

Dear Editor,

Regarding the minor corrections during proofreading, we kindly request your approval for the following changes. These modifications are purely for mathematical rigor and the correction of typographical errors, and none of them alter the raw data, scientific results, or conclusions of this study.

****For TS1 & TS2:****

As previously communicated with you, modifications regarding the matrix decomposition are permitted here. During our self-check, we found that the original description of the background error correlation matrix $\mathbf{C}$ skipped the Cholesky factorization step, which made the mathematical logic incomplete. Furthermore, there was a typographical error describing the retained triangular matrix.

The correct mathematical process should be: first, decompose $\mathbf{C}$ via Cholesky factorization into a lower triangular matrix $\mathbf{C}_z^{\frac{1}{2}}$; then, decompose $\mathbf{C}_z^{\frac{1}{2}}$ into the tensor product of directional components. The change from "upper" to "lower" triangular matrix accurately reflects this mathematical principle. This correction ensures the strictness and transparency of our variational assimilation theory.

Original sentence (page 5, line 75-79; TS1):

Furthermore, the background error correlation matrix $\mathbf{C}$ is decomposed into the tensor product of correlation matrices along the longitude, latitude, and vertical directions, i.e., $\mathbf{C} = \mathbf{C}_x \otimes \mathbf{C}_y \otimes \mathbf{C}_z$ (Derber and Rosati, 1989; Li et al., 2008).

Revised sentence (TS1):

Furthermore, the background error correlation matrix $\mathbf{C}$ is first decomposed via Cholesky factorization as $\mathbf{C} = \mathbf{C}_z^{\frac{1}{2}}(\mathbf{C}_z^{\frac{1}{2}})^T$, where $\mathbf{C}_z^{\frac{1}{2}}$ is a lower triangular matrix. The matrix $\mathbf{C}_z^{\frac{1}{2}}$ is then further decomposed into the tensor product of correlation matrices along the longitude, latitude, and vertical directions, i.e., $\mathbf{C}_z^{\frac{1}{2}} = \mathbf{C}_x^{\frac{1}{2}} \otimes \mathbf{C}_y^{\frac{1}{2}} \otimes \mathbf{C}_z^{\frac{1}{2}}$ (Derber and Rosati, 1989; Li et al., 2008).

Page 8, line 8 (TS2): "upper" changed to "lower" for consistency.

TS3: This is a typographical transcription error made during the manuscript preparation. The correct per-iteration time is 0.92 s, not 0.99 s. To clarify this, we have provided a screenshot of Fig. 14a from the proofreading stage (with blue boxes highlighting the values 5.47 and 0.92). The correct value of 0.92 s is explicitly shown in the data labels of this figure. This is strictly a text correction to match the figure and has no impact on the overall performance trends, speedup ratios, or any scientific conclusions.

We assure you that these corrections are strictly for factual accuracy and mathematical clarity. Thank you for your time and understanding.

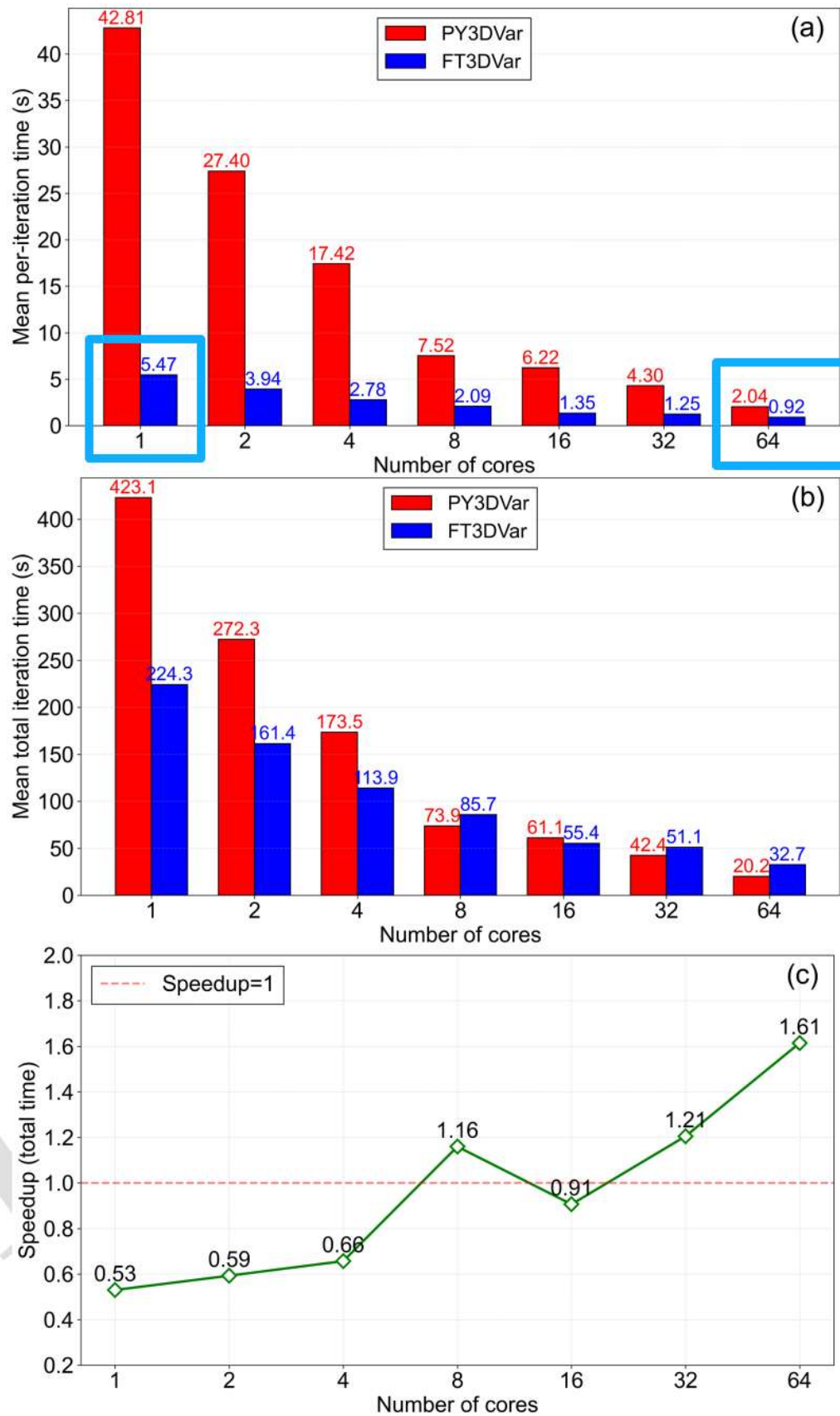


Figure 14. Parallel scalability of FT3DVar (blue) and PY3DVar (red) across CPU core counts. (a) Mean per-iteration time. (b) Mean total iteration time. (c) Speedup of PY3DVar relative to FT3DVar. Dashed line in (c): speedup = 1.

We assure you that these corrections are strictly for factual accuracy. If you approve these changes, please kindly instruct the typesetter to proceed, as I have already clearly indicated all the specific modifications in my previous response/PDF annotations. If these changes cannot be accepted, please let me know, and I will revise them accordingly.

Sincerely,

Hancheng Ye (on behalf of all authors)