

Author comment in reply to the referee report on egusphere-2026-3823

The fractional Rothermel model: Mittag-Leffler kinetics of wildfire front propagation from a stochastic hierarchical parcel cascade, with application to heading fire collapse

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I thank the referee for the time given to the manuscript. The report examines a single sentence of Section 3.2, concludes that “the claimed solution is not the correct one”, and on that basis alone recommends against publication. I show below that this conclusion is incorrect. The function used in the manuscript, $\Phi(t) = E_\alpha(-(t/\tau)^\alpha)$, is the correct survival function; the referee’s own calculation confirms it; the function the referee proposes in its place is not a survival function at all; and the consequence the referee draws for the Rothermel power law rests on a misreading of where that power law enters the model. No result of the manuscript is affected.

1. The manuscript’s function is the correct one, and the report confirms it

The quantity in question is defined in Section 3.1, before any fractional operator appears: $\Phi(t) = \Pr(T > t)$ is the survival function of the ignition waiting time T , whose density has the heavy tail $\psi(t) \sim t^{-(1+\alpha)}$. A survival function must satisfy $\Phi(0) = 1$, $0 \leq \Phi \leq 1$, and $\Phi = 1 - \int_0^t \psi$. For the heavy-tailed (Mittag-Leffler) renewal process, $\tilde{\psi}(s) = 1/(1 + (s\tau)^\alpha)$, and therefore

$$\tilde{\Phi}(s) = \frac{1 - \tilde{\psi}(s)}{s} = \frac{\tau^\alpha s^{\alpha-1}}{1 + (s\tau)^\alpha} \implies \Phi(t) = E_\alpha(-(t/\tau)^\alpha), \quad \Phi(0) = 1. \quad (1)$$

This is the standard result of Hilfer and Anton (1995), Metzler and Klafter (2000) and Mainardi (2010), the last being the source the referee cites. The referee’s own second Laplace computation obtains $\mathcal{L}\{E_\alpha(-t^\alpha)\} = s^{\alpha-1}/(1 + s^\alpha)$, which is exactly (1). The report thus verifies that the manuscript’s function is the survival function of the process the manuscript describes.

2. The function proposed by the referee is not a survival function

The referee’s alternative, $\Phi(t) = -\frac{d}{dt}E_\alpha(-t^\alpha)$, does not follow from the physics of the problem. It follows from an initial condition the referee chooses, in the report’s words “if I set $[J^{1-\alpha}\Phi]_{t=0} = 1$ ”. That choice is not available here, for three reasons.

(i) *It is singular.* Explicitly,

$$-\frac{d}{dt}E_\alpha(-t^\alpha) = t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha) \sim \frac{t^{\alpha-1}}{\Gamma(\alpha)} \rightarrow \infty \quad (t \rightarrow 0^+),$$

so it violates both $\Phi(0) = 1$ and $\Phi \leq 1$. A probability cannot be unbounded.

(ii) *It is the density, not the survival.* Its Laplace transform is $1/(1 + s^\alpha) = \tilde{\psi}(s)$, and it integrates to one over $(0, \infty)$. It is the waiting-time density $\psi = -d\Phi/dt$ of the very process whose survival function is $E_\alpha(-t^\alpha)$. The report replaces a probability by its own derivative and calls the derivative the correct solution.

(iii) *The initial condition is unphysical for this problem.* The condition $[J^{1-\alpha}\Phi]_{t=0} = 1$ prescribes a fractional integral of the unburned fraction at the initial instant. The physical datum is that all fuel is unburned at $t = 0$, that is $\Phi(0) = 1$, and for any bounded Φ one has $[J^{1-\alpha}\Phi]_{t=0} = 0$, not 1.

The two functions are therefore not a “wrong” and a “correct” solution of one problem. They are the survival function and the density of a single renewal process, and the manuscript uses each in its proper role: Section 3.1 states the density tail $t^{-(1+\alpha)}$, and Section 3.2 states the survival tail $t^{-\alpha}/\Gamma(1-\alpha)$. That these tails differ, which the referee presents as a consequence of an error, is simply the relation $\psi = -\Phi'$ and is already in the manuscript.

3. What the referee’s calculation actually shows

What the calculation in the report establishes is narrower than what the report concludes. The Riemann-Liouville derivative of the constant initial value does not vanish, ${}^{RL}D_t^\alpha 1 = t^{-\alpha}/\Gamma(1-\alpha)$, so that

$${}^{RL}D_t^\alpha E_\alpha(-(t/\tau)^\alpha) = -\tau^{-\alpha}E_\alpha(-(t/\tau)^\alpha) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \quad (2)$$

The one-line statement of the kinetic equation in Section 3.2 was written in compressed form without displaying this initial-value term. The complete Riemann-Liouville statement of the relaxation problem with the physical datum $\Phi(0) = 1$ is

$${}^{RL}D_t^\alpha [\Phi - \Phi(0)] = -\tau^{-\alpha}\Phi \iff \frac{d\Phi}{dt} = -\tau^{-\alpha} {}^{RL}D_t^{1-\alpha}\Phi, \quad \Phi(0) = 1, \quad (3)$$

whose unique solution is $E_\alpha(-(t/\tau)^\alpha)$. The referee states this second form at the end of the report and agrees that the Mittag-Leffler function solves it. The report therefore ends by conceding the point at issue: the function in the manuscript is the solution of the Riemann-Liouville fractional relaxation equation with the physical initial condition. The remedy is to display the initial-value term in one sentence. It is not to change the function, and nothing in the report supports changing it.

