



Sunlit and Unlit: The Limitations of Albedo Prediction in Energy Balance Models

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Abstract. Energy balance models (EBMs) comprise a simplified physical approach to climate modeling relative to more advanced general circulation model (GCM) counterparts. In particular, EBMs simplify numerous radiative features of Earth's surface-atmosphere system through a singular parameterization of albedo which defines the proportion of reflected to total incoming solar radiation. The literature provides some basic albedo models. However, these models are known to be limited in their real-world application. This paper takes a *model identification* approach to predicting the rate of change in top-of-atmosphere (TOA) albedo given real climate data. First, we design and implement several regression models for the prediction of the rate of change in albedo. Namely, we consider a variety of polynomial and kernel ridge regression (KRR) solutions, comparing these predictors to benchmark functions from the EBM literature. Second, we evaluate these predictors through a series of case studies in which we justify the selection of simulated temporal resolutions. Third, we quantify the performance of individual albedo predictors and test the impact of coupled temperature and albedo updates. Finally, this work finds that KRR with a radial basis function (RBF) kernel can provide a robust predictor of albedo, but requires careful consideration of tuning, spatial resolution, and input feature selection. In short, the paper delivers a thorough assessment of the limitations of albedo model identification and prediction.

1 Introduction

In projecting Earth's future climate, models must balance physical fidelity with computational tractability. Comprehensive general circulation models (GCMs) represent coupled atmospheric, oceanic, and surface processes and serve as benchmark tools in coordinated model development and intercomparison efforts Abdelmoaty et al. (2021); Tilmes et al. (2018). However, their computational expense can limit their use in large ensembles and long-horizon simulations Jones et al. (2005); Otto-Bliesner et al. (2017); Smith and Gregory (2012). Energy balance models (EBMs), by contrast, provide reduced-order approximations of Earth's temperature evolution by modeling radiative imbalance and simplified heat transport North (1975). Although EBMs cannot resolve the full set of coupled processes represented in GCMs, they remain valuable because they provide interpretable and computationally efficient approximations of climate response. Fig. 1 depicts this conceptual contrast between EBMs and GCMs, while noting that individual models within each class may vary substantially in resolution, parameterization, and computational cost.

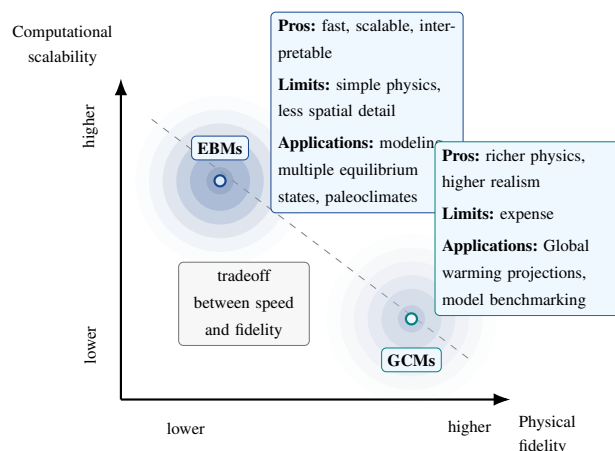


Figure 1. Contrast between EBMs and GCMs and their application—physical fidelity versus computational tractability.

25 A central challenge in improving EBM fidelity is the representation of unresolved surface and radiative processes, particularly albedo. In GCMs, albedo is distributed between land surface type Rowe (1993), reflectivity by clouds Hunt (1982), and radiative scattering by aerosols Petty (2006). In EBMs, these effects are typically compressed into a single parameter which may be modeled in a variety of forms, depending upon simplified models of temperature and surface types. While these approaches preserve the computational efficiency of EBMs, they limit the ability of the model to represent spatially and temporally varying changes in top-of-atmosphere (TOA) albedo. Rather than prescribing albedo solely through an assumed functional form, we learn dynamic TOA albedo updates from reanalysis data and evaluate their coupled effect on EBM temperature evolution. In particular, we compare observation-informed albedo models with literature-based parameterizations, integrate each within a one-dimensional EBM, and examine how errors in albedo prediction propagate into temperature errors, as well as how temperature errors affect subsequent albedo evolution. The following literature review contrasts albedo treatments in GCM and EBM studies to contextualize the parameterizations used as benchmarks in this work.

Albedo Representations by Model Type. Albedo is integral to the temporal development of Earth’s temperature field, defining the proportion of incoming shortwave radiation (insolation) reflected by the surface-atmosphere system. In higher-fidelity climate models, its representation must account for feedbacks between surface states, radiation, and temperature. A prominent example is the *ice-albedo feedback loop*, in which warming reduces snow and ice cover, exposing darker underlying surfaces that absorb more radiation and further amplify warming North and Kim (2017). Because this feedback depends on the evolution of the underlying cryospheric extent and evolving surface properties, its representation becomes complex. GCM-based studies propose several classes of methods to address this problem. One class of methods decomposes the net climate response into contributions from individual feedback mechanisms Stocker (2011); Rohling et al. (2012). The study Booth et al. (2024), for example, incorporates ice-albedo feedback as an additional contribution, estimating its radiative effect using proxy data from previous ice ages. A second class of approaches uses observable present-day behavior to constrain future feedback strength.



The study Thackeray and Hall (2019) relates seasonal variations in present-day ice cover to projected future feedback across a GCM ensemble, reducing the influence of models with unrealistic estimates of sea ice thickness. Other studies improve the physical representation of albedo directly through sea ice and melt pond parameterizations which more clearly inform nuance in cryospheric cover Diamond et al. (2024); Hunke et al. (2013). These efforts increase the physical detail with which higher-fidelity models represent surface-radiation interactions, but also introduce computational and parametric complexity when surface properties evolve over time.

In contrast to the process-level albedo representations used in higher-fidelity climate models, EBM typically represent albedo through compact, interpretable parameterizations. A common approach is to prescribe albedo as a function of temperature, assuming temperature as a proxy for snow and ice extent. Sellers' early EBM formulation incorporates this idea through a temperature-dependent albedo law with prescribed lower and upper temperature thresholds Sellers (1969). Later works opt for smooth approximations of this transition, often through hyperbolic tangent functions centered near a freezing-temperature threshold North and Kim (2017); Del Sarto et al. (2024). These functions provide a tractable mechanism for representing ice-albedo feedback and are useful for studying equilibrium structure and climate sensitivity North (1984); Huang and Bowman (1992). However, because they are typically univariate, they assign albedo from temperature alone and therefore do not explicitly account for additional variables such as surface composition or solar geometry.

A second class of EBM approaches increases physical detail by assigning albedo according to prescribed surface categories. The model development in Zhuang et al. (2017) considers a two-dimensional EBM using surface classifications such as land, sea, snow, and sea ice, allowing albedo to spatially vary with the underlying surface type. This type of formulation enables an efficient comparison with observed climatology and provides a more spatially resolved treatment of surface-radiation interactions than purely temperature-dependent forms. This model is extended through *TransEBM*, incorporating transient parameterizations of orbital configuration, atmospheric carbon dioxide, and insolation for long-horizon paleoclimate applications Ziegler and Rehfeld (2021). These models enhance the physical fidelity of EBM methodology, but omit the positive (amplifying) feedback resulting from cryospheric losses as the Earth warms.

Other EBM-based studies introduce nonlinear or transient surface behavior more directly, adding nuance to both temperature evolution and cryospheric extent. More recent work Samuelsberg and Jakobsen (2025) analyzes a one-dimensional EBM with nonlinear albedo feedback using boundary integral methods, showing how threshold-like albedo transitions can affect equilibrium structure and spatial climate response. While this approach preserves the analytical advantages of an idealized EBM, the albedo law remains intentionally simplified. Related energy balance approaches have also been developed for cryosphere-specific applications. The study Krebs-Kanzow et al. (2021) introduces the *diurnal Energy Balance Model* (dEBM), a computationally efficient surface mass balance scheme for ice sheets in Earth-system modeling; dEBM defines glaciated-surface reflectivity to depend on varying states of snow and ice, while also incorporating radiative and atmospheric forcing information.

These efforts show that EBM-style models can move beyond purely temperature-dependent parameterizations of albedo, while retaining their scalability. The progression from temperature-dependent methods to surface-aware and cryosphere-focused energy balance modeling suggests advantages for regression and machine learning approaches for albedo prediction.



80 **Empirical Albedo Models.** The field of remote sensing proposes numerous approaches to inform predictions of surface-level albedo. Surface-level reflectivity comprises the highly variable influences of vegetation, land use, and moisture Rowe (1993), as opposed to the compacted albedo assumed in EBM. The study Li and Garand (1994) presents land surface albedo as a succinct parameterization of solar zenith angle, top-of-atmosphere (TOA) albedo, and total precipitable water. Their estimates are shown to agree well with analysis data sets produced at the time of publication. More advanced methods involve the use of machine learning (ML), with particular use drawn from neural networks. The study Liang et al. (1999) applies a feed-forward neural network to separate the surface albedo from its TOA counterpart, where results are shown to be independent of atmospheric conditions. Further, the study Zhang et al. (2020) considers land surface albedo prediction for two study regions, applying a *nonlinear autoregressive neural network* (NARnet) to construct time series estimates of albedo. Their approach shows better agreement in snow-free regions with changes in the land surface, snowfall, and melting requiring changes to their model's parameters. For EBM development, these studies suggest that there is a methodological precedent for multivariate albedo prediction, incorporating solar geometry, surface class, and seasonality into their formulations.

Paper's Contribution and Organization. This paper's contributions are as follows:

- We formulate a one-dimensional EBM in state-space form in which TOA albedo is treated as an evolving state variable rather than a prescribed diagnostic field. The novelty of this formulation is the introduction of an inferred albedo-update model which estimates the rate of change in albedo from local temperature, albedo, and insolation. We then numerically integrate the coupled temperature-albedo system forward in time, providing a reduced-order setting for studying how latitudinal reflectivity evolves jointly with temperature and how errors in the albedo update propagate through the EBM.
- We analyze how temporal resolution affects calibrated EBM parameters and the inferred heat-transport requirements of the model. Using observed radiative balance, temperature, and albedo fields, we compute a residual term that quantifies the unresolved transport and additional flux required for the EBM to remain consistent with real-world observations. This analysis provides a basis for assessing whether daily time steps offer a meaningful improvement over coarser temporal discretizations.
- We develop and compare various forms for predicting the temporal derivatives of albedo and temperature using various forms of polynomial regression and Gaussian kernel ridge regression. For albedo, we evaluate polynomial model order, latitude-grouped kernel predictors, and Laplacian-informed smoothing to determine how local and neighboring-latitude information affect predictions. We then extend this analysis to temperature prediction by replacing the physical EBM update with a vector of temperature kernels and coupling the resulting temperature model with the selected albedo models. This comparison identifies when additional spatial information improves prediction and when it introduces unnecessary complexity.

Section 2 details the governing equations and various methods implemented for calculating changes in temperature and albedo. Section 3 describes each of the case studies proposed in this paper and the associated climate data that are required in their evaluation. In each subsection, the results of the corresponding case study will be presented, informing the approach of the subsequent case studies. Section 4 concludes the paper with remarks on the performance of the proposed model forms, limitations of the paper, and future work. Paper acronyms and notation are given in Appendix A.



2 Methodology

115 This section details the methodology in the paper. Some of the developments in this section, like some of the governing equations, are standard in the literature. We reproduce them herein so that the paper's developments are concise, while offering an educational background for the readers who might be unfamiliar with the EBM-related models, starting from the PDEs and their approximations. A knowledgeable reader may skip the first few paragraphs of the EBM development.

2.1 EBM Formulation

120 Assume the Earth's surface may be parameterized by longitude, $\theta \in [-\pi, \pi)$, and latitude, $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Define $\mathbf{r} = [\varphi, \theta]^\top$, as the coordinate pair of a given point on Earth's spherical surface. Intuitively, the EBM may be summarized as a rate of change in local heat storage at $\mathbf{r}$. This rate of change is driven by absorbed shortwave radiation, Q_{SW} , reduced by outgoing longwave radiation, Q_{LW} , and redistributed laterally by diffusive heat transport, Q_{diff} . The rate of change in local heat storage may be rewritten as the product of the thermal heat capacity, $C(\mathbf{r}, t)$, and rate of change in temperature, $\frac{\partial u(\mathbf{r}, t)}{\partial t}$. The general form of the 2D EBM may be written:

$$C(\mathbf{r}, t) \frac{\partial u(\mathbf{r}, t)}{\partial t} = Q_{\text{SW}}(\mathbf{r}, t) - Q_{\text{LW}}(\mathbf{r}, t) + Q_{\text{diff}}(\mathbf{r}, t), \quad (1a)$$

$$Q_{\text{SW}}(\mathbf{r}, t) = S(\mathbf{r}, t)(1 - \alpha(\mathbf{r}, t)), \quad (1b)$$

$$Q_{\text{LW}}(\mathbf{r}, t) = \varepsilon(\mathbf{r}, t)\sigma_0 u(\mathbf{r}, t)^4, \quad (1c)$$

$$Q_{\text{diff}}(\mathbf{r}, t) = \nabla \cdot (D(\mathbf{r}, t) \nabla u(\mathbf{r}, t)). \quad (1d)$$

130 We define C to be the effective heat capacity in $\text{J}/(\text{m}^2 \text{K})$, u to be the temperature in K , S to be the insolation in W/m^2 , α to be the albedo, ε to be Earth's emissivity, σ_0 to be the Stefan-Boltzmann constant in $\text{W}/(\text{m}^2 \text{K}^4)$, and D to be the thermal conductivity in $\text{W}/(\text{m}^2 \text{K})$. Averaging across all θ , (1a)–(1d) are converted to a one-dimensional EBM of the following form:

$$C(\varphi, t) \frac{\partial u(\varphi, t)}{\partial t} = Q_{\text{SW}}(\varphi, t) - Q_{\text{LW}}(\varphi, t) + Q_{\text{diff}}(\varphi, t) \quad (2a)$$

$$Q_{\text{SW}}(\varphi, t) = S(\varphi, t)(1 - \alpha(\varphi, t)) \quad (2b)$$

135 $Q_{\text{LW}}(\varphi, t) = \varepsilon(\varphi, t)\sigma_0 u(\varphi, t)^4, \quad (2c)$

$$Q_{\text{diff}}(\varphi, t) = \mathcal{L}(u(\varphi, t)), \quad (2d)$$

where $\mathcal{L}(u)$ represents the Laplace-Beltrami operator for 1D heat transport, considering only meridional flux. This formulation builds on the Budyko-Sellers energy balance framework, where temperature evolution is governed by radiative imbalance, and adopts the North-type diffusion model for horizontal heat transport Budyko (1969); Sellers (1969); North (1975). A general overview of the gridded EBM is provided in Fig. 2, relating Earth's parameterized fields and their relations to individual forcings. We now define each term of (2a), depicting the assumptions made in these equations and the supporting literature.

From (2b), the absorbed shortwave radiation is determined by the incoming shortwave radiation and the local albedo. We compute the incoming shortwave radiation using the daily-mean insolation formulation in Berger (1978), which relates Earth's

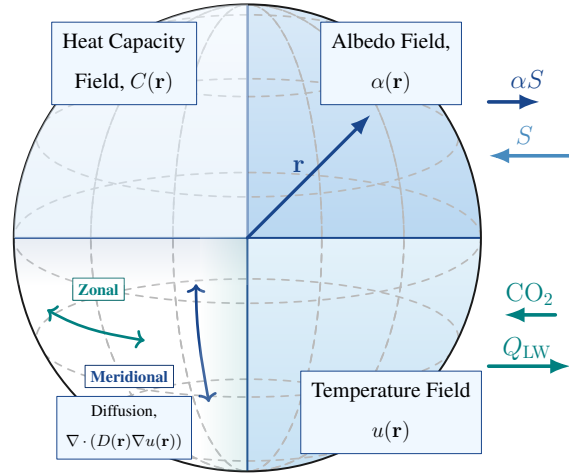


Figure 2. Schematic of the 2D EBM depicted in (1a)–(1d). Each quadrant represents a term of the governing equation.

orbital configuration to the latitudinal distribution of solar forcing. In the one-dimensional EBM, this forcing is zonally averaged and therefore independent of longitude. For a given latitude φ and time t , the daily-mean insolation falls into one of three regimes: polar night (PN), in which the Sun remains below the horizon over the full day; polar day (PD), in which the Sun remains above the horizon over the full day; and the intermediate case, in which sunrise and sunset occur during the day. Following the formulation refined in Ziegler and Rehfeld (2021), the zonally averaged incoming shortwave radiation is written as:

$$S(\varphi, t) = \begin{cases} 0, & \text{PN} \\ \frac{s_0}{\psi(t)^2} (\sin \varphi \sin \delta), & \text{PD} \\ \frac{s_0}{\psi(t)^2 \pi} (h_0 \sin \varphi \sin \delta + \cos \varphi \cos \delta \sin h_0), & \text{else} \end{cases} \quad (3a)$$

$$\psi(t) = \left(\frac{1 + \epsilon \cos(\varsigma(t) - \varsigma_0)}{1 - \epsilon^2} \right)^2, \quad (3b)$$

$$\delta(t) = \sin^{-1}(\sin \varpi \sin \varsigma(t)), \quad (3c)$$

$$h_0(t) = \cos^{-1}(-\tan \varphi \tan \delta(t)), \quad (3d)$$

$$\text{PN} : \tan \varphi \tan \delta(t) \leq -1, \quad (3e)$$

$$\text{PD} : \tan \varphi \tan \delta(t) \geq 1, \quad (3f)$$

where $s_0 = 1361 \text{ W/m}^2$ is the solar constant Kopp and Lean (2011), $\delta(t)$ is the declination angle, $h_0(t)$ is the absolute hour angle of the Sun, $\varsigma(t)$ is the angle counterclockwise from Earth's perihelion, ς_0 is the longitude of the perihelion, ϵ is the eccentricity of Earth's orbit, ϖ is Earth's obliquity angle, and $\psi(t)$ is the normalized Earth-Sun distance. The orbital parameters ς_0 , ϖ , and ϵ are short-term constants gathered from Laskar et al. (2004). Conditions PN and PD are defined by (3e) and (3f),



160 denoting the orbital conditions which represent polar night and polar day, respectively.

We next evaluate the local outgoing longwave radiation (OLR), embodied by Q_{LW} , the primary mechanism for global cooling. Earth emits energy as a blackbody in space, following the Stefan-Boltzmann law in (2c). While physically justified, the quartic dependence on temperature produces highly nonlinear behavior which may be difficult to contain in transient simulations. For this purpose, many EBMs opt instead for the linearization proposed in Budyko (1969):

$$165 \quad Q_{\text{LW}}(\varphi, t) \approx A(t) + B(u(\varphi, t) - 273.15), \quad (4)$$

where $A(t)$ denotes the OLR at 0°C in W/m^2 , and B denotes the slope, or radiative damping in $\text{W}/(\text{m}^2\text{ }^\circ\text{C})$, that relates OLR forcing to changes in surface temperature. To capture the effect of the greenhouse gas effect, we parameterize A in terms of global average CO_2 concentrations Myhre et al. (1998), hindering the reference OLR as carbon concentrations increase. This allows the intercept, A , to vary with time:

$$170 \quad A(t) = A_{\text{ref}} - S_{\text{CO}_2} \ln \left(\frac{\text{CO}_2(t)}{\text{CO}_{2,\text{ref}}} \right), \quad (5)$$

where S_{CO_2} is the scaling factor in W/m^2 , $\text{CO}_{2,\text{ref}}$ is the reference atmospheric carbon dioxide concentration in parts per million (ppm), and A_{ref} is the reference OLR given reference carbon concentrations at 0°C . The values of A_{ref} and B are developed from tuning with wide ranges of accepted values proposed in the literature North and Kim (2017). This additional forcing is a simplified representation of the greenhouse gas effect, inhibiting OLR as more carbon is emitted. Outside of seasonal variations in insolation, increases to CO_2 drive the EBM into disequilibrium.

