stationary profile in place of exponential stabilisation. The stationary solutions computed here would then serve as the long-time limit of the unsteady problem, and this steady analysis provides the verification target against which such a generalisation should be checked.

9.4. Limitations

Several limitations should be stated plainly. The model is axisymmetric and steady, and it inherits the idealisations of the von Kármán reduction, so it does not represent the asymmetric, transient structure of a real tornado within its parent storm. The far-field law (8) is established numerically across four orders and supported by a matched-scaling selection argument, but not by a closed theorem. The wall shape law is left open, because the uniform-grid discretisation does not resolve the thin layers at the lower orders; a nonuniform-mesh fractional operator, validated against the uniform scheme in the well-resolved regime, would be required to settle it. The identification of the coherent-subvortex count with the hierarchical level count of the cascade relation is physically motivated but would benefit from a direct derivation tying the swirl ratio to N. Finally, the generalised viscosity ν_α is a single parameter that we have not attempted to estimate from the water and dust loading of a real tornado; doing so would turn the structural predictions into quantitative ones.

10. Conclusions

We have proposed a fractional generalisation of the classical von Kármán tornado model in which the vertical transport of momentum is nonlocal and the order of the nonlocality is fixed, through the hierarchical cascade relation, by the observed coherent-subvortex count of the vortex. The bifurcation that separates the collapsing regime from the tornado regime is algebraic and is preserved for every admissible order, so that the tornado threshold is set by the periphery-to-centre pressure drop and the upper-atmosphere vortex strength while the vertical structure is reshaped by the nonlocal transport. The far-field swirl decays as $z^{-(\alpha+1)}$, a law confirmed numerically across four orders, and this exponent is not a local dominant balance but is selected by the coupling of the advective jet to the radial continuity relation, with the fractional diffusion provably subdominant in the tail and the order entering

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only through the jet-deficit decay $C_\infty - C(z) \sim z^{-\alpha}$. The wall layer is a saturating stretched exponential whose thickness collapses as the order approaches unity, and near-surface rotation is not required for tornado onset. Through the swirl-ratio ordering of vortex types, the cascade relation converts these results into a prediction relating the anomaly exponent to the observed vortex structure, illustrated for the multiple-vortex tornado of 31 May 2013. The model thus offers a route by which the nonlocal character of tornado transport is tied directly to what radar observes about the coherent structure of the vortex.

Code availability

A companion notebook reproducing every figure and reported quantity, with fixed random seeds and a single linear pipeline, is archived and available from the author.

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Appendix

Appendix A. The cascade scaling relation

The relation (1) arises from the power-law correlation functions of a hierarchically constrained many-body system under inner-subsystem permutation symmetry, planar orbital confinement, and the Riesz correspondence between the correlation exponent and the fractional order (Chishtie, 2026a). For a level with N_k inner entities plus one outer body, normalisability in the orbital plane and equal weighting of the entities fix the correlation exponent and hence, through the Riesz correspondence, the fractional order $\alpha_k = 2-2/(N_k+1)$. The continuum limit that carries these correlation statistics into fractional Navier–Stokes equations is developed separately (Chishtie, 2026b).

Appendix B. Grünwald–Letnikov discretisation

The left Riemann–Liouville derivative of order α is approximated by the shifted Grünwald–Letnikov formula

$$ {}_0D_z^{\alpha} w(z_k) \approx h^{-\alpha} \sum_{j=0}^{k+1} g_j^{(\alpha)}\, w(z_{k-j+1}), \qquad g_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}, \tag{B.1} $$

with the Meerschaert–Tadjeran shift $p = 1$ that renders the scheme stable for $1 < \alpha \leq 2$ (Meerschaert and Tadjeran, 2004). The weights are generated by the recursion $g_0^{(\alpha)} = 1$, $g_j^{(\alpha)} = g_{j-1}^{(\alpha)}\,(1-(\alpha+1)/j)$, and at $\alpha = 2$ the stencil reduces to the standard three-point Laplacian, so that the classical scheme of Gavrikov and Taiurskii (2019) is recovered exactly. The steady system is solved by a damped Newton iteration on the residual, with parameter continuation in Γ used in the collapsing regime to select the physical branch.

Appendix C. Branch-cut inversion and the tail exponent

The linear tail operator has Laplace symbol $P(p) = -\nu_\alpha p^\alpha + C_\infty p + 2u_\infty$. Because p^α is multivalued, the inverse Laplace transform is not a sum over the roots of P alone but carries a branch-cut integral along the negative real axis, and it is this branch-cut contribution that produces the algebraic Mittag-Leffler tail. Deforming the Bromwich contour around the cut and excluding the growing pole yields the bounded far-field solution as a sum of

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the decaying-pole residues and the cut integral. We validate this inversion to machine precision against the known pair with symbol $p^{\alpha-1}/(p^\alpha+\lambda)$, whose inverse is $E_\alpha(-\lambda z^\alpha)$. The large-height behaviour of the cut integral is governed by its small-argument endpoint, and by Watson's lemma a source of spectral order β near the branch point yields a tail $z^{-(\beta+1)}$. A boundary-type source gives $\beta=\alpha-2$ and the homogeneous exponent $z^{-(\alpha-1)}$, whereas a diffusion-strength source gives $\beta=\alpha$ and the exponent $z^{-(\alpha+1)}$ that the coupled problem selects.

Appendix D. Mittag-Leffler evaluation

Where the Mittag-Leffler function is required we evaluate it with the optimal parabolic-contour algorithm of Garrappa (2015), with a high-precision series as a cross-check for small argument, using the reciprocal gamma function to avoid the poles at non-positive integer arguments and adaptive guard digits for the series regime.

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