4. The Rothermel power law is not a temporal decay

The report’s only stated consequence is that, since the tails of the two functions differ and “reproducing the phenomenological power-law displayed by the Rothermel model was the starting idea, the analysis should be rearranged.” This misreads the manuscript in two respects.

First, the Rothermel power law is a power law in midflame wind speed, $\phi_w = (U/U_1)^B$, with exponent B a tabulated fuel property. It is not a power law in time and is not produced by the long-time tail of any relaxation function. In the manuscript it is recovered from the first-order term of the Mittag-Leffler series in the dimensionless wind drive $x = (U/U_*)^B$,

$$\phi_w^{\text{ML}} = \phi_\infty [1 - E_\alpha(-x)] = \frac{\phi_\infty}{\Gamma(1+\alpha)} x + O(x^2),$$

for every $\alpha \in (0, 1]$, with $U_* = U_1(\phi_\infty/\Gamma(1+\alpha))^{1/B}$. The temporal tail exponent, the quantity on which the referee’s objection turns, appears nowhere in this recovery.

Second, the starting idea of the manuscript is not to generate the power law by fractional calculus, as the report’s summary has it. The power law is Rothermel’s and is taken as given. The purpose of the manuscript is the opposite: to bound it, replacing unbounded growth by a saturating plateau, and to prove the consequences for heading fire collapse.

5. Nothing downstream depends on the sentence examined

The wind factor of Section 3.3 is a constitutive saturation law in the wind variable, introduced, in the manuscript's words, "in analogy with" the Mittag-Leffler survival. Everything that follows uses only analytic properties of the function $E_\alpha(-c)$: its power series, its large-argument asymptotics and its complete monotonicity (Pollard's theorem). This covers the definition of $h(c)$ and of the critical plateau ϕ^* , the auxiliary-ratio lemma, all four parts of the critical-plateau theorem, the recovery of the classical threshold U_c , the fuel table, the comparison with the grassland model of Cheney et al. (1998), and every figure. None of these invokes a differential equation in time. The referee's function could not be substituted in any case: it diverges at zero argument, so the resulting wind factor would not vanish at zero wind.

The report does not comment on any of this material: the stochastic cascade and the origin of the order, the lemma and theorem on heading fire collapse, the front framework, or the validation and fuel analysis. A recommendation against publication based on the wording of one sentence that no result uses is, I respectfully submit, not supported by the report's own content.

6. A minor remark: the case $\alpha = 1$

The referee remarks that $\alpha = 1$ should be considered separately. No separate treatment is needed. The Riemann-Liouville derivative is defined for every real order $\alpha > 0$ by

$${}^{RL}D_t^\alpha f(t) = \frac{d^n}{dt^n} J^{n-\alpha} f(t), \quad n = \lfloor \alpha \rfloor + 1,$$

which at $\alpha = 1$ gives ${}^{RL}D_t^1 f = \frac{d^2}{dt^2} J^1 f = f'$, and which is also the continuous limit $\alpha \rightarrow 1^-$ of the expression displayed in the manuscript, since $J^{1-\alpha} f \rightarrow f$.

7. Clarifications in the revised manuscript

1. Section 3.2: derive Φ from the renewal process as in (1), state the kinetic equation in the complete form (3) with $\Phi(0) = 1$ and the identity (2) displayed, and give the waiting-time density $\psi(t) = \tau^{-\alpha} t^{\alpha-1} E_{\alpha,\alpha}(-(t/\tau)^\alpha)$ alongside it, so that the two functions cannot be confused.
2. Section 3.2: display the general Riemann-Liouville definition for $\alpha > 0$, of which the formula in the manuscript is the $n = 1$ case. The Riemann-Liouville operator is retained throughout.
3. Section 3.3: state explicitly that the wind factor is a constitutive law in the wind variable and that the lemma and theorem on heading fire collapse are independent of the time-domain equation.

In summary, the report's central claim, that the manuscript uses the wrong function, is incorrect: the function is the survival function of the stated process, as the referee's own transform shows, and the proposed replacement is its density, which is unbounded and cannot be a probability. The stated consequence for the Rothermel power law does not arise, because that power law is in wind speed and is recovered independently of any temporal tail. What remains is the display of an initial-value term in one sentence, which I will supply. I ask the editor to weigh the recommendation accordingly, and I would welcome the referee's assessment of the results of the manuscript, on which the report is silent.

References

Hilfer, R., Anton, L., 1995. Fractional master equations and fractal time random walks. *Physical Review E* 51, R848–R851.

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