Whereas Q_{LW} and Q_{SW} govern local radiation budgets, Q_{diff} , the diffusive heat transport, defines how this imbalance is distributed from the equator to the poles. In (2d), we define $\mathcal{L}(u(\varphi, t))$ to represent the meridional heat transport in 1D spherical coordinates. This operator is expanded:

$$\mathcal{L}(u) = \frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} \left(D \cos \varphi \frac{\partial u}{\partial \varphi} \right).$$

180 For convenience, let $\mu = \sin \varphi$ North and Kim (2017); North (1975):

$$\mathcal{L}(u) = \frac{\partial}{\partial \mu} \left(D(1 - \mu^2) \frac{\partial u}{\partial \mu} \right). \quad (6)$$

This form automatically satisfies the Neumann boundary conditions of no-flux at the poles, $\mu = \pm 1$:

$$(1 - \mu^2) \frac{\partial u}{\partial \mu} \Big|_{\mu=\pm 1} = 0. \quad (7)$$

If D is independent of latitude, it may be removed from the encompassing derivative with respect to μ . Changes to the evaluation of D as a function of latitude are applicable to models which demonstrate spatial heterogeneity or hemispheric asymmetry in heat transport Ziegler and Rehfeld (2021).

Having defined the 1D EBM in continuous time and continuous space, the following section details this problem in terms of discrete space, where the Earth system comprises a finite number of latitudinal bands. Appendix D (Tab. D1) outlines the units, corresponding discrete-space form, and methods of determination for each quantity proposed in (2)–(6).



190 2.2 State-Space Solution for the 1D EBM

This paper presents a state-space approach in which the climate system is represented by a finite-dimensional set of state variables, standardizing the identification of a candidate albedo model. If the state variables and disturbances of a system are known, the governing equations can be used to predict its future evolution. Consider a spatial discretization of the Earth system into N latitudinal bands, where each latitude is indexed by φ_i for $i = 1, 2, \dots, N$. The continuous-time, spatially discrete state vector is written as $\mathbf{x}(t) = [\mathbf{u}^\top(t), \boldsymbol{\alpha}^\top(t)]^\top \in \mathbb{R}^{2N}$. The state variables considered in this work are $\mathbf{u} \in \mathbb{R}^N$, the temperature at all latitudes of the system, and $\boldsymbol{\alpha} \in \mathbb{R}^N$, the albedo at all latitudes of the system. To represent exogenous forcings on the system, we write $\mathbf{w}(t) = [\mathbf{s}^\top(t), \mathbf{f}_{\text{CO}_2}^\top(t)]^\top \in \mathbb{R}^{2N}$, the vector of disturbances, where $\mathbf{s}(t) \in \mathbb{R}^N$, is the insolation vector, and $\mathbf{f}_{\text{CO}_2}(t) \in \mathbb{R}^N$ is the vector of CO_2 forcings. The state-space model is written as

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{\mathbf{u}}(t) \\ \dot{\boldsymbol{\alpha}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u(\mathbf{u}(t), \boldsymbol{\alpha}(t), \mathbf{w}(t); t, \boldsymbol{\varrho}_u) \\ \mathbf{f}_\alpha(\mathbf{u}(t), \boldsymbol{\alpha}(t), \mathbf{w}(t); t, \boldsymbol{\varrho}_\alpha) \end{bmatrix}, \quad (8a)$$

$$\mathbf{w}(t) = \begin{bmatrix} \mathbf{s}(t) \\ \mathbf{f}_{\text{CO}_2}(t) \end{bmatrix}, \quad (8b)$$

where $\mathbf{f}_u$ and $\mathbf{f}_\alpha$ are the functions governing the continuous-time evolution of $\mathbf{u}(t)$ and $\boldsymbol{\alpha}(t)$, respectively. The parameters comprising $\mathbf{f}_u$ and $\mathbf{f}_\alpha$ are denoted by the vectors $\boldsymbol{\varrho}_u$ and $\boldsymbol{\varrho}_\alpha$, respectively.

From the dynamical system posed in (8), the objective of this paper is to create a systematic approximation of Earth's reduced-order physics through an energy balance lens. Given real-world discrete-time observations of temperature $\mathbf{u}$ and albedo, $\boldsymbol{\alpha}$, we develop numerically integrable approximations for the true $\mathbf{f}_u$ and $\mathbf{f}_\alpha$ governing the one-dimensional system. The remainder of the methodology proceeds in two steps. First, we derive the form of the partially known rate of change in temperature, $\mathbf{f}_u$, from the spatially discretized 1D EBM and the radiative models developed in (2)–(7). This term is partially known because its radiative-balance and diffusive-transport structure are prescribed by the EBM, while its evaluation still depends on the albedo field $\boldsymbol{\alpha}(t)$ and the exogenous forcings $\mathbf{w}(t)$. Second, after obtaining a numerically integrable form of $\mathbf{f}_u$, we identify candidate forms of the unknown rate of change in albedo, $\mathbf{f}_\alpha$. Thus, the temperature equation provides the known physical scaffold of the state-space model, while the albedo equation defines the model-identification problem considered in this work.

2.2.1 Physical Temperature Model

It is known that $\mathbf{f}_u$ must equate to $\dot{\mathbf{u}}(t)$, the vector defining the temporal derivative of $\mathbf{u}$. Using the governing formulation in (2) and the linearization in (4):

$$\begin{aligned} \dot{\mathbf{u}}(t) = & \text{diag}(\mathbf{c}(t))^{-1} [\mathbf{s}(t) \odot (\mathbf{1} - \boldsymbol{\alpha}(t)) \\ & - (A(t)\mathbf{1} + B\mathbf{u}(t)) + \mathbf{L}(t)\mathbf{u}(t)], \end{aligned} \quad (9)$$



where $\mathbf{c} \in \mathbb{R}^N$ is the vector of heat capacities in $\text{J}/(\text{m}^2 \text{K})$, $\mathbf{1} \in \mathbb{R}^N$ is the N -dimensional vector of ones, $A(t) \in \mathbb{R}$ is the scalar
calculated in (5), and $\mathbf{L} \in \mathbb{R}^{N \times N}$ is the tridiagonal matrix encoding meridional heat transport between latitudes. The proposed
220 form of $A(t)$ assumes a globally uniform forcing, $\mathbf{f}_{\text{CO}_2}$:

$$A(t)\mathbf{1} = A_{\text{ref}}\mathbf{1} - \mathbf{f}_{\text{CO}_2}(t).$$

The term $\mathbf{L}\mathbf{u}(t)$ evaluates the transfer of heat between φ_i and its neighboring latitudinal bands, approximating (6) in discrete
space, while enforcing the boundary condition in (7).

Given that (9) defines a continuous-time ODE, a numerical time-integration scheme is required to advance the state from
225 $\mathbf{u}(t_k) = \mathbf{u}^k$ to $\mathbf{u}(t_{k+1}) = \mathbf{u}^{k+1}$ for a discrete time step size Δt and time $t_k = t_0 + k\Delta t$, where $k \in \mathbb{Z}$. In this paper, we opt
for a Crank-Nicolson discretization as it provides a second-order time-centered approximation while retaining unconditional
stability for linear problems. This implicit stability of Crank-Nicolson is useful for EBMs integrated with daily or coarser
climate forcings, where explicit methods such as Forward Euler can require restrictive time steps Stocker (2011). This stability
is beneficial in comparing how the selection of time step, Δt , affects the EBM solution. We now derive the discrete-time,
230 discrete-space Crank-Nicolson update for the proposed 1D EBM.

2.2.2 Meridional Flux Model

Calculating absorbed shortwave radiation and OLR at time t_k remains fairly straightforward. Using (3), the insolation at time t_k ,
 s^k may be calculated at all latitudes. For brevity, we reserve the formulation of the insolation model in discrete time and discrete
space for Appendix B. The corresponding albedo, α^k , may be numerically integrated from some discrete time estimator of $\dot{\alpha}$
235 and an existing state α^k , and $\mathbf{f}_{\text{CO}_2}$ can be predicted using either observational data or baseline emissions trajectories. However,
to numerically model the diffusive transport, Q_{diff} , we must evaluate the form of $\mathbf{L}^k \mathbf{u}^k$. Heat transport connects individual
latitudinal climates so that the temperature of a given latitude is not inconsistent with those of its immediate neighbors. From
(6), heat transport between latitudes must be evaluated with respect to $\mu = \sin \varphi$. The difference $\Delta\mu = \mu_{i+1} - \mu_i$ is assumed to
be constant for all $i = 1, 2, \dots, N-1$. Next define $\mu_{i+\frac{1}{2}} = \frac{1}{2}(\mu_i + \mu_{i+1})$, indicative of the interpolation between the latitudinal
240 sines μ_i and μ_{i+1} . Let the subscript of i refer to the evaluation of any variable evaluated at μ_i , e.g., $u_i = u(\mu_i)$.

Note, this assumption of uniform spacing in μ is used only in the diffusive operator, where it improves the numerical
behavior near the poles. All radiative balance and temperature data are evaluated on the original uniformly spaced latitude grid
of a constant resolution, $\Delta\varphi = \varphi_{i+1} - \varphi_i$, to avoid the interpolation of observational inputs that are provided as functions of
latitude.

245 Meridional flux is implemented so that (a) only the immediate neighbors to μ_i , μ_{i-1} and μ_{i+1} , are considered, and (b) at the
poles the diffusive flux through the exterior boundary is set to zero, so each polar grid point exchanges heat only with its single
interior neighbor. The latter assumption satisfies the boundary condition outlined in (7). Let

$$\boldsymbol{\omega}^k = \mathbf{L}^k \mathbf{u}^k, \tag{10}$$



where $\omega^k \in \mathbb{R}^N$ is the diffusive heat transport in W/m^2 at time t_k . Then the entries of ω^k are given by

$$\omega_i^k = \begin{cases} d_1^k(u_2^k - u_1^k), & i = 1, \\ -d_{N-1}^k(u_N^k - u_{N-1}^k), & i = N, \\ d_i^k(u_{i+1}^k - u_i^k) - d_{i-1}^k(u_i^k - u_{i-1}^k), & \text{else,} \end{cases} \quad (11)$$

where d_i^k is the interface coefficient unique to μ_i at time t_k . We then define the interface coefficient to be:

$$d_i^k = \frac{1 - \mu_{i+1/2}^2}{(\Delta\mu)^2} D_{i+1/2}^k, \quad i = 1, \dots, N-1, \quad (12)$$

where $D = D(\mu, t_k)$ is the weight of the diffusion as a function of μ and k .

2.2.3 Spatially Varying Diffusion

Following Chang and Merlis (2023), we relax the assumption of a globally constant diffusion coefficient, D . In particular, Chang and Merlis (2023) emphasizes that meridional heat transport can depend on the climate state, including temperature-dependent processes. As a proxy, we adopt the prescribed space-time parameterization $D(\mu, t_k)$ that serves as a reduced-order proxy for these unresolved variations. Spatial dependence in $\mu = \sin \varphi$ permits different latitudinal regions to assign different weights to meridional heat transport, while temporal dependence provides a simple seasonal proxy to model the unresolved effects of temperature on thermal conductivity. To retain hemispheric asymmetry, the seasonal correction is taken to depend on the signed coordinate μ , so that the Northern and Southern Hemispheres experience opposing seasonal phases.

We partition the domain $\mu \in [-1, 1]$ into N_d regions, assigning a unique regional multiplier η_n to each region. Each $\chi_n(\mu)$ is constructed to be approximately one within its assigned domain and approximately zero outside that band, with smooth transitions near neighboring regional boundaries. This smoothing avoids discontinuous jumps in the diffusion coefficient, while preserving the interpretation of η_n . We reserve the explicit construction of χ_n for Appendix C. The diffusion coefficient, $D(\mu, t_k)$, is proposed to be

$$D(\mu, t_k) = D_0 \left(\sum_{n=1}^{N_d} \eta_n \chi_n(\mu) \right) \left(1 + \frac{\mu}{2} \cos \nu^k \right), \quad (13)$$

where $\nu^k \in [0, 2\pi)$ is the phase angle of the year, D_0 is the global diffusion scale, and η_n is the dimensionless regional diffusion multiplier such that $D_0 \eta_n$ represents the effective thermal conductivity in the interior of the n -th region. By convention, $\nu^k = 0$ corresponds to 1 January. The factor $\mu \cos \nu^k$ damps the seasonal correction near the equator and gives opposite seasonal signs in the Northern and Southern Hemispheres, consistent with stronger seasonal variability in temperature toward the poles. The regional multipliers are tuned as a one-dimensional simplification of the hemispheric and surface-oriented parameterizations used in Zhuang et al. (2017) and Ziegler and Rehfeld (2021). With the proposed heat transport framework summarized in (10), we may now combine all radiative balance components into a discrete time numerical integration scheme.



275 2.2.4 Crank-Nicolson Numerical Integration Scheme

Let the sum of the absorbed shortwave radiation and OLR terms be represented by a single forcing term

$$\mathbf{g}^k = \mathbf{s}^k \odot (\mathbf{1} - \boldsymbol{\alpha}^k) - A^k \mathbf{1} - B\mathbf{u}^k,$$

where $\mathbf{g} \in \mathbb{R}^N$ represents the net radiative imbalance in W/m^2 at each latitude. This term collects the local radiative contributions to the energy balance, while the remaining transport contribution is represented separately through the diffusive model in
280 (10) and (11). At time t_k , we write the Crank-Nicolson scheme in the format of the governing equation in (9):

$$\text{diag}(\mathbf{c}^k) \frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} = \frac{1}{2}(\boldsymbol{\omega}^{k+1} + \boldsymbol{\omega}^k + \mathbf{g}^{k+1} + \mathbf{g}^k),$$

where the averaged sum of radiative imbalance and diffusion between t_k and t_{k+1} represents a trapezoidal approximation of the EBM solution over the time interval $[t_k, t_{k+1}]$. These changes are evaluated using information from both the current and next time levels. The terms are then rearranged to express the update for $\mathbf{u}^{k+1}$:

$$285 \quad \mathbf{u}^{k+1} = \mathbf{u}^k + \frac{\Delta t}{2} \text{diag}(\mathbf{c}^k)^{-1} (\boldsymbol{\omega}^{k+1} + \boldsymbol{\omega}^k + \mathbf{g}^{k+1} + \mathbf{g}^k). \quad (14)$$

The above formulation implicitly evaluates $\mathbf{u}^{k+1}$ using fixed-point iteration. The methods of determination for each component comprising (14) and its supporting parameterizations are summarized in Appendix D (Tab. D1).

2.2.5 Empirical Temperature Prediction

The prediction of $\mathbf{f}_u$ proposed in (9) and numerically integrated in (14) comprises this paper’s physical approach to temperature modeling. An alternative problem arises in the empirical modeling of the system’s temperature evolution, comparing its
290 predictive capabilities and limitations relative to those of (9). We consider a high-level overview of temperature prediction to inform later frameworks in generalized kernel solutions and albedo prediction.

Let $\hat{\mathbf{u}}(t_k) \in \mathbb{R}^N$ denote the discrete-time predictor of the rate of change in temperature corresponding to the k -th time interval. If this rate of change is known, we may write the numerical time progression of the system as

$$295 \quad \mathbf{u}^{k+1} = \mathbf{u}^k + \Delta t \hat{\mathbf{u}}(t_k). \quad (15)$$

Given a time step Δt , the temperature derivative is approximated by the discrete difference between successive states. In (15), we use a Forward Euler update, motivated by its use in neural-network-based dynamical models and its computational simplicity relative to higher-order schemes such as Runge-Kutta methods Long et al. (2018); Zhang et al. (2019). The output of interest is the rate of change in temperature, learned from real-world temperatures and the associated input data. From the
300 supporting parameterizations of the EBM presented in (9), we consider a regression model that accepts inputs of temperature, albedo, and insolation at t_k . For brevity, we omit the inclusion of CO_2 , opting for a short-term simulation horizon. The proposed regression may be trained on the same data set that informs the tuned-parameter evaluations summarized in Appendix D (Tab. D1).



Given the numerically integrable forms of $\mathbf{f}_u$ in (14) and (15), our focus now turns to the model forms which inform $\hat{\mathbf{u}}$ and the corresponding model identification of $\mathbf{f}_\alpha$. We begin by introducing a Gaussian kernel, a model which accepts latitudinal observations of temperature, albedo, and insolation, informing our estimates of the rate of change in albedo over time. This introduction is intended to familiarize the reader with the notation and methodology that inform our prediction of $\mathbf{f}_\alpha$ and additional regression-based models of $\mathbf{f}_u$.

2.3 Gaussian Kernel Ridge Regression

310

This paper uses Gaussian kernel ridge regression (KRR) to approximate functions $\mathbf{f}_u$ and $\mathbf{f}_\alpha$ from observed climate variables. In particular, we use KRR to predict the rate of change in temperature or albedo from the local radiative and thermodynamic state. Ridge regression is a regularized regression method that balances agreement with the training data against a penalty on model complexity, reducing sensitivity to noise and limiting overfitting Hoerl and Kennard (1970). KRR extends this idea to nonlinear regression by using a kernel function to compare input states, rather than requiring an explicit parametric form for the unknown function Schölkopf and Smola (2002).

315

2.3.1 Formulation of Gaussian KRR

The objective of this subsection is to define a unique latitudinal approximation for a generic output. Let $\mathbf{y} \in \{\mathbf{u}, \alpha\}$ denote either temperature or albedo, and let $\dot{\mathbf{y}}$ denote the corresponding rate of change to be learned. This rate of change at the k -th time step is computed as

320

$$\dot{\mathbf{y}}(t_k) = \frac{\mathbf{y}^{k+1} - \mathbf{y}^k}{\Delta t}, \quad (16)$$

where $\mathbf{y}^k \in \mathbb{R}^N$ is the output vector evaluated across N latitudes and Δt is the time step. Collecting all $\dot{\mathbf{y}}(t_k)$ over $M - 1$ time intervals gives $\dot{\mathbf{Y}} \in \mathbb{R}^{N \times (M-1)}$, whose i -th row contains the training targets for latitude φ_i .

Regression is performed independently at each latitude band. For latitude φ_i and time step k , the input feature vector is defined as

325

$$\mathbf{v}_i^k = \begin{bmatrix} u_i^k & \alpha_i^k & s_i^k \end{bmatrix}^\top \in \mathbb{R}^3, \quad (17)$$

where u_i^k , α_i^k , and s_i^k denote temperature, albedo, and insolation, respectively. These inputs describe the local climate state used to predict the selected output, $\dot{\mathbf{y}}$. Before evaluating the kernel, each feature is standardized using the temporal mean and standard deviation at the corresponding latitude:

$$\mathbf{z}_i^k = \text{diag}(\tilde{\mathbf{v}}_i)^{-1} (\mathbf{v}_i^k - \bar{\mathbf{v}}_i), \quad (18)$$

330

where $\bar{\mathbf{v}}_i \in \mathbb{R}^3$ and $\tilde{\mathbf{v}}_i \in \mathbb{R}^3$ are the respective vectors of the feature-wise temporal mean and standard deviation at φ_i . This normalization places temperature, albedo, and insolation on comparable scales before computing Euclidean distances in feature space.



