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Title: The fractional Rothermel model: Mittag-Leffler kinetics of wildfire front propagation from a stochastic hierarchical parcel cascade, with application to heading fire collapse

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In this manuscript the author studies the suitability of the fractional relaxation as a foundational approach to the Rothermel model for the rate of spread of wildfires. The starting idea is the phenomenological power-law dependence of the rate of spread on the midflame wind speed. Since fractional calculus is a very successful approach for generating a power-law growth, the author explores such suitability.

Besides and first of any considerations about the consistency of the proposed approach, I would like to report that, unfortunately, there is a mistake in the mathematical part. I will illustrate this below.

Thus, because of this failure, I cannot recommend the publication.

In any case, I consider that this kind of research is necessary for the development of a mathematical theory of wildfire propagation, which for sure can help in the understanding and forecasting of forest fires. Whence, I would like to push the author to consider my notes below and rearrange accordingly the results of this research.

Remark

In subsection 3.2, the author introduces the time-fractional derivative in the Riemann–Liouville sense in formula (3), i.e.,

$$D^\alpha f(t) = \frac{d}{dt} \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau \right] = \frac{d}{dt} J^{1-\alpha} f(t), \quad 0 < \alpha < 1,$$

where J^μ , with $\mu > 0$, is the fractional integral of order μ . The case $\alpha = 1$ should be considered separately. Two lines below, the author states the main equation

$$D^\alpha \Phi = -\Phi,$$

where, without loss on generality of my argument, I set the time-scale equals to 1. Then, the author claims that the solution of this fractional equation is the Mittag–Leffler function $E_\alpha(-t^\alpha)$

$$\Phi(t) = E_\alpha(-t^\alpha) = \sum_{n=0}^{\infty} \frac{(-t^\alpha)^n}{\Gamma(\alpha n + 1)}.$$

Unfortunately, *this is not correct*. The author can verify what I am writing by passing through Section 1.3 of the book "Fractional Calculus and Waves in Linear Viscoelasticity" by Francesco Mainardi.

By applying the Laplace transform

$$\mathcal{L}\{f(t); s\} = \tilde{f}(s) = \int_0^{\infty} e^{-st} f(t) dt,$$

we have that

$$\begin{aligned}\mathcal{L}\{D^\alpha f; t\} &= \mathcal{L}\left\{\frac{d}{dt}[J^{1-\alpha}f]; t\right\} \\ &= s\mathcal{L}\{J^{1-\alpha}f; t\} - [J^{1-\alpha}f]_{t=0} \\ &= s^\alpha \tilde{f}(s) - [J^{1-\alpha}f]_{t=0},\end{aligned}$$

where I used the fact that, since $J^\mu f$ is a convolution integral, then

$$\mathcal{L}\{J^\mu f; s\} = \mathcal{L}\left\{\frac{t^{\mu-1}}{\Gamma(\mu)}; s\right\} \tilde{f}(s) = \frac{\tilde{f}}{s^\mu}.$$

Therefore, by transforming in Laplace the main equation, we have

$$s^\alpha \tilde{\Phi}(s) - [J^{1-\alpha}\Phi]_{t=0} = -\tilde{\Phi}(s),$$

which gives

$$\tilde{\Phi}(s) = \frac{1}{1+s^\alpha} = -\left[s \frac{s^{\alpha-1}}{1+s^\alpha} - 1\right],$$

if I set $[J^{1-\alpha}\Phi]_{t=0} = 1$, such that

$$\Phi(t) = -\frac{d}{dt}E_\alpha(-t^\alpha).$$

In fact, the Laplace transform of the Mittag-Leffler function $E_\alpha(-t^\alpha)$ is

$$\begin{aligned}\mathcal{L}\{E_\alpha(-t^\alpha); s\} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\alpha n + 1)} \mathcal{L}\{t^{\alpha n}; 1\} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\alpha n + 1)} \frac{\Gamma(\alpha n + 1)}{s^{\alpha n + 1}} \\ &= \frac{1}{s} \sum_{n=0}^{\infty} \left(-\frac{1}{s^\alpha}\right)^n \\ &= \frac{1}{s} \frac{1}{1+s^{-\alpha}} = \frac{s^{\alpha-1}}{1+s^\alpha}, \quad \text{Re } s > 1.\end{aligned}$$

As a consequence of the fact that the claimed solution is not the correct one, we also have that the power-law decaying of the correct solution $-\frac{d}{dt}E_\alpha(-t^\alpha)$ differs from the wrongly claimed solution $E_\alpha(-t^\alpha)$. By reminding that reproducing the phenomenological power-law displayed by the Rothermel model was the starting idea, the analysis should be rearranged.

At the same time, we have indeed that the Mittag-Leffler function $E_\alpha(-t^\alpha)$, with $0 < \alpha < 1$, is the solution of the equation

$$\frac{d\Phi}{dt} = -D^{1-\alpha}\Phi, \quad \Phi(0) = 1,$$

or analogously

$$D_C^\alpha \Phi = -\Phi, \quad \Phi(0) = 1,$$

where D_C^α is the time-fractional derivative in the Caputo sense and it is defined as

$$D_C^\alpha f = J^{1-\alpha} \frac{df}{dt}, \quad 0 < \alpha < 1.$$