The similarity between two normalized input states, $\mathbf{z}_i^m$ and $\mathbf{z}_i^n$ for $m, n \in \{1, 2, \dots, M-1\}$, is measured using the Gaussian
335 radial basis function (RBF) kernel. The kernel function and corresponding kernel matrix at φ_i are defined by

$$\kappa(\mathbf{z}_i^m, \mathbf{z}_i^n) = \exp(-\gamma \|\mathbf{z}_i^m - \mathbf{z}_i^n\|^2), \quad (19a)$$

$$\mathbf{K}_{mn}(\varphi_i) = \kappa(\mathbf{z}_i^m, \mathbf{z}_i^n), \quad (19b)$$

where, $\gamma > 0$ is the kernel bandwidth parameter. This isotropic kernel depends only on the Euclidean distance between two
normalized input vectors, assigning greater similarity to nearby climate states and smaller similarity to more distant states. The
340 resulting kernel matrix $\mathbf{K}_i \in \mathbb{R}^{(M-1) \times (M-1)}$ is symmetric. Let

$$\dot{\mathbf{y}}_i = \begin{bmatrix} \dot{y}_i(t_1) & \dot{y}_i(t_2) & \cdots & \dot{y}_i(t_{M-1}) \end{bmatrix}^\top \in \mathbb{R}^{M-1}$$

be the vector of observed finite-difference targets for the selected output variable at latitude φ_i . The KRR coefficients are
obtained by solving

$$\boldsymbol{\beta}_i = (\mathbf{K}_i + \lambda \mathbf{I})^{-1} \dot{\mathbf{y}}_i, \quad (20)$$

345 where $\lambda > 0$ is the ridge regularization parameter and $\mathbf{I} \in \mathbb{R}^{(M-1) \times (M-1)}$ is the identity matrix, and $\boldsymbol{\beta}_i \in \mathbb{R}^{M-1}$. Given a new
input state $\mathbf{v}_i^*$, we apply the same normalization to obtain $\mathbf{z}_i^*$. The prediction of the output at latitude φ_i is then

$$\hat{y}_i(\mathbf{v}_i^*) = \mathbf{k}_i(\mathbf{z}_i^*)^\top \boldsymbol{\beta}_i, \quad (21)$$

where $\mathbf{k}_i(\mathbf{z}_i^*) \in \mathbb{R}^{M-1}$ is the vector of kernel evaluations between the new normalized input state and each training state at
latitude φ_i , and $\hat{y}_i(\mathbf{v}_i^*)$ is the scalar KRR predictor of $\dot{y}_i$ given $\mathbf{v}_i^*$. The same construction is used for both $\dot{\mathbf{u}}$ and $\dot{\alpha}$, with
350 separate target vectors but a common set of input features. For brevity in tuning, the hyperparameters γ and λ are taken to be
constant across latitude bands.

2.3.2 Spatially Naïve Gaussian KRR Solution

The proposed kernel regression may be numerically integrated via Forward Euler, following the definition of $\dot{\mathbf{y}}_k$ in (16). This
integration from the state at $\mathbf{y}^k$ to that of $\mathbf{y}^{k+1}$ is evaluated using a vector $\hat{\mathbf{y}}_{\text{raw}}(t_k) \in \mathbb{R}^N$ whose entries are calculated in (21).
355 The Forward Euler solution is

$$\hat{\mathbf{y}}_{\text{raw}}(t_k) = \begin{bmatrix} \hat{y}_1(\mathbf{v}_1^k) & \cdots & \hat{y}_N(\mathbf{v}_N^k) \end{bmatrix}^\top, \quad (22a)$$

$$\mathbf{y}^{k+1} = \mathbf{y}^k + \Delta t \hat{\mathbf{y}}_{\text{raw}}(t_k), \quad (22b)$$

where all entries of $\mathbf{y}^k$ are bounded between a constant floor parameter, $y_{\min}$, and a constant ceiling parameter, $y_{\max}$. We note
that whereas the physical EBM relies on the heat transport framework defined in (10)–(13) to make temperatures consistent
360 between latitudes, the proposed KRR at φ_i has no knowledge of its neighboring predictions, assuming a model form which is
naïve to its surrounding states. We refer to the numerical solution of $\mathbf{y}^{k+1}$ in (22) as the naïve KRR solution.



2.3.3 Spatially Informed Gaussian KRR

In a dynamical systems setting, Song et al. (2025) improves KRR by incorporating the intrinsic geometry of the data through a diffusion maps kernel, allowing the regression model to account for nonuniform sampling of input states. Motivated by this idea, we define a simpler spatially informed Gaussian KRR in which each latitude-wise predictor is given access to neighboring climate states. For latitude φ_i , the informed input vector is defined as

$$\mathbf{v}_{\text{inf},i}^k = \begin{cases} [u_i^k, u_{i+1}^k, \alpha_i^k, \alpha_{i+1}^k, s_i^k]^\top, & i=1, \\ [u_{i-1}^k, u_i^k, \alpha_{i-1}^k, \alpha_i^k, s_i^k]^\top, & i=N, \\ [u_{i-1}^k, u_i^k, u_{i+1}^k, \alpha_{i-1}^k, \alpha_i^k, \alpha_{i+1}^k, s_i^k]^\top, & \text{else.} \end{cases} \quad (23)$$

Thus, $\mathbf{v}_{\text{inf},i}^k \in \mathbb{R}^7$ for interior latitudes, while the boundary inputs satisfy $\mathbf{v}_{\text{inf},1}^k, \mathbf{v}_{\text{inf},N}^k \in \mathbb{R}^5$. We implement the spatially augmented input vector in (23) as compared to (17). Building upon $\hat{\mathbf{y}}_{\text{raw}}(t_k)$ in (22), we apply a Laplacian smoothing step across latitudes:

$$\hat{\mathbf{y}}_{\text{inf}}(t_k) = (\mathbf{I} + \ell \mathbf{L}_s)^{-1} \begin{bmatrix} \hat{y}_1(\mathbf{v}_{\text{inf},1}^k) \\ \vdots \\ \hat{y}_1(\mathbf{v}_{\text{inf},N}^k) \end{bmatrix} \quad (24)$$

where $\ell \geq 0$ controls the strength of the spatial smoothing, $\mathbf{L}_s \in \mathbb{R}^{N \times N}$, and $\mathbf{I} \in \mathbb{R}^{N \times N}$. The tridiagonal matrix $\mathbf{L}_s$ is defined:

$$\mathbf{L}_s = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

This smoothing step is motivated by the tridiagonal diffusion operator used in the physical EBM in (10). From (24), the informed KRR solution can then be written using Forward Euler:

$$\mathbf{y}^{k+1} = \mathbf{y}^k + \Delta t \hat{\mathbf{y}}_{\text{inf}}(t_k). \quad (25)$$

From this solution, we may compare the advantage of spatial information gained relative to the formulation in (22).

2.3.4 Kernel Model Reduction

The spatially naïve and spatially informed KRR predictors defined in (22) and (25), respectively, assign an individual Gaussian KRR model to each latitude. For a one-dimensional latitudinal EBM, this latitude-wise construction may be computationally



feasible. However, the marginal improvement relative to simpler predictors may not justify the cost when the spatial discretization is refined, or when the approach is extended to two-dimensional global grids parameterized by $\mathbf{r}$. To reduce the number of kernel models while retaining latitude-dependent albedo states, we introduce a regional KRR approximation.

We define a region to be a collection of latitudes. Let $a(i)$ represent the function that maps the index of $i \in \{1, \dots, N\}$ to its corresponding regional index $a \in \{1, \dots, N_r\}$, where $N_r \leq N$. In the regional mesh, we define the number of latitudes comprising one region to be the same as that of its neighbors. We train one predictor for each region by uniformly averaging the latitude-wise observations within that region at each time step, producing one temporal series of regional feature vectors, $\{\mathbf{v}_{a(i)}^k\}_{k=1}^{M-1}$. This averaging preserves the temporal kernel matrix size, $\mathbf{K}_i \in \mathbb{R}^{(M-1) \times (M-1)}$, while reducing the number of kernel models that must be trained from N to N_r . Latitudes in the same region share the same learned kernel model, but each latitude supplies its own local input vector. The notation $\hat{y}_{a(i)}(\mathbf{v}_i^k)$ denotes the output-rate prediction obtained by evaluating the regional kernel predictor indexed by $a(i)$ using the latitude-specific feature vector $\mathbf{v}_i^k$. The rate of change produced by the regional KRR solution is as follows:

$$\begin{aligned}\hat{\mathbf{y}}_{\text{raw}}(t_k) &= \left[\hat{y}_{a(1)}(\mathbf{v}_1^k) \cdots \hat{y}_{a(N)}(\mathbf{v}_N^k) \right]^\top. \\ \hat{\mathbf{y}}_{\text{inf}}(t_k) &= (\mathbf{I} + \ell \mathbf{L}_s)^{-1} \left[\hat{y}_{a(1)}(\mathbf{v}_{\text{inf},1}^k) \cdots \hat{y}_{a(N)}(\mathbf{v}_{\text{inf},N}^k) \right]^\top.\end{aligned}$$

Here, the regional solutions of $\hat{\mathbf{y}}_{\text{raw}}$ and $\hat{\mathbf{y}}_{\text{inf}}$, may be directly applied to the Forward Euler schemes in (22) and (25), respectively. When $N_r = N$, the reduced model recovers the original latitude-wise KRR formulation. We vary the selection of N_r to assess whether latitude-wise kernels provide significant improvement in prediction over coarser regional approximations.

That being said, Algorithm 1 defines our approach to the training and prediction of a predictor, $\hat{\mathbf{y}}(t_k)$, which approximates $\mathbf{f}_u$ and $\mathbf{f}_\alpha$ at t_k . This algorithm incorporates all steps of the procedure, considering a general regional reduction of $N_r \leq N$. Provided training data $\{\mathbf{u}^k, \alpha^k, \mathbf{s}^k\}$ and known parameterizations of Δt , γ , λ , and ℓ , we may train N_r kernels. These kernels differentiate procedures depending upon the naïve and informed cases, defining the different input samples comprising a regional model, $\hat{y}_{a(i)}$. From these regional models, we may calculate individual latitudinal predictions. We then concatenate these predictions by latitude to form $\hat{\mathbf{y}}_{\text{raw}}$ and $\hat{\mathbf{y}}_{\text{inf}}$, comprising the numerically integrable predictors of albedo in (22) and (25), respectively.

Having defined all assumptions in our approximations of $\mathbf{y}^{k+1}$ using Gaussian KRR, we now focus our methodology on two regression-based predictors of $\mathbf{f}_\alpha$. Once these models and their corresponding assumptions are defined, we can enumerate a selected set of benchmark functions from the literature to compare their performance.

2.4 Model Identification of Albedo Dynamics

We now consider candidate regression models for the rate of change of albedo, represented by the state-space function $\mathbf{f}_\alpha$. The objective of this section is to approximate how α evolves from local climate information, using temperature, albedo, and insolation as predictors. Temperature provides information about snow, ice, and other thermodynamic conditions that influence reflectivity, while insolation provides a seasonal radiative forcing signal at each latitude. The current albedo state is also included as a predictor, allowing the model to account for persistence in surface and atmospheric reflectivity. These variables



Algorithm 1: Regional kernel reduction approach to Gaussian KRR for state-space models $\mathbf{f}_\alpha$ and $\mathbf{f}_u$ in (8).

Input: Training data $\{\mathbf{u}^k, \boldsymbol{\alpha}^k, \mathbf{s}^k\}_{k=1}^M$, time step Δt , kernel bandwidth $\gamma > 0$, regularization parameter $\lambda > 0$, spatial awareness mode $\in \{\text{naïve}, \text{informed}\}$, smoothing parameter $\ell \geq 0$, regional reduction N_r , and region map $a(i) \in \{1, \dots, N_r\}$.

Output: Gaussian KRR prediction vector $\hat{\mathbf{y}}$ for $\mathbf{y} \in \{\mathbf{u}, \boldsymbol{\alpha}\}$.

Select target output $\mathbf{y} \in \{\mathbf{u}, \boldsymbol{\alpha}\}$.

Compute observed finite differences $\dot{\mathbf{y}}(t_k)$ using (16).

Training:

for $j = 1, \dots, N_r$ **do**

if mode = *naïve* **then**

 Assemble latitudinal input samples using (17).

 Average inputs over region j to obtain $\{\mathbf{v}_j^k\}_{k=1}^{M-1}$.

else

 Assemble latitudinal input samples using (23).

 Average inputs over region j to obtain $\{\mathbf{v}_j^k\}_{k=1}^{M-1}$.

 Average all $\dot{\mathbf{y}}_i$ satisfying $a(i) = j$ to obtain $\dot{\mathbf{y}}_j$.

 Compute feature-wise training statistics $\bar{\mathbf{v}}_j$ and $\tilde{\mathbf{v}}_j$.

 Standardize inputs to obtain $\{\mathbf{z}_j^k\}_{k=1}^{M-1}$ using (18).

 Form the regional kernel matrix $\mathbf{K}_j$ using (19).

 Solve for $\boldsymbol{\beta}_j$ using (20).

 Store regional model $\{\bar{\mathbf{v}}_j, \tilde{\mathbf{v}}_j, \boldsymbol{\beta}_j, \{\mathbf{z}_j^k\}_{k=1}^{M-1}\}$.

Prediction:

for $i = 1, \dots, N$ **do**

if mode = *naïve* **then**

 Assemble new input state $\mathbf{v}_i^k$ using (17).

else

 Assemble new input state $\mathbf{v}_{\text{inf},i}^k$ using (23).

 Standardize $\mathbf{v}_i^*$ to obtain $\mathbf{z}_i^*$ using $\bar{\mathbf{v}}_{a(i)}$ and $\tilde{\mathbf{v}}_{a(i)}$.

 Form $\mathbf{k}_{a(i)}(\mathbf{z}_i^*)$ using the stored regional model.

 Predict the raw local rate of change using (21).

if mode = *informed* **then**

 Concatenate all local predictors by latitude.

 Smooth local predictions using (24), $\hat{\mathbf{y}}_{\text{inf}}(t_k)$

 Set $\hat{\mathbf{y}} \leftarrow \hat{\mathbf{y}}_{\text{inf}}$.

else

 Concatenate the predictions by latitude in (22), $\hat{\mathbf{y}}_{\text{raw}}(t_k)$.

 Set $\hat{\mathbf{y}} \leftarrow \hat{\mathbf{y}}_{\text{raw}}$.

return $\hat{\mathbf{y}}$



define a reduced-order representation of albedo dynamics that can be fit independently at each latitude. We propose a host
415 of regression models featuring their own individual advantages and limitations. One type of predictor considers polynomial
regression, opting for both (i) a univariate sum of polynomials and (ii) a general polynomial which couples the interactions
between variables across multiple orders.

2.4.1 Polynomial Predictors

We first consider polynomial models for the local rate of change of albedo. This model class is advantageous because of its
420 interpretability and its computational scalability as a simple regressor. For each latitude φ_i , the regression target is the time
series

$$\dot{\alpha}_i = \left[\dot{\alpha}_i(t_1) \quad \dot{\alpha}_i(t_2) \quad \cdots \quad \dot{\alpha}_i(t_{M-1}) \right]^\top \in \mathbb{R}^{M-1},$$

where $\dot{\alpha}_i(t_k)$ is computed from the observed albedo sequence. The predictors at latitude φ_i are the local albedo α_i , temperature
 u_i , and insolation s_i . We compare two polynomial representations. The first is a full interaction, or cross-term (CT), polynomial,
425 defined as

$$\text{CT: } \hat{\alpha}_i = \sum_{p=0}^P \sum_{q=0}^Q \sum_{r=0}^R \rho_{i,pqr} \alpha_i^p u_i^q s_i^r, \quad (27)$$

where ρ is the coefficient of ordinary least squares regression, and P , Q , and R are the maximum degrees assigned to albedo,
temperature, and insolation, respectively. The formulation in (27) includes pure polynomial terms, pairwise interactions, and
higher-order interactions among the three predictors, providing a flexible representation of the local albedo dynamics. However,
430 as proposed, each latitude has $(P+1)(Q+1)(R+1)$ unique regression coefficients, motivating a lower-dimensional alternative.
We propose the univariate polynomial (UP) sum to be

$$\text{UP: } \hat{\alpha}_i = \rho_{i,0} + \sum_{p=1}^P \rho_{i,p}^{(\alpha)} \alpha_i^p + \sum_{q=1}^Q \rho_{i,q}^{(u)} u_i^q + \sum_{r=1}^R \rho_{i,r}^{(s)} s_i^r. \quad (28)$$

Unlike (27), this model excludes cross terms between albedo, temperature, and insolation. The additive form is less flexible, but
it reduces the number of fitted coefficients to $P+Q+R+1$. This provides a simpler baseline for evaluating whether interaction
terms improve predictions of the albedo update. Both polynomial forms are fit independently at each latitude. Let

$$\Phi_i = \left[\phi_i^1 \quad \cdots \quad \phi_i^{M-1} \right] \in \mathbb{R}^{b \times (M-1)},$$

denote the design matrix formed by evaluating the selected polynomial basis b across the $M-1$ training times, where $\phi_i^k \in \mathbb{R}^b$
represents the selected basis vector at time t_k . Next, let ρ_i denote the corresponding coefficient vector. The local regression
435 problem can be written as

$$\begin{aligned} \dot{\alpha}_i &= \Phi_i^\top \rho_i, \\ \rho_i^* &= \arg \min_{\rho_i} \left\| \dot{\alpha}_i - \Phi_i^\top \rho_i \right\|^2, \end{aligned}$$



where $\rho_i^* \in \mathbb{R}^b$ is the local vector of optimal regression coefficients. For the full interaction model in (27), $b = (P + 1)(Q + 1)(R + 1)$.

For the additive model in (28), $b = P + Q + R + 1$. In either case, the corresponding polynomial basis evaluations are collected
440 in ϕ_i^k , so that the predicted albedo rate of change at latitude φ_i and time t_k is written compactly as

$$\hat{\alpha}_i(t_k) = (\rho_i^*)^\top \phi_i^k.$$

Concatenating all $\hat{\alpha}_i(t_k)$ by latitude, the albedo state is then advanced using a Forward Euler update,

Polynomial: $\alpha^{k+1} = \alpha^k + \Delta t \left[(\rho_1^*)^\top \phi_1^k \quad \dots \quad (\rho_N^*)^\top \phi_N^k \right]^\top.$ (30)

The proposed polynomial regressions summarize the first category of empirical models under consideration.

445 2.4.2 Gaussian KRR Predictors

We next specify the Gaussian KRR formulation derived in Section 2.3 to the dynamics of albedo by setting $\mathbf{y} = \alpha$. For each latitude φ_i , the target vector is the observed rate of change of albedo, $\dot{\alpha}_i$, computed according to (16). The Gaussian KRR predictor then approximates the local albedo dynamics by mapping the local inputs of temperature, albedo, and insolation to a scalar predictor, $\hat{\alpha}_i$ using the kernel matrix $\mathbf{K}_i$ in (19b) and the coefficient solution, β_i , in (20). The regularization
450 parameter λ in (20) penalizes model complexity through the ridge term $\lambda \mathbf{I}$, reducing sensitivity to noisy observations and limiting overfitting.

Two KRR input structures are considered. In the spatially naïve case, the input at latitude φ_i is the local state vector in (17), so the predictor depends only on α_i^k , u_i^k , and s_i^k . We define the general form of the raw KRR prediction of the albedo update:

$$\hat{\alpha}_{\text{raw}}(t_k) = \begin{bmatrix} \hat{\alpha}_{a(1)}(\mathbf{v}_1^k) \\ \vdots \\ \hat{\alpha}_{a(N)}(\mathbf{v}_N^k) \end{bmatrix}.$$

455 In the spatially informed case, the input vector is replaced by the augmented state in (23), allowing the kernel evaluation to depend on neighboring states of temperature and albedo. This informed counterpart implements an initial prediction and subsequent smoothing as in (24):

$$\hat{\alpha}_{\text{inf}}(t_k) = (\mathbf{I} + \ell \mathbf{L}_s)^{-1} \begin{bmatrix} \hat{\alpha}_{a(1)}(\mathbf{v}_{\text{inf},1}^k) \\ \vdots \\ \hat{\alpha}_{a(N)}(\mathbf{v}_{\text{inf},N}^k) \end{bmatrix}.$$

The state is then advanced with the Forward Euler update:

460 **Naïve KRR:** $\alpha^{k+1} = \alpha^k + \Delta t \hat{\alpha}_{\text{raw}}(t_k),$ (31a)

Informed KRR $\alpha^{k+1} = \alpha^k + \Delta t \hat{\alpha}_{\text{inf}}(t_k),$ (31b)

where (31a) and (31b) are the respective numerical integration schemes of the naïve and informed cases of albedo prediction.



2.4.3 Treatment of Polar Night

A complication in fitting the proposed albedo models is the presence of polar night. At sufficiently high latitudes, there are periods of the year when no incoming shortwave radiation reaches the surface. This creates missing entries in observations of albedo that must be treated before constructing any regression targets or design matrices. We divide the handling of polar night into two problems: (i) data selection and (ii) model progression in these periods.

In the literature, observations of polar albedo are often restricted to months or periods for which sufficient sunlight is available Pistone et al. (2014), Möller and Möller (2017). This restriction allows albedo fields and associated metrics to be constructed from physically interpretable observations. Motivated by this practice, we define a validity mask that removes polar night observations before fitting the regression model. Let $\mathcal{M}_i$ denote the set of valid training indices at latitude φ_i , defined by the availability of incoming shortwave radiation and valid albedo observations. The corresponding design matrix and target vector are then restricted to these valid times: $\Phi_i \in \mathbb{R}^{M'_i \times b}$ and $\dot{\alpha}_i \in \mathbb{R}^{M'_i}$ where $M'_i = |\mathcal{M}_i|$ is the number of valid training samples at latitude φ_i . Thus, each latitudinal predictor is trained only on physically defined albedo observations, although the number of training samples becomes latitude dependent. The validity mask must also be consistent with the finite-difference target used to approximate the albedo rate. For example, if $\dot{\alpha}_i(t_k)$ is computed using a forward difference between t_k and t_{k+1} , then both α_i^k and α_i^{k+1} must be valid for the sample to be included. This prevents the regression model from being trained on albedo changes involving undefined or unsupported observations.

During simulation, however, the coupled state-space model must still be advanced through polar night intervals. This creates a support mismatch: the learned albedo-rate model is trained only on sunlit, observationally supported samples, but the numerical model evolves over the full latitude-time grid. Similar issues arise in two-stage machine-learning approaches for zero-inflated or regime-dependent data, where a first stage identifies the active or valid regime and a second stage performs prediction within that regime Rožanec et al. (2025). In the present setting, the first stage is not a learned classifier, because the orbital conditions in (3) directly identify whether $s_i^k = 0$ at each latitude and time. We apply the learned model only when insolation is nonzero.

When $s_i^k = 0$, the learned predictor is temporarily bypassed and the albedo state is assigned a prescribed polar night value $\alpha_{PN} \in [0, 1]$. The value α_{PN} is therefore a numerical background value rather than an observed albedo measurement. It is chosen to be representative of polar albedo values and is used only to propagate the coupled model through intervals where shortwave-based albedo observations are unsupported. Once the grid point returns to the sunlit regime, the learned albedo-rate model is again evaluated using the updated state variables. Having summarized the candidate predictors of $\mathbf{f}_\alpha$ and their challenges, we now propose accepted parameterizations of albedo in the literature. These explicit models will serve as benchmarks for the performance of the proposed regression models when jointly integrated with the physical EBM in (14).

2.4.4 Benchmark Functions

This paper considers two functions from which to benchmark the proposed models of albedo: (i) a constant albedo field as implemented in Zhuang et al. (2017) and Ziegler and Rehfeld (2021) and (ii) a hyperbolic tangent approximation as implemented



in Del Sarto et al. (2024). First consider a constant albedo field parameterized by an underlying surface on a discrete grid in $\mathbf{r}$. Following Ziegler and Rehfeld (2021) and Zhuang et al. (2017), we identify four surface types: land ($j = 1$), ocean ($j = 2$), sea ice ($j = 3$), and snow ($j = 4$). Each node on the grid may be identified as one of these four surface types. Let $\vartheta_{i,j}$ represent the fraction of the j -th surface type comprising the latitudinal band φ_i . Then the vector of constant albedo at any time t_k may be represented:

Constant: $\alpha = \left[\sum_{j=1}^4 \vartheta_{1,j} \alpha_{1,j} \cdots \sum_{j=1}^4 \vartheta_{N,j} \alpha_{N,j} \right]^\top$, (32a)

$$\alpha_{i,j} = \begin{cases} 0.32 + 0.05P_2(\varphi_i), & j = 1, \\ 0.29 + 0.08P_2(\varphi_i), & j = 2, \\ 0.6, & j = 3, \\ 0.7, & j = 4, \end{cases} \quad (32b)$$

$$P_2(\varphi_i) = \frac{1}{2} (3 \sin^2 \varphi_i - 1), \quad (32c)$$

where $\sum_{j=1}^4 \vartheta_{i,j} = 1$ at φ_i and $P_2(\varphi_i)$ is the second-order Legendre polynomial evaluated at φ_i . The constant case as depicted in (32) is advantageous for short-term model simulations under the assumption of equilibrium, avoiding the difficulty of polar night periods and additional computational efforts from the proposed dynamical albedo models. For longer simulation horizons, however, this model lacks the positive feedback that amplifies warming as albedo is lost. This limitation will allow us to evaluate both the extent to which this lack of amplifying feedback affects temperature predictions and the extent to which the regression-based updates are able to capture this change.

Next consider the hyperbolic tangent function of the existing albedo state:

tanh: $\alpha^k = \alpha_H + \left(\frac{\alpha_L - \alpha_H}{2} \right) \odot (1 + \tanh(w(\mathbf{u}^k - u_{\text{ref}} \mathbf{1})))$, (33)

where $\alpha_H \in \mathbb{R}^N$ is the vector of high albedo values, $\alpha_L \in \mathbb{R}^N$ is the vector of low albedo values, w is the slope parameter in K^{-1} , and u_{ref} is the globally uniform reference temperature in K. The proposed dependence on temperature is formed from an assumption of cryospheric decline resulting from some increase to temperature North and Kim (2017). As applies to the existing EBM in (14), this explicit formulation allows for latitudinal temperature $\mathbf{u}^k$ to explicitly model the corresponding latitudinal albedo α^k without the requirement of joint numerical integration.

In Del Sarto et al. (2024), the parameterizations of α_H and α_L are assumed to be uniform across all latitudes. For this paper, however, we define these vectors with knowledge of the underlying surface composition and existing data. From real-world observations, α_H is defined as the maximum observed albedo at each latitude. From the surface fractions implemented in (32), we define α_L as an elementwise weighted average:

$$\alpha_L = \left[\alpha_L(\varphi_1) \cdots \alpha_L(\varphi_N) \right]^\top, \quad (34a)$$

$$\alpha_L(\varphi_i) = (\vartheta_{i,1} + \vartheta_{i,4})\alpha_{\text{land}} + (\vartheta_{i,2} + \vartheta_{i,3})\alpha_{\text{ocean}}, \quad (34b)$$

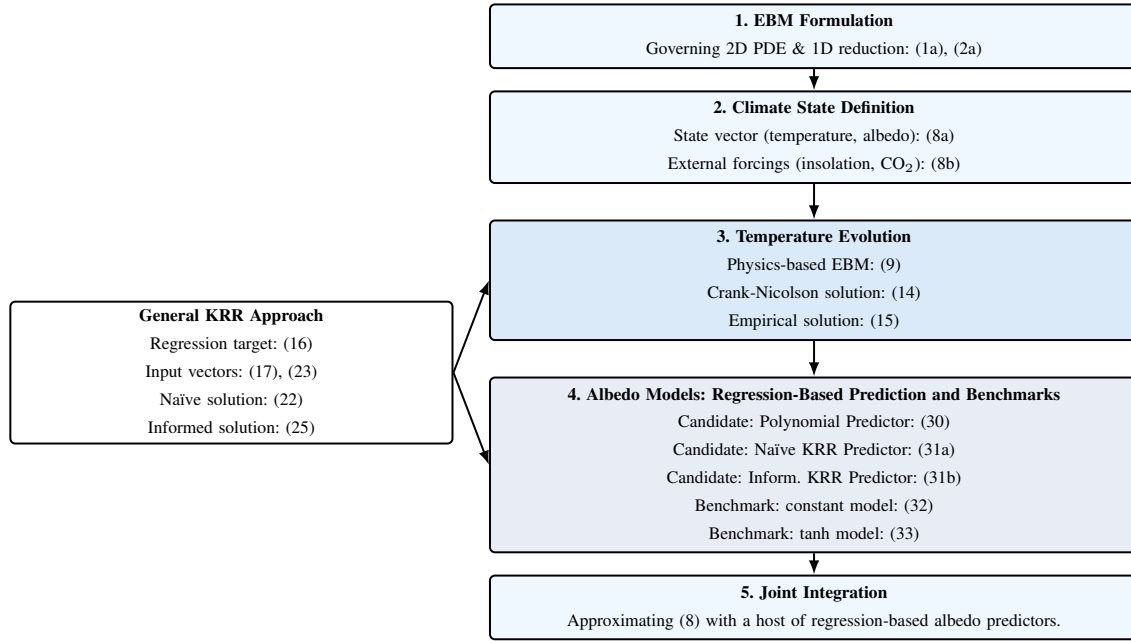


Figure 3. Methodological summary of Section 2. The paper begins from the EBM PDE, recasts the model in state-space form, defining the physical and observation-informed updates for state variables of temperature and albedo.

where α_{land} and α_{ocean} are fixed parameters representing land and ocean albedo. The calculation of α_{H} assumes no cryospheric cover, summing land and ocean surface fractions with those of snow and sea ice, respectively. The proposed forms of α_{L} and α_{H} allow the explicit model in (33) to operate within feasible albedo boundaries at each latitude. In contrast, neither the polynomial nor KRR models utilize the latitudinal boundaries defined by α_{L} and α_{H} . For the proposed regression models to predict comparably to this benchmark, they must also update albedo within this feasible range.

Having defined the benchmark albedo parameterizations and the proposed regression-based approximations of f_{α} , we now have a complete state-space formulation for the dynamical system in (8). In this formulation, f_u is specified by the discretized EBM, while f_{α} is represented by candidate models whose predicted rate of change in albedo, $\hat{\alpha}$, may be embedded within the coupled dynamical system. Fig. 3 summarizes the proposed methodology, including the construction of the physics-based temperature model, the general form of the Gaussian kernel, and the evaluation of the resulting coupled state-space approximations. We now implement a series of case studies designed to assess the functionality of the EBM framework and the performance of the proposed albedo predictors relative to specified benchmarks.



535 3 Case Studies

In understanding the foundation of EBMs and the formulations which constitute their numerical solutions, this paper proposes three case studies which investigate the performance of EBMs and the candidate forms of a dynamical albedo model. These case studies answer the following high level research questions:

540 **Q1.** How does the temporal resolution of an EBM affect calibrated model parameters, inferred heat transport, and agreement with observed latitudinal temperature evolution? (Section 3.2)

Q2. To what extent can albedo be predicted from real-world observations, and how do the proposed formulations compare against observed albedo fields and established EBM albedo parameterizations? (Section 3.3)

Q3. How does spatial information influence learned climate-state predictions? Does the inclusion of neighboring states increase a model’s predictive accuracy? (Sections 3.3, 3.4)

545 **Q4.** How do prediction errors in one model influence errors in the other? (Sections 3.3, 3.4)

We first consider the tuning problem, evaluating the relevant parameterizations in Appendix D (Tabs. D1 and D2) across different temporal resolutions. This allows us to evaluate the extent to which the tuned solutions and the diffusive behavior of the EBM change between time steps. Once the model is tuned, we select one time step from which to progress the dynamical system and couple the individual albedo models outlined in Section 2.4 to the EBM. This allows us to evaluate the limitations of individual models, noting whether prediction errors are largely spatial or seasonal. After evaluating the coupled performance of a physical EBM and candidate albedo models, we opt for a regression-based predictor of temperature as presented in (15). This predictor is coupled with the best-performing albedo models, evaluating their coupled predictive capabilities and potential drawbacks given additional information. We begin by defining the sources of data and computational resources which support our case studies.

555 3.1 Experimental Setup

We acquire surface maps, radiative data, and temperature data from ERA5’s “Hourly data on single levels from 1940 to present” reanalysis Hersbach et al. (2023). This data set defines spatial resolutions of $0.25^\circ \times 0.25^\circ$ with hourly temporal resolutions. We collect hourly data from 1 January to 31 December for years 2001-2005, 2010-2011, and 2023. Lower temporal resolution data are formed from uniformly averaging hourly data into one-day, three-day, and six-day intervals. Data are averaged uniformly across longitudes for the proposed 1D solutions. Atmospheric CO_2 concentrations are assumed to be globally uniform as in (4) and (9). Concentrations are sourced from Mauna Loa Observatory on daily resolutions in ppm Keeling et al. (1976), Lan and Keeling (2026). All experiments are performed in Python 3.14 on a MacBook Pro with an Apple M3 Pro chip, an 11-core CPU, and 18 GB RAM.

565 This paper uses ERA5 variables *skt*, *sd*, *siconc*, *lsm*, *tisr*, and *tsr*. The skin temperature field, *skt*, evaluates the theoretical temperature (K) required to satisfy the energy budget at the uppermost surface layer. TOA radiative data are quantified by *tisr*, the total incident shortwave radiation, and *tsr*, net (positive) inward shortwave radiation, each provided in J/m^2 and accumulated



over hourly periods. We calculate albedo at any time by $(t_{isr} - t_{sr})/t_{isr}$. Surface data are quantified by sd , $siconc$, and lsm . The snow depth, sd , is defined in meters, identifying the equivalent quantity of snow melt. The sea ice concentration, $siconc$, refers to the fraction of sea ice cover comprising a given node's surrounding area. Similarly, lsm , refers to the fraction of surrounding surface which comprises oceanic or land-based surfaces. ERA5 variable descriptions and applications are summarized in Appendix E (Tab. E1).

We consider two surface categorizations: (i) a constant surface corresponding to the constant albedo benchmark in (32) and (ii) a dynamic surface which varies with sea ice extent. The former model is quantified by the lsm , $siconc$, and sd at 00:00 of 1 January 2001. Using the identifiers in Appendix E (Tab. E2), we can map the underlying surface types and their corresponding albedo with the surface averaging technique depicted in (32). The implementation of this surface does not consider any change to radiative data by t_{isr} nor t_{sr} . The latter dynamic surface informs the hyperbolic tangent benchmark in (34) and a time-varying heat capacity field, c^k , motivated by the real-world seasonal ebb and flow of Earth's sea ice extent. The heat capacity is modeled by

$$c^k = \frac{1}{w_c} \left[\sum_{j=1}^4 \vartheta_{1,j}^k c_j \quad \cdots \quad \sum_{j=1}^4 \vartheta_{N,j}^k c_j \right]^\top, \quad (35)$$

where w_c is the globally uniform heat capacity correction factor determined from tuning, c_j is the heat capacity associated with the j -th surface type sourced from Ziegler and Rehfeld (2021), and $\vartheta_{i,j}^k$ is the fraction of φ_i comprising the j -th surface type at t_k . We maintain lsm and sd as in the constant surface case, but we gather hourly data for $siconc$ from 2001-2005. This seasonal dependence is motivated by the observed seasonal variability of sea ice extent discussed in Thackeray and Hall (2019). For long-term simulations extending beyond 31 December 2005, we prescribe $siconc$ using a repeating cycle of observations starting from 1 January 2001. This limits the amount of required reanalysis data informing the model, but also limits the fidelity of our proposed surface method. Surface identification and heat capacity mapping is performed at each time in the simulation, using the values summarized in Appendix E (Tab. E2).

We define the performance of all predictors of temperature using the root mean square error (RMSE) of temperature across all latitudes. Allow the subscript 'true' in any variable (e.g., $\mathbf{u}_{true}$) to represent the real-world evaluation of that variable. The considered RMSE measures can be written as follows:

$$RMSE(t_k) = \frac{1}{\sqrt{N}} \|\mathbf{u}_{true}^k - \mathbf{u}^k\|, \quad (36a)$$

$$\overline{RMSE} = \frac{1}{M} \sum_{k=1}^M RMSE(t_k), \quad (36b)$$

where (36a) denotes the error metric of the model across latitudes at time t_k and (36b) denotes the temporal mean of the error metric for a selected range of times. We assume that all latitudes may be weighted equally as a ceiling error estimate that gives preference to polar latitudes. Albedo error metrics are given by mean absolute error (MAE) in the same manner:

$$MAE(t_k) = \sum_{i=1}^N \frac{|\alpha_{true,i}^k - \alpha_i^k|}{N}, \quad \overline{MAE} = \sum_{k=1}^M \frac{MAE(t_k)}{M}.$$



Albedo errors are evaluated using MAE as opposed to RMSE because albedo is bounded and often exhibits localized discontinuities associated with transition periods such as those of polar night. Having defined the data and the general metrics for comparison between individual performances by predictors, we now consider the tuning problem for the physical EBM
600 proposed in (14).

3.2 EBM Parameterization and Performance

Appendix D (Tab. D1) summarizes the primary quantities appearing in the physical EBM formulation and identifies whether each quantity is prescribed, parameterized from data, or determined through tuning. Parameters with fixed or observation-based parameterizations are listed in Appendix E (Tab. E3), while the remaining free parameters are calibrated by minimizing the
605 temperature error defined in (36b). In particular, the tunable parameter set consists of $\{A_{\text{ref}}, B, w_c, D_0, \eta_1, \dots, \eta_6\}$, where A_{ref} and B control the linear outgoing longwave radiation response, w_c controls the prescribed surface/heat-capacity weighting, D_0 sets the global magnitude of meridional diffusion, and η_n introduces regional modulation of the diffusion profile in (13). The regional diffusion multipliers are defined over $N_d = 6$ latitude bands with transition locations at $\mu = -0.35, -0.2, 0, 0.34, 0.64$, together with the polar boundaries. The ordering is taken from south to north, so that η_1 corresponds to the southern polar
610 region and η_6 corresponds to the northern polar region. These boundaries are selected to allow hemispheric asymmetry in the effective meridional heat transport while retaining a low-dimensional tuning problem. They also provide separate control over broad polar, mid-latitude, and equatorial regimes, where land-sea distribution, sea ice extent, and seasonal heat transport can produce distinct temperature responses. In total, this problem entails the selection of ten parameters.

We tune parameters with knowledge of the corresponding ranges which they are known to occupy in the literature. As A_{ref}
615 and B are ubiquitous among numerous EBMs, we are able to compare to a larger body of literature. North and Kim (2017), Zhuang et al. (2017), Ziegler and Rehfeld (2021), Chang and Merlis (2023). For parameters such as D_0 and its corresponding regional multipliers, we directly compare our results to those of Ziegler and Rehfeld (2021), evaluating the proposed diffusion coefficient in (13) by latitude and time of year. We perform a simple parameter sweep by changing individual values that minimize $\overline{\text{RMSE}}$ over the 2001-2005 period. While this change occurs, we hold all other parameters constant. For comparison,
620 we also implement a differential evolution, allowing parameters to evolve jointly. The results do not differ significantly between the two methods. In both approaches, we assume albedo varies with ERA5 observations.

3.2.1 Daily Tuning Results

The tuning results are tabulated in Tab. 1 and shown in Fig. 4, including the selected diffusivity parameters, heat-capacity correction, and linearized OLR parameters, A_{ref} and B . The most influential parameter is A_{ref} , which sets the baseline outgoing
625 longwave radiation and therefore controls the net energy surplus or deficit available for meridional heat transport. The tuned value, $A_{\text{ref}} = 180.3 \text{ W/m}^2$, is lower than the values $A_{\text{ref}} = 210.2, 210.3 \text{ W/m}^2$ reported in Zhuang et al. (2017) and Ziegler and Rehfeld (2021), and substantially lower than $A_{\text{ref}} = 281.67 \text{ W/m}^2$ in Chang and Merlis (2023). The same trend holds for B , with the tuned value $B = 1.62 \text{ W/m}^2 \text{ }^\circ\text{C}$ falling below $B = 2.13, 2.15 \text{ W/m}^2 \text{ }^\circ\text{C}$ in Zhuang et al. (2017) and Ziegler and Rehfeld (2021), and below $B = 1.8 \text{ W/m}^2 \text{ }^\circ\text{C}$ in Chang and Merlis (2023). These differences are likely due to the joint

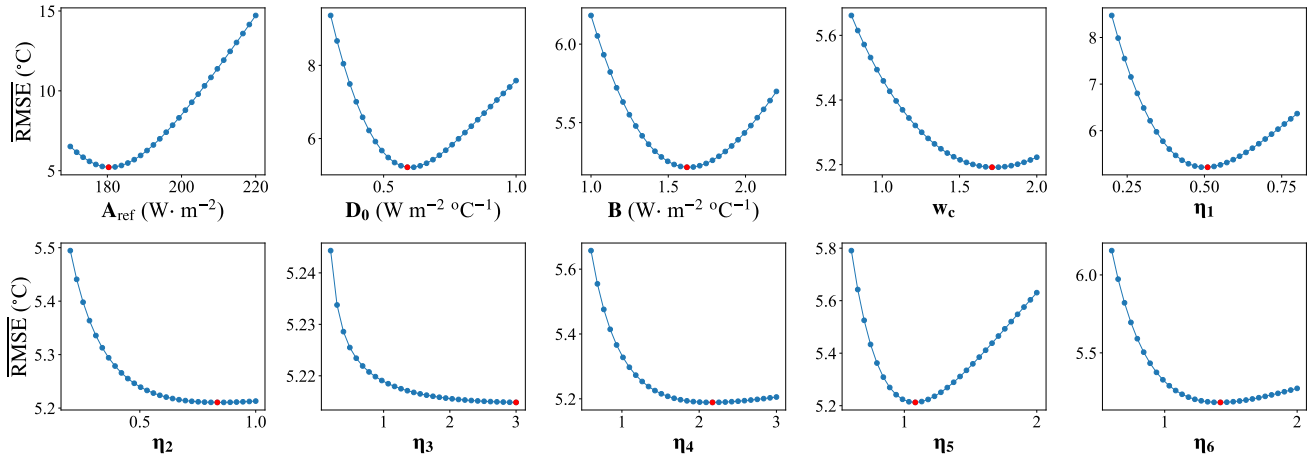


Figure 4. Sensitivity plots for all A_{ref} , B , D_0 , w_c , and all η_n for $N_d = 6$ regions using a daily time step in a 1D EBM. Tuning performed from 1 January 2001 – 31 December 2005. Optimal parameterizations are highlighted in red.

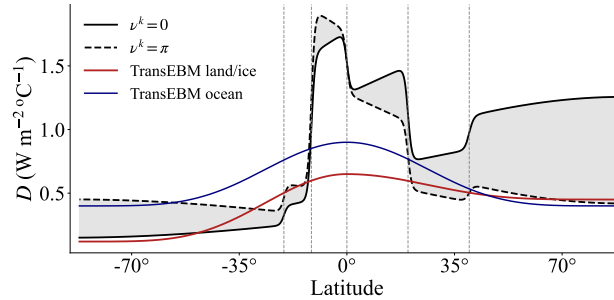


Figure 5. Comparative evaluation of tuned $D(\mu, t_k)$ and the surface-dependent diffusion function as proposed in Ziegler and Rehfeld (2021) in $\text{W m}^{-2} \text{K}^{-1}$. Evaluations of $\nu^k = 0, \pi$ reflect the boundaries which $D(\mu, t_k)$ may occupy based on the latitude and time of year. Dashed gray lines denote regional partitions for each η_n with $n \in \{1, 2, \dots, 6\}$ from left to right.

630 calibration of OLR parameters and the heat-capacity correction w_c , which together determine the model's radiative damping and transient thermal response.

The optimal $w_c = 1.71$ conveys an increase in the damping of $\dot{\mathbf{u}}$ relative to those heat capacities found in Tab. E2 evaluating equivalent heat capacities of $1.56 \text{ MJ}/(\text{m}^2 \text{K})$ for land surfaces, $231.14 \text{ MJ}/(\text{m}^2 \text{K})$ for oceanic surfaces, and $3.29 \text{ MJ}/\text{m}^2 \text{K}$ for sea ice surfaces. Note, this tuning takes into consideration seasonal transitions along latitudinal bands where transitions between
635 sea ice and ocean surfaces affect the corresponding heat capacity. This change allows for an implicit model of the underlying surface based on the observed transitions between sea ice and oceanic surfaces. The diffusion constant, D_0 in $\text{W m}^{-2} \text{K}^{-1}$, displays a significant impact on the underlying RMSE, providing the second highest evaluation of $\overline{\text{RMSE}}$ around $D_0 = 0.3 \text{ W m}^{-2} \text{K}^{-1}$. For $D_0 < 0.5 \text{ W m}^{-2} \text{K}^{-1}$, the model underestimates the scaling of the necessary heat transport, resulting in



higher $\overline{\text{RMSE}}$. Overestimation in the range of $D_0 > 0.7 \text{ W m}^{-2} \text{ K}^{-1}$ does not result in the same increase to $\overline{\text{RMSE}}$. This is
640 indicative of greater error arising from insufficient diffusion, resulting in overcooling near the poles and overheating in the
tropics. The regional scaling factors in η_n for $N_d = 6$ regions display sensitivities in line with the scale of their region. The
scaling in the Southern hemisphere corresponding to η_1 displays the largest sensitivity among all η_n , with underestimations
of heat transport amplifying the rate of cooling in Antarctic climates. Note, the optimal η_3 is constrained at the boundary of
 $\eta_3 = 3.0$ as the extent to which η_3 impacts $\overline{\text{RMSE}}$ is insignificant compared to the other regional multipliers. The resulting
645 diffusion function, $D(\mu, t_k)$, is plotted by latitude in Fig. 5 alongside the surface-dependent diffusion functions proposed in
Ziegler and Rehfeld (2021).

Fig. 5 displays agreement in the proposed range of thermal conductivities in the Southern Hemisphere and the increasing
trend towards the equator. $D(\mu, t_k)$ produces twice the expected equatorial diffusivity to those values proposed in TransEBM
as a result of the truncated estimation of η_3 . From Fig. 4, we note that the selection of η_3 is of minimal impact to $\overline{\text{RMSE}}$. The
650 equatorial overestimations posed by η_3 and η_4 are attributable to this localization by latitude in tuning according to a global
objective. The Northern Hemisphere displays consistent overestimation by $D(\mu, t_k)$ relative to the TransEBM solution for the
majority of the year. From the $\mu \cos \nu^k$ term in (13), the range of values proposed above 40° latitude displays the greatest
seasonal variability between $\nu^k = 0$ and $\nu^k = \pi$. From the proposed tuning, we now evaluate the EBM performance across
latitude and time.

655 3.2.2 EBM Performance

The latitudinal performance of the EBM with all optimal parameters is provided in Fig. 6(a) where underestimations are in blue
and overestimations are in red. ERA5 measurements are provided in Fig. 6(b) and may be compared to the EBM prediction in
Fig. 6(c). The maximum magnitude of overestimation is 24.0°C and the maximum magnitude of underestimation is 28.9°C .
Both instances are observed in polar latitudes. The simulation from 1 January 2001 – 31 December 2005 took 4.3 seconds. The
660 temperature model is shown to encapsulate the banding from the insolation angle in the mid-latitudes, following the observed
seasonality induced by changes to insolation. The model performs best in the equatorial latitudes and conversely performs
worse towards the poles. The simulation displayed in Figure 6 operates with a temporal mean RMSE of 5.22°C .

Evident in Fig. 6(d), this model's latitudinal profile shows agreement with that of ERA5. Towards the equator, the prediction
appears to plateau with some small hemispheric asymmetry in temperature favoring warmer climates in the Southern tropics.
665 The EBM displays a general linearization of the temperature as the the latitudes approach the poles and mid-latitudes. This lin-
earization diverges most from the ERA5 measurements in the Southern Hemisphere, particularly between -60° and 40° , where
the EBM consistently underestimates the temperatures in the Southern Ocean. The Northern Hemisphere, despite displaying
consistent cycles of underestimation and overestimation appears to agree well with ERA5 in the expected latitudinal profile.
This is indicative of errors by phase lag in the model relative to ERA5.

670 We identify three principal sources of error in the EBM: the phase lag of the model relative to real-world observations,
the resulting heat transport between poles and subpolar latitudes, and oscillatory and local behaviors at the poles. Following
Ziegler and Rehfeld (2021), this paper defines the phase lag as the mean of the number of days between the maximum insola-



tion observed at the summer solstice (Dec 21 in Southern Hemisphere, Jun 21 in the Northern Hemisphere) and the maximum temperature observed at each latitude in the EBM for the five years provided. We deploy sinusoidal fits to the mean temperature evolution at each latitude, averaging the resulting sines for each year in 2001-2005. This evaluation is a simplification of the phase lag discussed in North and Kim (2017), resulting from cyclic forcing by insolation. This lag is presented as a function of latitude for both EBM and ERA5 outputs in Fig. 6(e). We omit the equatorial latitudes in both ERA5 and EBM solutions as the problem is ill-defined towards the tropics. Latitudinal bands of truncation are selected to omit those latitudes which produce poor sinusoidal fits or negative estimates of phase lag. Towards the poles, this phase lag is much smaller, reflecting the near immediate impact of a change in insolation on temperature. Towards the tropics, the phase lag grows larger, reflecting changes to surface types and the resulting increases in heat capacity. There is general agreement between the model and observations on the latitudinal trend with the EBM smoothing those observed lag phases. Towards the mid-latitudes, the EBM largely overestimates this lag phase relative to ERA5. This effect may be attributable to the selection of heat capacities and the assumption of a global correction, w_c , as opposed to individual correction factors for each type of surface considered in Tab. E2.

Additionally, the smoothing by the proposed tridiagonal diffusion operator, $\mathbf{L}$, in (14) may propagate these errors in phase, extending cooling from polar night periods in the Northern and Southern hemispheres to subpolar latitudes. This prolongs the duration of colder periods, delaying the expected seasonal behavior of the model relative to the skin temperature, and rippling out towards the mid-latitudes. This effect is particularly prevalent in the Southern hemisphere where latitudes -40° to -70° observe smoother temperature transitions in the EBM than those of ERA5, resulting in underestimation by these latitudes towards the end of the year.

The maximum and minimum errors displayed within the model are most prevalent at the poles. Evident in Fig. 6(a), there are temperature fluctuations on the order of 30°C , displaying temperature trends which cannot be answered by insolation changes, alone. Whereas a latitudinal model allows for consistent evaluations of trends in tropical and subtropical regions representing larger global surface area, the disadvantage of such a model arises from the evaluation of the poles which represent a smaller proportion of Earth's surface area. Towards the poles, the latitudinal band under evaluation grows more local, requiring that the EBM compare a global methodology to a single region of the globe. Provided the state variable of insolation, the state variable $\mathbf{u}$ is informed of seasonality induced by orbital forcings. The Crank-Nicolson solver in (14) is then able to predict the general trend rather than the local behavior. Such physical predictors of polar fluctuations in temperature are beyond the scope of this paper and require additional knowledge of local atmospheric conditions.

From the proposed parameterizations and model performance, we may now compare the impacts of temporal discretization on the dynamical system. This allows us to evaluate the sensitivity of parameters with time and, subsequently, justify the use of time steps on the scale of days.

3.2.3 EBM Performance Across Time Steps

In many dynamical systems, it is the case that as the discretizations become more fine, the model is better able to predict local behaviors of its underlying system. With climate models, however, it is often the case that such high resolution investigations are limited in spatial range or computationally prohibitive. Many EBM works simulate paleoclimates with time stepping on

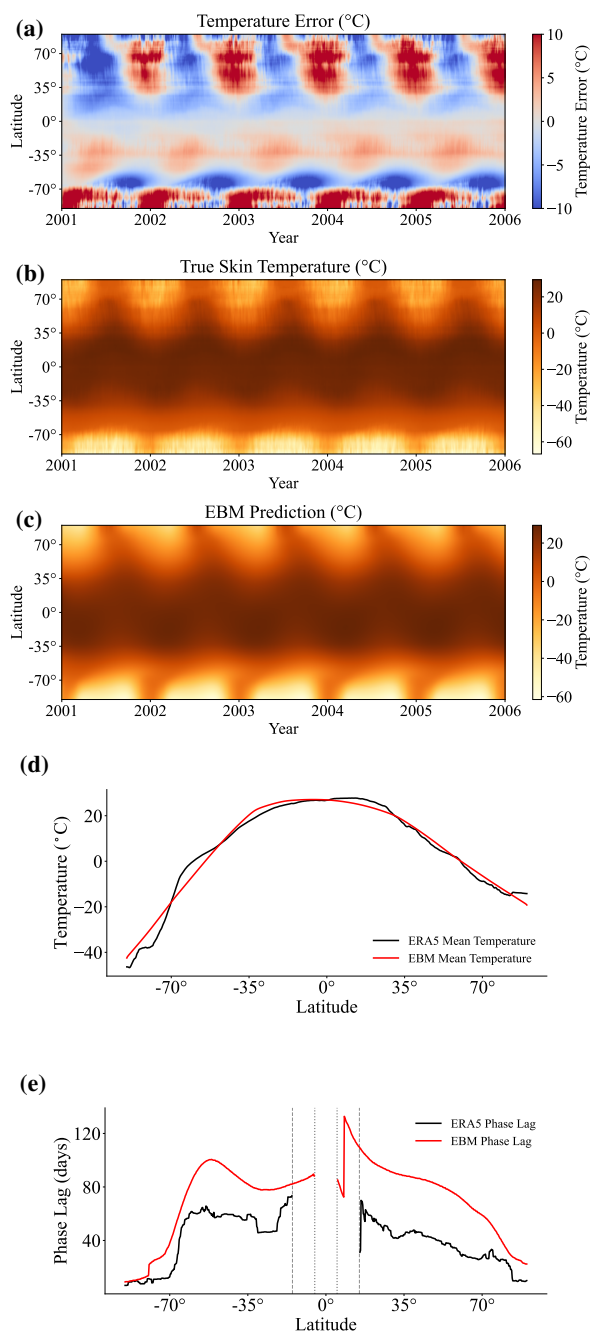


Figure 6. Summary of 1D EBM performance against ERA5 data from 2001–2005. (a) Latitudinal temperature error using optimal $\overline{\text{RMSE}}$ parameters ($\Delta t = 1$ day) and $\pm 10^\circ\text{C}$ truncation in prediction error. (b) ERA5 skin temperature and (c) EBM temperature predictions using observed albedo. (d) Mean latitudinal temperature profiles. (e) Phase lag (days) by latitude, excluding $\pm 5^\circ$ (EBM) and $\pm 15^\circ$ (ERA5).



the order of weeks to months due to the scale of the simulation horizon. To investigate a compromise between the differing objectives, this case study simulates (14) for various Δt . We implement the same tuning methods as in the daily case.

The tuned parameters are summarized in Tab. 1. We note that the objective 2001-2005 mean RMSE varies little between EBM simulations with daily time steps or larger. Similarly, their optimal parameterizations exhibit minimal change with nearly all parameters being consistent between time steps. In contrast, the hourly case displays noticeably different values, particularly in the diffusion scale and regional diffusion multipliers. This suggests that the effective heat-transport depends strongly on temporal resolution when the model is pushed to sub-daily scales. Rather than indicating a change in physical meridional transport alone, this behavior indicates that the diffusion term compensates for unresolved high-frequency processes that are not explicitly modeled.

To examine this effect directly, we define a diagnostic term representing the heat-transport contribution required for the observed ERA5 fields to satisfy the discrete EBM balance exactly at a given time step. Let

$$\omega_{\text{req}}(t_k, \Delta t) = \frac{\mathbf{c}^k}{\Delta t} \odot (\mathbf{u}_{\text{true}}^{k+1} - \mathbf{u}_{\text{true}}^k) - \mathbf{g}_{\text{true}}^k, \quad (37)$$

where $\mathbf{g}_{\text{true}}^k \in \mathbb{R}^N$ is the observed radiative imbalance evaluated from the ERA5 temperature, albedo, and insolation fields using the radiative terms in (14). The quantity ω_{req} should not be interpreted as a directly observed meridional heat transport. Instead, it measures the residual transport-like term that the reduced-order EBM would need to reproduce the observed temperature field at temporal resolution Δt . Fig. 7 reports

$$\max_{k,i} |\omega_{\text{req},i}(t_k, \Delta t)|$$

over 1 January 2001-31 December 2005 for time steps ranging from one hour to six days under the assumption of daily tuning values. The required magnitude of heat transport decreases sharply as the time step is increased. At hourly resolution, the EBM balance requires residual magnitudes approaching 10^5 W/m², indicating that the model is being forced to absorb sub-daily variability that is outside the intended scope of the diffusive heat-transport parameterization. By daily resolution, this maximum decreases by at least an order of magnitude, and for coarser multi-day resolutions the curve becomes comparatively stable.

This behavior supports the interpretation of the tuning results in Tab. 1. The similarity of the daily, three-day, and six-day parameterizations indicates that the EBM is relatively insensitive to temporal aggregation once high-frequency variability has been filtered. Conversely, the distinct hourly parameterization suggests that sub-daily simulation requires the diffusion term to compensate for unresolved processes rather than simply representing large-scale meridional heat transport. As such, the daily time step provides a practical compromise, remaining sufficiently resolved to capture seasonal evolution while avoiding the unstable flux required by hourly variability.

Given the optimal parameterizations for an EBM progressed on a daily time step, we may now consider alternative albedo model forms which approximate $\mathbf{f}_\alpha$. All future case studies implement a daily time step with the selected parameterizations of the EBM in Appendix E (Tab. E3) and Tab. 1.

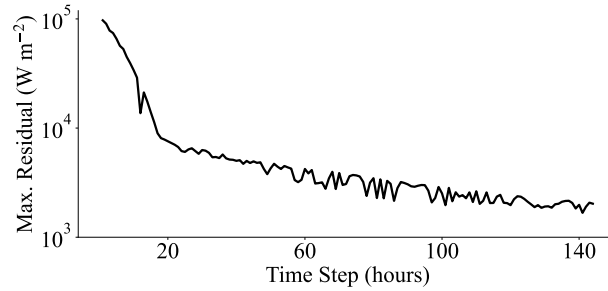


Figure 7. Maximum magnitude of required heat transport as calculated in (37) evaluated over various Δt . Each time step is progressed from 1 January 2001 to 31 December 2005.

Table 1. Optimal OAT parameterizations and 2001-2005 mean RMSE across specified Δt .

Parameter	Units	1 hour	1 day	3 days	6 days
A_{ref}	(W/m ²)	175.17	180.34	180.34	180.34
B	(W/m ² °C)	1.66	1.62	1.62	1.62
w_c	–	2.00	1.71	1.71	1.67
D_0	(W/m ² K)	0.64	0.61	0.59	0.59
η_1	–	0.45	0.51	0.51	0.51
η_2	–	0.37	0.83	0.83	0.83
η_3	–	3.00	3.00	3.00	3.00
η_4	–	3.00	2.17	2.17	2.17
η_5	–	1.61	1.08	1.08	1.08
η_6	–	2.00	1.42	1.42	1.42
RMSE	°C	7.23	5.22	5.19	5.24

3.3 Albedo Model Forms

We now evaluate models of α^k for joint integration with the tuned EBM from Section 3.2. In this case study, the coupled temperature-albedo system is initialized on 1 January 2001 and progressed through 31 December 2023 using the Crank-Nicolson temperature model in (14). Model performance is evaluated over three periods: the training period 2001-2005, the validation period 2010-2011, and the long-term testing period 2023. The latter period is included to assess the stability of the EBM solution over an extended period of time. Fig. 3 lists the albedo cases under consideration and Table E4 in Appendix E lists the prescribed parameterizations used in this case study.

We first consider the polynomial predictors defined in (30). The coefficients are estimated by ordinary least squares using observations from 1 January 2001 through 31 December 2005. We test polynomial orders $P, Q, R \in \{3, 4, 5\}$ and compare both the additive univariate polynomial (UP) model in (28) and the cross-term (CT) model in (27). For compactness, the results are



reported for equal-order cases with $P = Q = R$, with representative plots shown for $P = 3$ and $P = 5$. These cases allow us to
750 compare polynomial orders while distinguishing between models that exclude and include cross-variable interactions among
temperature, albedo, and insolation.

We next consider the Gaussian KRR predictors defined by the naïve and informed cases in (31a) and (31b), respec-
tively. To examine the tradeoff between local resolution and computational cost, we define regional predictor counts $N_r \in$
 $\{18, 36, 72, 144, 721\}$. When $N_r = 721$, a distinct predictor is assigned to each latitude, allowing the most localized represen-
755 tation of albedo evolution. In the naïve case, each regional prediction is independent of neighboring latitudes. In the informed
case, neighboring states are included in each predictor’s vector of inputs, and smoothed to penalize sharp spatial inconsistencies
in predicted albedo between adjacent latitudes.

The Gaussian KRR predictors use globally uniform hyperparameters γ and λ , where γ denotes the RBF kernel bandwidth
and λ denotes the regularization parameter, as summarized in Appendix D (Tab. D2). For each naïve and informed KRR case,
760 the model is trained on 2001–2005 data and the hyperparameters are selected by minimizing coupled EBM temperature error
over a held-out tuning period from 1 January 2010 through 31 December 2011. Specifically, we perform a coarse-to-fine grid
search over $\gamma \in [0.01, 1.0]$ and $\lambda \in [10^{-6}, 10^{-3}]$. After identifying the best-performing pair (γ, λ) on the coarse grid, the search
is restricted to a neighborhood of that pair and refined twice for brevity.

The regression-based albedo models are compared against the benchmark cases listed in Fig. 3. These benchmark models
765 provide reference behaviors for assessing whether the proposed polynomial and KRR predictors improve the coupled EBM
simulation relative to accepted albedo parameterizations. All candidate models are integrated over the same 2001–2023 hori-
zon, evaluating the RMSE over the specified training, validation, and long-term testing windows. All albedo models are clipped
between 0.2 and 0.75, representing the general upper and lower bounds noted by daily-averaged ERA5 observations.

3.3.1 Coupled Model Simulations

770 Fig. 8 displays the temperature RMSE of each coupling of the albedo model and the EBM solution. The base case, representing
the tuned solution in Section 3.2 is plotted only from 2001–2005 for continuity with the selected ranges of ERA5 albedo data.
Abbreviations UP3, UP5, CT3, and CT5, refer to the labeled polynomial predictors where $P = Q = R = 3$ and 5, respectively.
This figure plots only the naïve albedo predictor where $N_r = 721$.

It is evident that the constant and kernel models perform well in this period, agreeing with the predictions of the base case in
775 the 2001–2005 period. Even in later years, the cases perform with similar ranges of $\overline{\text{RMSE}}$, indicating insignificant divergence
for the selected simulation horizon. The same may be stated for the hyperbolic tangent benchmark, where explicit prediction
of albedo does not display diverging RMSE over time. This case overestimates losses in albedo towards the poles from the
selection of α_L , which represents an underlying surface with no cryosphere. This bias in prediction is indicative of the fixed
functional form assuming one monotonic temperature-dependent trajectory for albedo at each latitude. As a benchmark, it is
780 valuable for assessing superior predictors of albedo, but as an individual predictor it fails to capture changes to albedo which
are independent of temperature.

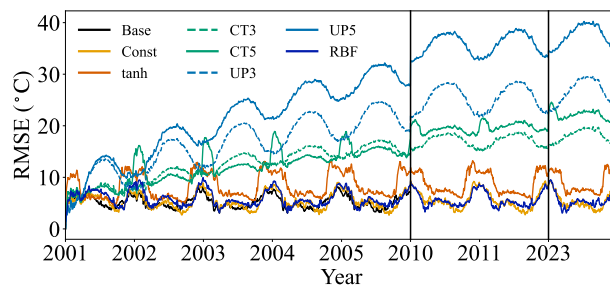


Figure 8. Comparison of RMSE across selected albedo model solutions. Black vertical lines denote partitions between 2001–2005 2010–2011, and 2023 evaluation periods. Regression model performances are depicted in shades of blue and those of the benchmarks are depicted in shades of orange.

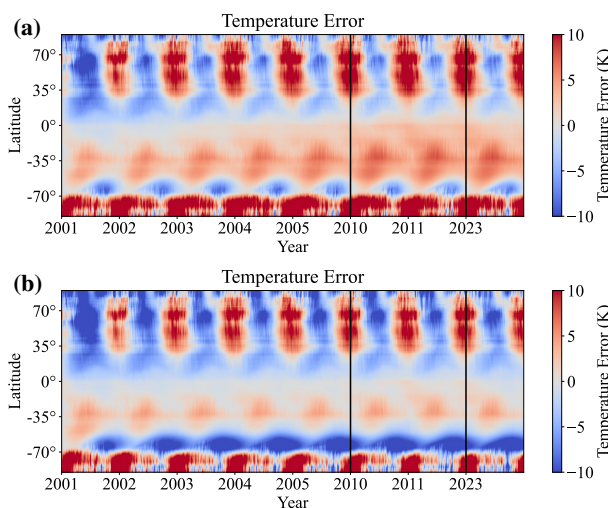


Figure 9. Latitudinal temperature error over 2001–2023 simulation for (a) constant albedo assumption and (b) RBF kernels across 721 latitudes. Black vertical lines denote partitions between 2001–2005 2010–2011, and 2023 evaluation periods.

The polynomial predictors exhibit the weakest performance among the tested albedo models, with long-horizon RMSE increasing upwards of 40°C in some cases. We note that this error is, in part, intrinsic to the assumption of a simple least squares regression. The coefficient of determination rarely exceeds 0.7 among the $N = 721$ latitudes in the system, indicating that the proposed input variables and corresponding basis functions are insufficient approximations of the rate of change in albedo. Latitudes with the highest coefficients of determination are largely concentrated about polar latitudes, which the EBM solution in Section 3.2 struggles to approximate. We note that this solution’s weakness may be exacerbated by various additional errors, including the use of Forward Euler as a simple numerical integration scheme and the use of a constant α_{PN} that transitions the albedo prediction from polar night to sunlit regimes.



790 The performance metrics among all albedo models are depicted in Tab. 2. On the basis of temporal mean temperature RMSE and albedo MAE, the constant and KRR cases provide the most acceptable predictions among the tested models. The constant case performs especially well in these aggregate metrics, which is not unexpected: a prescribed albedo field avoids the accumulation of error associated with an imperfect dynamic albedo update. However, these scalar metrics do not fully describe the long-horizon behavior of the coupled model.

795 Fig. 9 compares the latitudinal error profiles for the constant and naïve KRR cases under the same EBM parameterization. In Fig. 9(a), the constant-albedo case develops a gradual warm drift, most clearly visible in the tropics and mid-latitudes. This drift suggests that the fixed albedo field does not remain dynamically consistent with the radiative and diffusive balances imposed by the tuned EBM parameters over extended simulation horizons. We emphasize that this does not necessarily indicate failure of the constant-albedo assumption alone. Since the EBM parameters are tuned using observed albedo rather than this constant solution, some of the drift may reflect residual bias in the selected OLR or heat transport parameters.

800 This limitation is useful for interpreting the comparison of the two approaches. Both albedo models inherit the same tuned EBM parameters. As such, differences in the evolution of the temperature error field indicate differences in sensitivity to that fixed parameterization. Under this shared initialization, the Gaussian KRR solution in Fig. 9(b) exhibits a more temporally consistent latitudinal error structure, with less apparent accumulation of warm bias over time. While the constant case remains competitive in aggregate error metrics, the KRR model displays stronger long-horizon robustness in the coupled simulation.

810 This behavior is consistent with the structure of the KRR update. Unlike the polynomial predictors, which apply a fixed global set of basis functions and regression coefficients, the individual RBF kernel in (19b) evaluates the current state through a similarity metric evaluated with respect to its training data. The resulting update is therefore more locally conditioned on the evolving temperature, albedo, and insolation states. This state-dependent structure appears to reduce the amplification of existing parameter bias and helps preserve the stability of the coupled system over longer horizons of numerical integration.

Table 2. Temporal mean RMSE and mean average error (MAE) in albedo prediction across predictor forms. Error metrics evaluated uniformly across 2001-2005, 2010-2011, and 2023.

Predictor Case	Mean RMSE (°C)	Albedo MAE
Base	5.1919	—
Constant	5.6677	0.0385
UP(3,3,3)	19.7196	0.2213
UP(5,5,5)	26.5537	0.2411
CT(3,3,3)	14.0960	0.2262
CT(5,5,5)	15.2177	0.2402
tanh	8.5562	0.0980
RBF-721	5.8186	0.0405



3.3.2 Gaussian KRR Experiments

From Tab. 3, the constant and KRR cases provide the strongest performance among the tested cases of albedo prediction. The constant case remains a competitive benchmark in aggregate error because it avoids the accumulation of error associated with an imperfect dynamic albedo update. However, as shown in Fig. 9, the constant-albedo solution exhibits a gradual warm drift over the simulation horizon, suggesting sensitivity to residual bias in the fixed EBM parameterization. The RBF-KRR case does not exhibit the same degree of drift under identical EBM parameters, indicating improved long-horizon robustness of the coupled albedo update. Motivated by this behavior, we further examine a collection of KRR models under different choices of N_r and spatial information.

Tab. 3 summarizes the temporal mean temperature RMSE, albedo MAE, and simulation duration for the KRR configurations considered. Fig. 10 displays the albedo error across latitudes between the constant and RBF-72 cases. It is evident that the proposed constant assumption consistently underestimates albedo about the mid-latitudes, likely contributing to the temperature drift evident in Fig. 9(a). In the RBF-72 case, however, we show that the KRR predictor is capable of approximating changes to albedo with lower error relative to the constant case. We note that the predictor's errors are tied to those of the EBM's structure, overestimating the albedo in regions where the EBM in Fig. 6 is shown to overestimate. Occurrences of polar night show some overestimation by the selected α_{PN} at the North Pole once the model transitions from polar night to polar day. Over time, these occurrences of polar night are shown to influence the solutions of neighboring latitudes. By 2023, the KRR model has difficulty predicting albedo between 35° and 70° , indicating a need for further corrections between regimes of polar day and polar night.

Table 3. KRR simulation summaries for various groupings of N_r across naïve and informed cases. Model runs are progressed from 2001-2023 with error metrics recorded from 2001-2005, 2010-2011, and 2023. All informed models apply the Laplacian operator, $\mathbf{L}_s$, with $\ell = 0.3$ at each time step. Bold rows correspond to models which predict $\overline{\text{RMSE}}$ under that of the constant case.

Case- N_r	Model	$\overline{\text{RMSE}}$ ($^\circ\text{C}$)	Albedo MAE	Duration (s)
RBF-18	Naïve	5.70	0.0645	37.2
RBF-36	Naïve	5.62	0.0466	65.0
RBF-72	Naïve	5.51	0.0488	124.8
RBF-144	Naïve	5.79	0.0870	238.2
RBF-18	Informed	6.10	0.0968	42.3
RBF-36	Informed	5.66	0.0634	74.0
RBF-72	Informed	5.38	0.0333	135.2
RBF-721	Naïve	5.82	0.0405	1081.0
Const.	—	5.67	0.0385	17.5

We identify three primary findings. First, several grouped KRR models with $N_r < N$ outperform the full $N_r = 721$ latitude model in temperature RMSE. This suggests that the full-resolution KRR solution is not necessarily optimal under the present hyperparameter selection. In particular, the use of global hyperparameters γ and λ imposes the same kernel bandwidth and

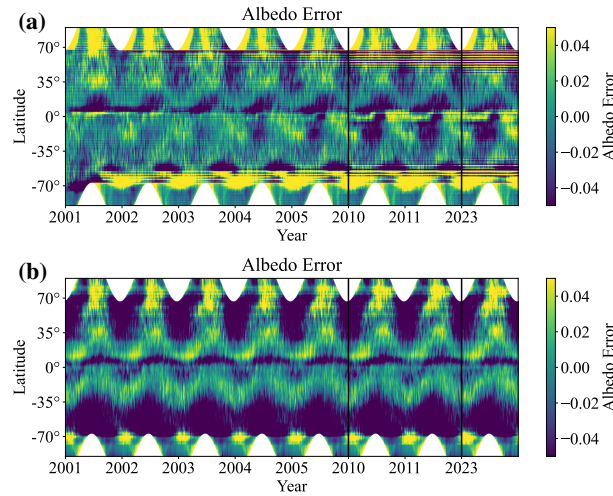


Figure 10. Latitudinal albedo error profiles for (a) informed RBF-72 and (b) constant cases. Black vertical lines denote partitions between 2001-2005, 2010-2011, and 2023 evaluation periods.

regularization strength across all latitudes, despite substantial differences in local seasonal variability, polar night behavior, and albedo-temperature sensitivity. As a result, the full-resolution model may be constrained by hyperparameters that are appropriate for some latitudes but poorly suited to others. Future refinements should therefore consider latitude-dependent or region-dependent hyperparameters, denoted for example by $\gamma_{a(i)}$ and $\lambda_{a(i)}$, where $a(i)$ maps latitude index i to its corresponding regional KRR model.

Second, the inclusion of spatial information does not uniformly improve predictive performance. For $N_r = 18$, the informed model increases both temperature and albedo error relative to the corresponding naïve case, while also increasing simulation time. For $N_r = 36$, the informed case again produces slightly larger error metrics than the naïve model. In contrast, the $N_r = 72$ informed model substantially improves both temperature RMSE and albedo MAE, and it is the only KRR configuration listed here that improves upon the constant case in both metrics. These results suggest that neighboring-state information and Laplacian smoothing are most beneficial when the regional discretization is fine enough to resolve meaningful latitudinal structure, but not so fine that each predictor becomes overly sensitive to local noise or poorly tuned global hyperparameters.

Third, simulation time increases with the number of regional predictors. This trend is expected, since increasing N_r increases the number of KRR evaluations required at each time step. The informed cases introduce only a modest additional cost relative to their naïve counterparts, indicating that the primary computational burden is associated with the number of regional KRR models rather than the inclusion of neighboring states or the application of the smoothing operator. The cost of high-resolution KRR is nontrivial. For global two-dimensional models parameterized by $\mathbf{r}$, where a latitudinal grid may be paired with a substantially larger longitudinal discretization, this additional cost may limit the practicality of the informed KRR as proposed. In regional climate applications, however, where high spatial resolution is often necessary and the computational domain is more restricted, the benefit of informed KRR updates may justify the added expense.

**Table 4.** Summary of coupled kernel model performances evaluated from 1 January 2001–31 December 2005.

Albedo Model	Hyperparameters		Naïve Temperature ($\lambda = 0.1, \gamma = 0.52$)			Informed Temperature ($\lambda = 0.094, \gamma = 0.01$)		
	λ	γ	Run Time (s)	RMSE ($^{\circ}\text{C}$)	MAE Albedo	Run Time (s)	RMSE ($^{\circ}\text{C}$)	MAE Albedo
RBF18-Naïve	0.0425	0.084	493.87	6.18	0.056	535.19	28.92	0.058
RBF36-Naïve	1.0000	0.400	487.59	4.54	0.038	538.88	28.94	0.045
RBF72-Naïve	0.0219	0.010	514.33	6.50	0.039	561.76	28.76	0.046
RBF144-Naïve	0.0021	0.957	696.46	3.87	0.039	580.99	27.78	0.069
RBF18-Informed	0.0018	0.010	698.03	8.50	0.179	595.60	28.21	0.086
RBF36-Informed	0.0703	0.010	701.31	7.01	0.079	576.23	26.70	0.081
RBF72-Informed	0.1000	0.010	604.23	6.56	0.069	555.06	25.92	0.073

Overall, the KRR results should be interpreted as evidence of conditional robustness rather than uniform superiority. The constant-albedo case remains an efficient and accurate benchmark for the proposed simulation horizon, but selected KRR configurations reduce long-horizon drift and can outperform the constant case when the regional resolution and spatial information are appropriately balanced. This supports the use of KRR as a flexible dynamic albedo model, while also indicating that its performance depends strongly on hyperparameter selection and the spatial scale at which the predictors are constructed.

3.4 Coupled Gaussian Kernel Model

Motivated by the performance of the KRR albedo predictor in Section 3.3, we extend the Gaussian KRR framework to temperature prediction as defined in (15). We do not propose this temperature update as a replacement for the physical EBM. Rather, this case study is intended to evaluate the behavior of a coupled KRR solution when the temperature update is learned from the same variables that inform the governing EBM in (9).

3.4.1 Temperature Kernel Tuning

For this experiment, we consider both naïve and informed temperature KRR predictors on the full latitudinal grid, so that $N_r = N = 721$. This choice provides an upper estimate of the computational cost associated with the coupled KRR framework, since one temperature predictor is trained and evaluated at each latitude. It also preserves the native discretization of the EBM: each latitude evolves according to a local temperature update constructed from the state variables at the corresponding grid point, with neighboring states included only in the informed formulation. Thus, the spatial discretization of the physical system determines the number of temperature predictors.

The informed coupled models include two smoothing parameters. The first, ℓ_α , is applied to the informed albedo update, while the second, ℓ_u , is applied to the temperature update. For all simulations using informed albedo predictors, we set $\ell_\alpha = 0.3$, consistent with the albedo experiments in Section 3.3. For the informed temperature predictor, we take $\ell_u = 0.3$ as the nominal case and additionally test $\ell_u = 0$ to assess whether post-prediction smoothing contributes meaningfully to the coupled temperature error.

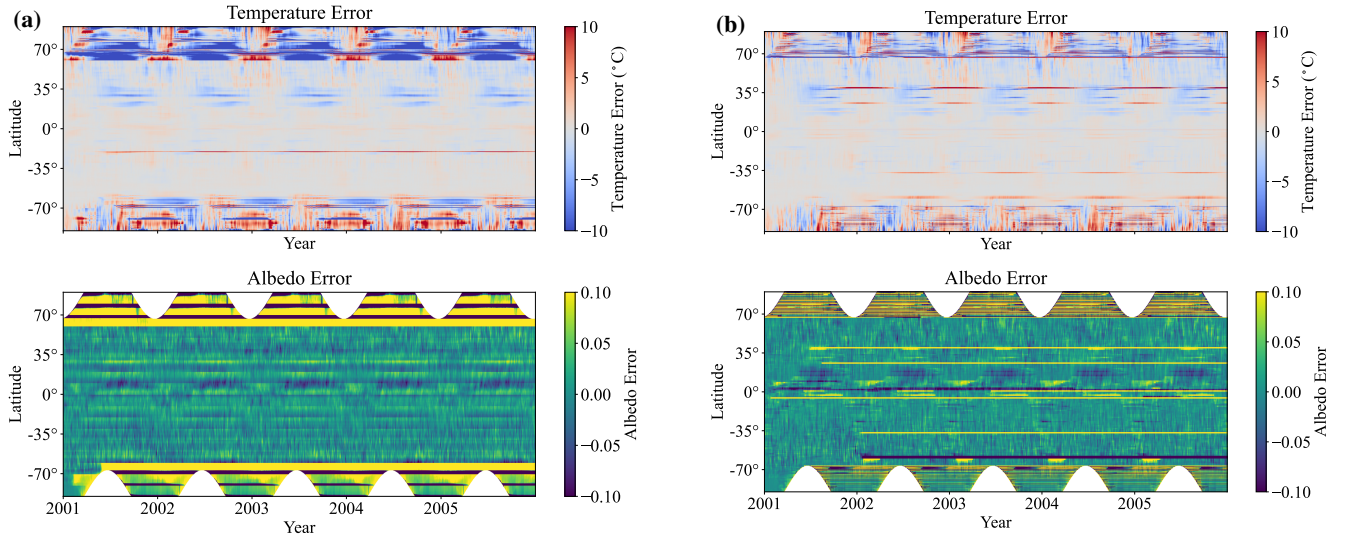


Figure 11. Comparison of temperature and albedo errors over latitude from 2001–2005, coupling (a) naïve temperature and RBF18-Naïve and (b) naïve temperature and RBF144-Naïve solutions. Polar night is shown in white.

Training and hyperparameter selection follow the procedure used for the albedo KRR models. Each temperature KRR predictor is trained using 2001–2005 data. The globally uniform hyperparameters (γ, λ) are then selected by minimizing the coupled temperature RMSE over the 2010–2011 tuning period using a coarse-to-fine grid search. Four grid-refinement iterations are used for the temperature models. During each tuning run, the coupled simulation is initialized on 1 January 2010, and the temperature KRR receives ERA5-observed albedo as an input. All temperature predictions are bounded between -100°C and 100°C .

This tuning procedure introduces an important limitation. In the fully coupled simulations, the temperature KRR no longer receives ERA5 albedo. The temperature kernel is tuned using α_{true} but deployed using the KRR update in (31). This mismatch can inflate apparent accuracy during the tuning period as the temperature predictor is selected under cleaner input conditions than those encountered in the coupled forecast. In deployment, errors in the albedo predictor may shift the temperature KRR inputs away from the distribution used for hyperparameter selection, allowing albedo error to propagate into the temperature update. This issue is especially relevant during polar night transitions.

As such, the results of this case study should be interpreted as an assessment of coupled KRR behavior under a partially idealized tuning procedure, rather than as a fully optimized coupled solution. A more rigorous extension would tune the temperature and albedo kernels jointly, or tune the temperature kernel using albedo inputs generated by the deployed albedo predictor. In total, we simulate 14 coupled kernel simulations between the temperature predictors and the albedo predictors implemented in Tab. 2.



890 3.4.2 Coupled Kernel Results

Tab. 4 summarizes the run times and mean performance metrics of the coupled kernel simulations. In no case does the informed temperature predictor improve upon its naïve counterpart. The informed temperature solutions produce substantially larger mean RMSE while yielding only minor changes in albedo MAE, indicating that the dominant degradation in performance occurs in the temperature update.

895 To assess potential sources of this behavior, we first vary the informed temperature smoothing strength from $\ell_u = 0.3$ to $\ell_u = 0$. The resulting changes in temperature RMSE and albedo MAE are negligible, indicating that the large informed-temperature errors are not attributable primarily to the smoothing step. Similarly, the use of globally uniform hyperparameters (λ, γ) does not, by itself, appear sufficient to explain the degradation, as the naïve temperature predictors use the same assumption without producing errors of comparable magnitude. These results suggest that the observed degradation is more closely associated
900 with the expanded informed input vector in (23), although additional ablation studies would be required to isolate the precise contribution of each input component. One plausible explanation is that the inclusion of neighboring temperature and albedo states introduces additional pathways for local and adjacent-latitude prediction errors to enter the similarity metric in (19). During coupled integration, errors in either the local state or neighboring states can shift the informed feature vector away from the training distribution. Under this hypothesis, the kernel may identify inappropriate training analogues and produce
905 updates that are inconsistent with the learned dynamics of the system.

Additionally, we note that in no case do the informed albedo models improve upon the MAE metric in comparison to their naïve counterparts. For $N_r = 18, 36$, this finding is consistent with the result obtained in Section 3.3, but differs for $N_r = 72$, where the informed predictor performs worse in both aggregate metrics. These results indicate that the addition of spatial information does not necessarily improve the predictive performance of an empirical temperature model. In the present
910 tuning configuration, the incorporation of neighboring-state information into the predictor form can instead propagate coupled prediction errors. This is shown to hinder both temperature and albedo performance relative to the corresponding coupling of naïve models.

We now narrow the focus of our discussion to these naïve temperature predictors. The coupled RBF144-Naïve model produces the most favorable aggregate error metrics among all simulations in Sections 3.3 and 3.4, outperforming the baseline
915 EBM in Section 3.2. To examine the spatial structure of these aggregated error metrics, Fig. 11 compares this case with the lower-resolution RBF18-Naïve albedo model. In both cases, the temperature and albedo error fields exhibit visibly related latitudinal structures, with the largest discrepancies concentrated near latitudes which experience polar night.

For the RBF18-Naïve case in Fig. 11(a), the coarse regional representation produces broad latitudinal bands of albedo error, visibly partitioning the latitudes into their respective regional regimes. Towards the poles, the bands are accompanied
920 by corresponding temperature-error structures, suggesting that errors in the albedo model are propagated through the recorded insolation and into the temperature update. This is evidenced by polar behavior above 65° , where individual latitudes of the same region consistently overestimate and underestimate during sunlit periods.



The RBF144-Naïve case in Fig. 11(b) resolves these structures at a finer spatial scale. Although the albedo and temperature errors are less regionally coarse, high-latitude sensitivity remains evident. Thus, increasing the number of albedo regions improves aggregate performance but does not fully remove coupled errors associated with polar night transitions and bounded albedo updates. These results suggest that the dominant issue is not simply the spatial resolution of the albedo predictor, but the interaction between albedo prediction error and temperature evolution in the coupled system. Jointly tuning the temperature and albedo kernels, especially in polar regions, may reduce this mismatch by exposing the temperature predictor to the albedo inputs encountered during coupled deployment rather than to ERA5- observed albedo alone.

3.4.3 Coupled Model Takeaways

The coupled Gaussian KRR experiments reveal two primary limitations of the proposed solutions by regression. First, polar night remains a central challenge for dynamic albedo prediction. At high latitudes, the albedo model must transition between prescribed polar night and sunlit values. These transitions introduce discontinuities and regime changes that are not fully resolved by increasing the number of albedo regions. Although we select a numerical placeholder from which to resume predictions, future work must consider both how albedo is parameterized in these periods and what metrics define the change in response between these regimes.

Second, the inclusion of spatial information does not necessarily improve coupled predictive performance. Although neighboring state information is intended to provide meridional context, the informed temperature predictors produce larger errors than their naïve counterparts under the present tuning strategy. This suggests that, in a fully coupled simulation, additional spatial features may also provide additional pathways for prediction error to propagate. As such, spatially informed kernels require additional considerations of feature design and tuning before they can be assumed to improve model performance.

The coupled KRR results suggest that Gaussian kernels can provide flexible solutions to a simplified model of latitudinal climate, but their performance depends strongly on how regime transitions, spatial information, and coupled-error propagation are handled. The most successful configurations are therefore not necessarily those with the greatest spatial complexity, but those whose learned updates remain consistent with the input distributions encountered during forward integration.

4 Conclusions

In this paper, we evaluate the latitudinal performance of a one-dimensional energy balance model under physical and empirical parameterizations. After tuning the EBM, we introduce a diagnostic metric for the heat transport required for the model to reproduce observed temperatures given real-world temperature, insolation, and albedo fields. This diagnostic motivates the use of daily time steps in the subsequent simulations, allowing the EBM and coupled albedo models to operate at a higher temporal resolution than is commonly used in many long-horizon EBM studies. Using the tuned EBM, we then numerically integrate a collection of benchmark and reanalysis-informed albedo models to assess their influence on coupled temperature prediction.

Among the most successful albedo models are the constant-albedo assumption and the Gaussian KRR predictors. The constant model provides a useful short-term benchmark because it preserves observed latitudinal ranges of albedo without in-



955 troduding additional dynamical error. However, its lack of seasonal and interannual variability makes it an increasingly poor
assumption for longer-term simulations, particularly when albedo change is expected to reflect evolving surface conditions.
By contrast, the KRR approach provides a dynamical update to albedo at a daily resolution and is able to produce temper-
ature predictions comparable to the constant model in several mid-latitude cases. In Section 3.3, various KRR models are
shown to outperform the constant benchmark’s mean temperature RMSE over time, with one coupled simulation exceeding
960 the benchmark’s predictive performance in both temperature and albedo.

The KRR results also clarify several limitations of the proposed regression-based formulation. First, the treatment of polar
night strongly influences future albedo prediction, since periods with little or no incoming shortwave radiation interrupt the
observational relationship between insolation, albedo, and temperature. This suggests that dynamical albedo models must
account not only for individual polar seasons, but also for transitions into and out of polar night. Second, the choice of predictor
965 inputs plays a significant role in coupled performance. In this paper, temperature, existing albedo, and insolation are assumed
to be sufficient predictors of the rate of change in albedo. We further test whether each latitude-specific predictor should
include information from neighboring latitudes. Although neighboring-state information is physically motivated, the spatially
informed formulation produced coupled errors as large as 27°C under the selected training resolution and tuning assumptions.
This result suggests that additional spatial information can amplify model error when the corresponding kernel predictors are
970 not sufficiently tuned, regularized, or trained across representative coupled states.

The kernel results also reveal a tradeoff between spatial resolution, predictive accuracy, and computational cost. Increasing
the spatial resolution of the kernel training data produces diminishing returns in albedo accuracy, while substantially increasing
the cost of simulation relative to explicit benchmark albedo functions. These costs may be mitigated through regional groupings
of latitude bands, but the selection of such groupings introduces an additional modeling decision. More broadly, the coupled
975 kernel results indicate that the robustness of a KRR albedo model depends not only on its standalone albedo error, but also on
how its errors interact with the temperature dynamics of the EBM.

Several limitations of this manuscript suggest clear routes for future work. The proposed diffusivity model captures latitudi-
nal structure and agrees qualitatively with trends reported in Ziegler and Rehfeld (2021), but it does not include the temperature-
dependent diffusivity recommended by Chang and Merlis (2023). Introducing such dependence could improve the dynamical
response of heat transport under changing climate conditions. The present work also omits moist static energy formulations,
980 which have been shown to better emulate polar amplification than surface-temperature EBMs alone Lu et al. (2022). In addi-
tion, both the temperature and albedo models are advanced using Forward Euler integration. Higher-order numerical methods,
such as Runge-Kutta schemes, may improve stability and accuracy, particularly when albedo evolves dynamically alongside
temperature.

985 A natural extension of this work is the development of a two-dimensional albedo-temperature dynamical system, incorpo-
rating variability across both latitude and longitude. Such a model would allow albedo evolution to depend more explicitly on
surface type, regional geography, and spatially heterogeneous surface transitions. This is especially important because albedo
is strongly affected by changes in the underlying surface, including sea ice melt, snow cover, melt ponds, and land-surface
transitions Hunke et al. (2013); Krebs-Kanzow et al. (2021). Future high-fidelity albedo predictors should therefore incorpo-



rate surface-type information directly into the state representation. In summary, the results of this study suggest that Gaussian KRR provides a promising but tuning-sensitive framework for dynamic albedo prediction. The joint integration is shown to work well when deployment inputs remain close to the training distribution, but polar night and coupled albedo-temperature errors are the principal mechanisms that break this condition. This manuscript emphasizes that the treatment of polar night, coupled-state training, and physically informed spatial structure are central challenges for future EBM-based albedo modeling.

Appendix A: List of Acronyms and Paper Notation

Here, we produce the acronyms used in the paper as well as the notation system.

Acronym	Meaning
CT	Cross-term polynomial model
EBM	Energy balance model
ERA5	ECMWF Reanalysis version 5
GCM	General circulation model
KRR	Kernel ridge regression
MAE	Mean absolute error
ML	Machine learning
OAT	One-at-a-time tuning
ODE	Ordinary differential equation
OLR	Outgoing longwave radiation
PD	Polar day
PDE	Partial differential equation
PN	Polar night
RBF	Radial basis function
RMSE	Root mean square error
TOA	Top-of-atmosphere
UP	Univariate polynomial model

The notation follows. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n \times m}$ represent the set of natural numbers, integers, the set of real numbers, the set of real-valued row vectors of dimension n , and the set of real-valued matrices of dimension $n \times m$, respectively. Let $\frac{\partial}{\partial t}$ denote the partial derivative of a variable with respect to time t . Alternatively, this derivative of any scalar or vector-valued variable, v , with respect to time may be denoted $\dot{v}$. Let lowercase bold notation, e.g. $\mathbf{x} \in \mathbb{R}^n$, represent an n -dimensional column vector, and uppercase bold notation, e.g. $\mathbf{X} \in \mathbb{R}^{n \times m}$, represent a matrix of dimension $n \times m$. Any scalar element of a vector $\mathbf{v} \in \mathbb{R}^n$ may be denoted v_i . Any scalar element of a matrix $\mathbf{X} \in \mathbb{R}^{n \times m}$ may be denoted X_{ij} for $1 \leq i \leq n$ and $1 \leq j \leq m$. The Euclidean norm of a vector is denoted as $\|\mathbf{x}\| = \sqrt{\sum_{i=1}^n x_i^2}$ for $\mathbf{x} \in \mathbb{R}^n$. Let ∇ denote the surface (horizontal) gradient operator on the unit sphere. For a scalar field $\phi(\varphi, \theta)$, it is given in spherical coordinates by $\nabla\phi = \frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi + \frac{1}{\cos\varphi}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta$, where



$\hat{e}_\varphi$ and $\hat{e}_\theta$ are the local orthonormal basis vectors tangent to the sphere. Quantity $\nabla \cdot \mathbf{a}$ represents the divergence operator such that $\nabla \cdot \mathbf{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$ for $\mathbf{a} \in \mathbb{R}^3$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the elementwise product is denoted by $\mathbf{x} \odot \mathbf{y}$, with components $(\mathbf{x} \odot \mathbf{y})_i = x_i y_i$ for $i = 1, \dots, n$. For a vector $\mathbf{x} \in \mathbb{R}^n$, $\text{diag}(\mathbf{x}) \in \mathbb{R}^{n \times n}$ depicts a real valued diagonal matrix where the diagonals are the entries of $\mathbf{x}$ and all other entries are zero. Given a true output of a scalar, vector, or matrix, a , any predictor of a may be written as $\hat{a}$.

Appendix B: State-Space Insolation Model

The insolation model formulated in (3) is advantageous as it serves as a proxy for the seasonality of the dynamical system in (8). From (3), the insolation model is defined in continuous time and continuous space. In discrete time and discrete space, the model must be reformulated as a vector-valued function for N latitudes. Define $\boldsymbol{\varphi} := [\varphi_1, \varphi_2, \dots, \varphi_N]^\top \in \mathbb{R}^N$ where the difference $\Delta\varphi = \varphi_{i+1} - \varphi_i$ is assumed to be constant for all $i = 1, 2, \dots, N-1$. Then the insolation at time t_k , $\mathbf{s}^k$, may be evaluated:

$$\mathbf{s}^k = \max \left\{ \frac{s_0}{\pi \psi(k)^2} (\sin \delta^k \text{diag}(\mathbf{h}_0^k) \sin \boldsymbol{\varphi} + \cos \delta^k \text{diag}(\cos \boldsymbol{\varphi}) \sin(\mathbf{h}_0^k), 0 \right\},$$

$$\psi^k = \left(\frac{1 + \epsilon \cos(\zeta^k - \zeta_0)}{1 - \epsilon^2} \right)^2,$$

$$\delta^k = \sin^{-1}(\sin \varpi \sin \zeta^k),$$

$$\mathbf{h}_0^k = \cos^{-1}(\max\{\min\{-\tan \delta^k \tan \boldsymbol{\varphi}, 1\}, -1\}),$$

where $\mathbf{h}_0^k \in \mathbb{R}^N$ is the vector of absolute hour angles at each latitude and all sinusoidal functions are elementwise operators. If time t_k and orbital parameters ϖ , ζ_0 , and ϵ are known, then the insolation at time t_k is known.

Appendix C: Smooth Regional Weight Functions

In (13), we define a diffusion coefficient, $D(\mu, t_k)$, which depends on both the sine of the latitude and the time of year. To represent nonuniform spatial models of D , we introduce regional parameters, η_n , where indices of $n \in \{1, \dots, N_d\}$ denote the region. This section evaluates $\chi_n(\mu) \in [0, 1]$, which smooths binary regional assignments. Define the regional partitions:

$$-1 < \mu'_1 < \dots < \mu'_{N_d-1} < 1,$$

where μ' denotes the interior boundaries separating the N_d latitudinal regions in the domain of $[-1, 1]$. To avoid discontinuities in the diffusion coefficient, each sharp regional boundary is replaced by a smoothed step function

$$H(x) = \frac{1}{2} \left[1 + \tanh \left(\frac{x}{w_d} \right) \right],$$



where $w_d = 0.02$ is the slope parameter. From $H(x)$, we may identify $\chi_n(\mu)$:

$$\chi_n(\mu) = \begin{cases} 1 - H(\mu - \mu'_1), & n = 1, \\ H(\mu - \mu'_{n-1}) - H(\mu - \mu'_n), & 2 \leq n \leq N_d - 1, \\ H(\mu - \mu'_{N_d-1}), & n = N_d. \end{cases}$$

This construction gives $\chi_n(\mu) \approx 1$ in the interior of the n -th region and $\chi_n(\mu) \approx 0$ outside that region, with smooth transitions
1035 across adjacent boundaries. For $N_d = 6$ regions, the behavior of this function with tuned $D_0, \eta_1, \dots, \eta_6$ is evaluated in Fig. 5.

Appendix D: State-Space Summary Tables

Tables D1 and D2 collect the state-space quantities and albedo-model operators used throughout the methodology. They are placed here to keep the main development focused while preserving the definitions needed to interpret the model equations.

Appendix E: Data and Parameter Tables

1040 Tables E1, E2, and E3 summarize the ERA5 variables, surface-type parameterizations, and fixed EBM parameters used in the case studies. These tables define the data sources and prescribed quantities that support the tuning and simulation results.



Table D1. State-space representations of energy balance model terms and operators: discrete dimensions, associated model quantities, units, sources of determination, and physical interpretations. Sources of determination are denoted by E, K, and T, corresponding to estimation, known parameterization, and tuning, respectively.

State-space Form	Discrete dimension	EBM Quantities	Units	Source	Represents	Notes
u	$\mathbb{R}^N$	u	[K]	E	Temperature	Evolves due to radiative imbalance and heat transport. Estimated through physical EBM and candidate albedo models.
α	$\mathbb{R}^N$	α	—	E	Albedo	Represents the fraction of incoming shortwave radiation that is reflected. Estimated through candidate albedo models.
s	$\mathbb{R}^N$	S	[W/m ²]	K	Insolation	Incident shortwave radiation predicted by formulation in Berger (1978), with orbital parameterizations from Laskar et al. (2004).
c	$\mathbb{R}^N$	C	[J/(m ² K)]	K	Effective heat capacity	Entries vary in space and time, reflecting changes in the underlying surface. Parameterizations sourced from Ziegler and Rehfeld (2021).
$A1$	$\mathbb{R}^N$	A_{ref}	[W/m ²]	T	OLR intercept	Globally uniform baseline longwave-emission intercept. The linear outgoing-longwave-radiation form follows Budyko (1969).
		S_{CO_2}	[W/m ²]	K	Scaling factor	Empirical unit forcing from increased global average CO ₂ concentrations determined in Myhre et al. (1998).
		$\text{CO}_{2,\text{ref}}$	[ppm]	K	Reference concentration	Empirical global average reference CO ₂ concentration determined in Myhre et al. (1998).
$B1$	$\mathbb{R}^N$	B	[W/(m ² °C)]	T	Radiative damping	Linearized OLR slope from Budyko (1969). This parameter controls the strength of temperature-dependent radiative cooling.
L	$\mathbb{R}^{N \times N}$	D_0	[W/(m ² K)]	T	Diffusion constant	EBM representation of the global thermal conductivity.
		η	—	T	Regional multiplier	Distinct multipliers embodying the spatially heterogeneous model of thermal conductivity.



Table D2. State-space representations of albedo models and operators: discrete dimensions, associated model quantities, units, sources of determination, and physical interpretations. Sources of determination are denoted by E, K, and T, corresponding to estimation, known parameterization, and tuning, respectively.

Albedo Model	State-Space Component	Discrete Dimension	Supporting Parameter	Source	Represents	Notes
Polynomial	ρ_i	$\mathbb{R}^b$	ρ	E	Regression Coefficient	Coefficients calculated from least squares regression. Solutions correspond to both cases of (27) and (28).
Polynomial	Φ_i	$\mathbb{R}^{M'_i \times b}$	–	E	Local Basis	Evaluation of polynomial basis defined by regressor case.
Gaussian KRR	$\hat{\mathbf{y}}_{\text{inf}}$	$\mathbb{R}^N$	ℓ	K	Smoothing strength	Numerical strength of tridiagonal smoothing between latitudinal predictors.
			γ	T	Kernel bandwidth	Defines the radius of the RBF kernel in (19). One value assumed for all kernels.
			λ	T	Regularization parameter	Numerical penalty on overfitting by kernel model. One value assumed for all kernels.
Constant	α	$\mathbb{R}^N$	α	K	Constant field	Fixed parameterization of albedo held at each latitude under a constant surface assumption.
			ϑ	K	Surface-type fraction	Defines the fraction of the underlying surface corresponding to a prescribed surface type.
Hyperbolic Tangent	α_L	$\mathbb{R}^N$	α_{land}	K	Land albedo	Low assumption of land albedo with no cryospheric cover.
			α_{ocean}	K	Ocean albedo	Low assumption of oceanic albedo with no cryospheric cover.
Hyperbolic Tangent	$w1$	$\mathbb{R}^N$	w	K	Slope parameter	Parameter defining the transition between low and high albedo states globally per K. Sourced from Del Sarto et al. (2024).
Hyperbolic Tangent	$u_{\text{ref}}1$	$\mathbb{R}^N$	u_{ref}	K	Reference temperature	Temperature (K) defining regime of changing albedo. Sourced from Del Sarto et al. (2024).



Table E1. ERA5 reanalysis variable descriptions, units, and applications to case studies. Tab. partitions categorize variables into surface, temperature, and radiative definitions.

Variable	Units	Description	Purpose
<i>lsm</i>	—	Land-sea mask identifying the underlying surface type. A value of zero denotes a purely oceanic surface, while a value of one denotes a purely land surface. Entries of 0 indicate oceanic surfaces and 1 indicate land surfaces.	The mask definition at 00:00 on 1 January 2001 defines the surface types as either being land or sea. This defines the surface fractions of ocean and land surfaces at each latitudinal band.
<i>sd</i>	[m]	Snow depth in meters of water equivalent	The mask definition at 00:00 on 1 January 2001 defines the snow surface types in the constant albedo case
<i>siconc</i>	—	Mask denoting the fraction of sea ice coverage on the underlying ocean surface.	This mask definition is evaluated from 1 January 2001 – 31 December 2005, defining the cycle of heat capacity changes. The map at 00:00 defines the sea ice coverage of the constant albedo field.
<i>skt</i>	[K]	Theoretical temperature of the uppermost Earth surface layer required to satisfy the surface energy balance.	Defines the target temperature used in model tuning, regression, and performance evaluations.
<i>tisr</i>	[J/m ²]	Total instantaneous incident solar radiation entering the Earth.	Defines TOA albedo at a given time. Defines insolation in regression models.
<i>tsr</i>	[J/m ²]	Total instantaneous net solar radiation entering the Earth.	Defines TOA albedo at a given time.

Table E2. Surface-type parameterizations for ERA5 identifiers, constant albedo, and heat capacity. Polynomial $P_2(\varphi)$ is defined in (32); heat capacity values follow Ziegler and Rehfeld (2021).

Surface Type	ERA5 Identifier	Constant Albedo	Heat Capacity, MJ/(m ² K)
Land	$lsm \geq 0.5$	$0.32 + 0.05P_2(\varphi)$	2.66
Ocean	$lsm < 0.5$	$0.29 + 0.08P_2(\varphi)$	395.26
Sea Ice	$siconc > 0.2$	0.6	5.62
Snow	$sd > 0.01$	0.7	2.66 ^a

^a We assume equivalent heat capacities for land and snow.



Table E3. Summary of known and prescribed EBM parameters.

Parameter	Value	Units	Notes
N	721	–	Number of latitudes from surface grid in Hersbach et al. (2023).
N_d	6	–	Number of regional multipliers from (13).
s_0	1361	W/m ²	Solar constant from Kopp and Lean (2011).
ς_0	1.796	rad	Longitude of Earth's perihelion from Laskar et al. (2004).
ϵ	0.0167	–	Eccentricity of Earth's orbit from Laskar et al. (2004).
ϖ	0.409	rad	Earth's obliquity angle from Laskar et al. (2004).
S_{CO_2}	5.35	W/m ²	Scaling factor for CO ₂ forcing from Myhre et al. (1998).
CO _{2,ref}	315	ppm	Reference atmospheric CO ₂ concentration from Myhre et al. (1998).

Table E4. Summary of known and prescribed albedo parameters.

Parameter	Value	Units	Notes
ℓ	0.3	–	Smoothing strength of $\mathbf{L}_s$.
α_{PN}	0.7	–	Polar night placeholder albedo for candidate albedo models.
α_{land}	0.3	–	Low assumption of land surface albedo in (34).
α_{ocean}	0.2	–	Low assumption of ocean surface albedo in (34).
w	0.1	K ^{−1}	Slope parameter of hyperbolic tangent benchmark in Del Sarto et al. (2024).
u_{ref}	275	K	Reference temperature of hyperbolic tangent benchmark in Del Sarto et al. (2024).



1045 *Code and data availability.* All radiative, surface, and temperature fields used in this study were obtained from the ERA5 data set “ERA5 hourly data on single levels from 1940 to present,” available at <https://doi.org/10.24381/cds.adbb2d47> Hersbach et al. (2023). Atmospheric CO₂ concentrations were prescribed using Mauna Loa observations from the NOAA Global Monitoring Laboratory Trends in Atmospheric Carbon Dioxide data portal, available at <https://gml.noaa.gov/ccgg/trends/data.html> Lan and Keeling (2026). All code supporting the results in this manuscript is archived on Zenodo at <https://doi.org/10.5281/zenodo.20967880> Blair et al. (2026). The archive includes scripts for downloading and preprocessing the ERA5 and Mauna Loa CO₂ data, implementing the kernel models, running the numerical experiments, and generating the figures presented in this paper.

1050 *Author contributions.* GKB contributed to conceptualization, methodology, software, validation, formal analysis, investigation, visualization, data curation, and writing the original draft. MHK contributed to conceptualization, methodology, investigation, visualization, and writing (review and editing). AFT contributed to conceptualization, methodology, supervision, visualization, and writing (review and editing). RB contributed to supervision and writing (review and editing). SM contributed to supervision and writing (review and editing).

Competing interests. The authors declare that they have no conflict of interest.

1055 *Acknowledgements.* The authors acknowledge the use of LLMs for the proofreading of the manuscript, giving comments on the consistency of the mathematical notation, and the coding. All LLM-generated code was manually checked and verified for correctness.



References

- Abdelmoaty, H. M., Papalexiou, S. M., Rajulapati, C. R., and AghaKouchak, A.: Biases Beyond the Mean in CMIP6 Extreme Precipitation: A Global Investigation, *Earth's Future*, 9, e2021EF002196, <https://doi.org/10.1029/2021EF002196>, 2021.
- Berger, A.: Long-Term Variations of Daily Insolation and Quaternary Climatic Changes, *Journal of Atmospheric Sciences*, 35, 2362 – 2367, [https://doi.org/10.1175/1520-0469\(1978\)035<2362:LTVODI>2.0.CO;2](https://doi.org/10.1175/1520-0469(1978)035<2362:LTVODI>2.0.CO;2), 1978.
- Blair, G., Kasma, M., Taha, A., Bennartz, R., and Mahadevan, S.: Sunlit and Unlit: The Limitations of Albedo Prediction in Energy Balance Models, <https://doi.org/10.5281/zenodo.20967880>, 2026.
- Booth, A. R., Goodwin, P., and Cael, B. B.: Ice sheet-albedo feedback estimated from most recent deglaciation, *Geophysical Research Letters*, 51, e2024GL109953, <https://doi.org/10.1029/2024GL109953>, 2024.
- Budyko, M. I.: The effect of solar radiation variations on the climate of the Earth, *Tellus*, 21, 611–619, <https://doi.org/10.1111/j.2153-3490.1969.tb00466.x>, 1969.
- Chang, C.-Y. and Merlis, T. M.: The Role of Diffusivity Changes on the Pattern of Warming in Energy Balance Models, *Journal of Climate*, 36, 7993–8006, <https://doi.org/10.1175/JCLI-D-23-0121.1>, 2023.
- Del Sarto, G., Bröcker, J., Flandoli, F., and Kuna, T.: Variational techniques for a one-dimensional energy balance model, *Nonlinear Processes in Geophysics*, 31, 137–150, <https://doi.org/10.5194/npg-31-137-2024>, 2024.
- Diamond, R., Schroeder, D., Sime, L. C., Ridley, J., and Feltham, D.: The significance of the melt-pond scheme in a CMIP6 global climate model, *Journal of Climate*, 37, 249–268, <https://doi.org/10.1175/JCLI-D-22-0902.1>, 2024.
- Hersbach, H., Bell, B., Berrisford, P., Biavati, G., Horányi, A., Muñoz Sabater, J., Nicolas, J., Peubey, C., Radu, R., Rozum, I., Schepers, D., Simmons, A., Soci, C., Dee, D., and Thépaut, J.-N.: ERA5 hourly data on single levels from 1940 to present, <https://doi.org/10.24381/cds.adbb2d47>, 2023.
- Hoerl, A. E. and Kennard, R. W.: Ridge Regression: Biased Estimation for Nonorthogonal Problems, *Technometrics*, 12, 55–67, <https://doi.org/10.1080/00401706.1970.10488634>, 1970.
- Huang, J. and Bowman, K. P.: A Semianalytic Climate Model with Multiple Equilibria, *Journal of the Atmospheric Sciences*, 49, 831–842, [https://doi.org/10.1175/1520-0469\(1992\)049<0831:ASCMWM>2.0.CO;2](https://doi.org/10.1175/1520-0469(1992)049<0831:ASCMWM>2.0.CO;2), 1992.
- Hunke, E. C., Hebert, D. A., and Lecomte, O.: Level-ice melt ponds in the Los Alamos sea ice model, *CICE, Ocean Modelling*, 71, 26–42, <https://doi.org/10.1016/j.ocemod.2012.11.008>, 2013.
- Hunt, B. G.: An Investigation with a General Circulation Model of the Climatic Effects of Cloud Albedo Changes Caused by Atmospheric Pollution, *Journal of Applied Meteorology and Climatology*, 21, 1071 – 1079, [https://doi.org/10.1175/1520-0450\(1982\)021<1071:AIWAGC>2.0.CO;2](https://doi.org/10.1175/1520-0450(1982)021<1071:AIWAGC>2.0.CO;2), 1982.
- Jones, C. G., Gregory, J. M., Thorpe, R. B., Cox, P. M., Murphy, J. M., Sexton, D. M. H., and Valdes, P. J.: Systematic optimisation and climate simulation of FAMOUS, a fast version of HadCM3, *Climate Dynamics*, 25, 189–204, <https://doi.org/10.1007/s00382-005-0045-8>, 2005.
- Keeling, C. D., Bacastow, R. B., Bainbridge, A. E., Ekdahl, C. A., Guenther, P. R., Waterman, L. S., and Chin, J. F. S.: Atmospheric carbon dioxide variations at Mauna Loa Observatory, Hawaii, *Tellus*, 28, 538–551, <https://doi.org/10.3402/tellusa.v28i6.11301>, 1976.
- Kopp, G. and Lean, J. L.: A new, lower value of total solar irradiance: Evidence and climate significance, *Geophysical Research Letters*, 38, <https://doi.org/https://doi.org/10.1029/2010GL045777>, 2011.



- Krebs-Kanzow, U., Gierz, P., Rodehacke, C. B., Xu, S., Yang, H., and Lohmann, G.: The diurnal Energy Balance Model (dEBM): a convenient surface mass balance solution for ice sheets in Earth system modeling, *The Cryosphere*, 15, 2295–2313, <https://doi.org/10.5194/tc-15-2295-2021>, 2021.
- 1095 Lan, X. and Keeling, R.: Trends in Atmospheric Carbon Dioxide: Mauna Loa CO₂ Data, NOAA Global Monitoring Laboratory and Scripps Institution of Oceanography, <https://gml.noaa.gov/ccgg/trends/data.html>, 2026.
- Laskar, J., Robutel, P., Joutel, F., Gastineau, M., Correia, A. C. M., and Levrard, B.: A long-term numerical solution for the insolation quantities of the Earth, *Astronomy & Astrophysics*, 428, 261–285, <https://doi.org/10.1051/0004-6361:20041335>, 2004.
- Li, Z. Q. and Garand, L.: Estimation of surface albedo from space: A parameterization for global application, *Journal of Geophysical Research: Atmospheres*, 99, 8335–8350, <https://doi.org/10.1029/94JD00225>, 1994.
- 1100 Liang, S., Strahler, A. H., and Walthall, C.: Retrieval of land surface albedo from satellite observations: A simulation study with a neural network, *Remote Sensing of Environment*, 69, 243–255, [https://doi.org/10.1016/S0034-4257\(98\)00112-8](https://doi.org/10.1016/S0034-4257(98)00112-8), 1999.
- Long, Z., Lu, Y., Ma, X., and Dong, B.: PDE-Net: Learning PDEs from Data, <https://arxiv.org/abs/1710.09668>, 2018.
- Lu, J., Zhou, W., Kong, H., Leung, L. R., Harrop, B., and Song, F.: On the Diffusivity of Moist Static Energy and Implications for the Polar Amplification Response to Climate Warming, *Journal of Climate*, 35, 7127 – 7146, <https://doi.org/10.1175/JCLI-D-21-0721.1>, 2022.
- 1105 Myhre, G., Highwood, E. J., Shine, K. P., and Stordal, F.: New estimates of radiative forcing due to well-mixed greenhouse gases, *Geophysical Research Letters*, 25, 2715–2718, <https://doi.org/10.1029/98GL01908>, 1998.
- Möller, M. and Möller, R.: Modeling glacier-surface albedo across Svalbard for the 1979–2015 period: The HiRSvaC500- α data set, *Journal of Advances in Modeling Earth Systems*, 9, 404–422, <https://doi.org/https://doi.org/10.1002/2016MS000752>, 2017.
- 1110 North, G. R.: Theory of Energy-Balance Climate Models, *Journal of the Atmospheric Sciences*, 32, 2033–2043, [https://doi.org/10.1175/1520-0469\(1975\)032<2033:TOEBCM>2.0.CO;2](https://doi.org/10.1175/1520-0469(1975)032<2033:TOEBCM>2.0.CO;2), 1975.
- North, G. R.: The small ice cap instability in diffusive climate models, *Journal of the Atmospheric Sciences*, 41, 3390–3395, [https://doi.org/10.1175/1520-0469\(1984\)041<3390:TSICII>2.0.CO;2](https://doi.org/10.1175/1520-0469(1984)041<3390:TSICII>2.0.CO;2), 1984.
- North, G. R. and Kim, K.-Y.: *Energy Balance Climate Models*, Wiley, Hoboken, NJ, ISBN 978-1-118-93980-2, 2017.
- 1115 Otto-Bliesner, B. L., Braconnot, P., Harrison, S. P., Lunt, D. J., Abe-Ouchi, A., Albani, S., Bartlein, P. J., Capron, E., Carlson, A. E., Dutton, A., Fischer, H., Goelzer, H., Govin, A., Haywood, A., Joos, F., LeGrande, A. N., Lipscomb, W. H., Lohmann, G., Mahowald, N., Nehrbass-Ahles, C., Pausata, F. S. R., Peterschmitt, J.-Y., Phipps, S. J., Renssen, H., and Zhang, Q.: The PMIP4 contribution to CMIP6 – Part 2: Two interglacials, scientific objective and experimental design for Holocene and Last Interglacial simulations, *Geoscientific Model Development*, 10, 3979–4003, <https://doi.org/10.5194/gmd-10-3979-2017>, 2017.
- 1120 Petty, G. W.: *A First Course in Atmospheric Radiation*, Sundog Publishing, Madison, WI, 2nd edn., ISBN 978-0-9729033-1-8, 2006.
- Pistone, K., Eisenman, I., and Ramanathan, V.: Observational determination of albedo decrease caused by vanishing Arctic sea ice, *Proceedings of the National Academy of Sciences*, 111, 3322–3326, <https://doi.org/10.1073/pnas.1318201111>, 2014.
- Rohling, E. J., Sluijs, A., Dijkstra, H. A., Köhler, P., van de Wal, R. S. W., Beerling, D. J., Berger, A., Bijl, P. K., Crucifix, M., DeConto, R. M., Drijfhout, S. S., Fedorov, A. V., Foster, G. L., Ganopolski, A., Hansen, J., Hönlisch, B., Hooghiemstra, H., Huber, M., and et al.: Making sense of palaeoclimate sensitivity, *Nature*, 491, 683–691, <https://doi.org/10.1038/nature11574>, 2012.
- 1125 Rowe, C. M.: Global land-surface albedo modelling, *International Journal of Climatology*, 13, 473–495, <https://doi.org/https://doi.org/10.1002/joc.3370130502>, 1993.



- Rožanec, J. M., Petelin, G., Costa, J., Cerar, G., Bertalaníč, B., Guček, M., Papa, G., and Mladeníč, D.: Dealing with zero-inflated data: Achieving state-of-the-art with a two-fold machine learning approach, *Engineering Applications of Artificial Intelligence*, 149, 110339, <https://doi.org/10.1016/j.engappai.2025.110339>, 2025.
- Samuelsberg, A. and Jakobsen, P. K.: Solving a North-type energy balance model using boundary integral methods, *Nonlinear Processes in Geophysics*, 32, 23–33, <https://doi.org/10.5194/npg-32-23-2025>, 2025.
- Schölkopf, B. and Smola, A. J.: *Learning with Kernels: Support Vector Machines, Regularization, Optimization, and Beyond*, MIT Press, 2002.
- Sellers, W. D.: A global climatic model based on the energy balance of the Earth–atmosphere system, *Journal of Applied Meteorology*, 8, 392–400, [https://doi.org/10.1175/1520-0450\(1969\)008<0392:AGCMBO>2.0.CO;2](https://doi.org/10.1175/1520-0450(1969)008<0392:AGCMBO>2.0.CO;2), 1969.
- Smith, R. S. and Gregory, J.: The last glacial cycle: Transient simulations with an AOGCM, *Climate Dynamics*, 38, 1545–1559, <https://doi.org/10.1007/s00382-011-1283-y>, 2012.
- Song, J., Huang, D., and Harlim, J.: Learning solution operator of dynamical systems with diffusion maps kernel ridge regression, version Number: 2, 2025.
- Stocker, T. F.: *Introduction to Climate Modeling*, Springer Textbooks in Earth Sciences, Geography and Environment, Springer, Berlin, Heidelberg, ISBN 978-3-642-00799-0, 2011.
- Thackeray, C. W. and Hall, A.: An emergent constraint on future Arctic sea-ice albedo feedback, *Nature Climate Change*, 9, 972–978, <https://doi.org/10.1038/s41558-019-0619-1>, 2019.
- Tilmes, S., Richter, J. H., Kravitz, B., MacMartin, D. G., Mills, M. J., Simpson, I. R., Glanville, A., Fasullo, J., Phillips, A., Lamarque, J.-F., Tribbia, J., Edwards, J., Mickelson, S., and Gosh, S.: CESM1(WACCM) Stratospheric Aerosol Geoengineering Large Ensemble (GLENS) Project, *Bulletin of the American Meteorological Society*, 99, 2361–2371, <https://doi.org/10.1175/BAMS-D-17-0267.1>, 2018.
- Zhang, G., Zhou, H., Wang, C., Xue, H., Wang, J., and Wan, H.: Forecasting Time Series Albedo Using NARnet Based on EEMD Decomposition, *IEEE Transactions on Geoscience and Remote Sensing*, 58, 3544–3557, <https://doi.org/10.1109/TGRS.2019.2958048>, 2020.
- Zhang, T., Yao, Z., Gholami, A., Keutzer, K., Gonzalez, J., Biros, G., and Mahoney, M.: ANODEV2: A Coupled Neural ODE Evolution Framework, <https://arxiv.org/abs/1906.04596>, 2019.
- Zhuang, K., North, G. R., and Stevens, M. J.: A NetCDF Version of the Two-Dimensional Energy Balance Model Based on the Full Multigrid Algorithm, *SoftwareX*, 6, 198–202, <https://doi.org/10.1016/j.softx.2017.07.003>, 2017.
- Ziegler, E. and Rehfeld, K.: TransEBM v. 1.0: description, tuning, and validation of a transient model of the Earth’s energy balance in two dimensions, *Geoscientific Model Development*, 14, 2843–2866, <https://doi.org/10.5194/gmd-14-2843-2021>, 2021.