

The authors have addressed all of my comments satisfactorily, with the exception of my main concern regarding the use and interpretation of the MCMC inversion framework.

The authors use Equations 5–7 to invert their samples in an underdetermined problem. As expected, the posterior distributions show strong correlations among parameters (for example, in Figures 5, 6, and S3). In this context, I do not understand what additional insight is gained from the MCMC inversion framework relative to direct exploration of the analytical solution. Since the authors present the analytical equation (Eq. 5), it seems that they could instead solve this equation directly over the range of parameters deemed reasonable. The resulting parameter trade-offs would then define the family of admissible solutions consistent with the measured nuclide concentrations.

My concern is therefore not necessarily with the use of MCMC itself, but with the interpretation of the posterior distributions in a problem that appears to be only partially identifiable. In such cases, MCMC may still be useful for mapping the admissible solution space, but posterior density variations should not automatically be interpreted as evidence for uniquely preferred or physically meaningful solutions. Instead, apparent maxima may arise from posterior geometry, prior truncation, or differences in parameter sensitivity and scaling, rather than from genuine resolution of the underlying parameters.

I suspect this may be happening in the present manuscript, which is otherwise well executed. I illustrate this concern with a figure I recently generated using an almost identical inversion framework.

Using a formulation equivalent to Equation 5, but simulating a step change in erosion from E_1 to E_2 at time t , I calculated the analytical solution for the combinations of parameters that explain the in situ ^{14}C and ^{10}Be concentrations of a single sample. I then ran a Hamiltonian gradient-based MCMC to estimate the posterior distribution. Similar to the results presented by the authors in their rebuttal, parameter recovery is generally good in the sense that the posterior follows the analytical solution closely. However, the posterior nevertheless shows a clear posterior mode. Based on the analytical solution, we know that this mode is not evidence of unique parameter identification. Rather, it is better interpreted as a pseudo-mode arising in a practically non-identifiable, or only partially identifiable, model with ridge-like geometry, potentially amplified by prior truncation and by differences in parameter sensitivity or scaling (e.g., the extent to which a change in one parameter affects the solution relative to a change in another) (Raue et al., 2009, 2013).

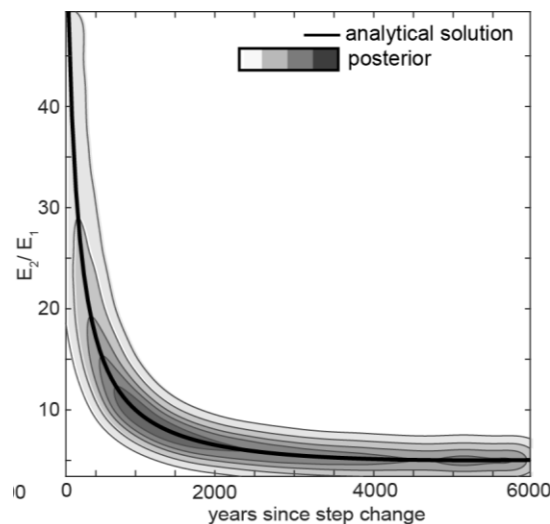


Figure: Ratio of old erosion rate (E_1) over new erosion rate (E_2) for a step change in erosion at different times, showing both the analytical solution and the MCMC posterior distribution.

A similar behaviour may be present in Figures 5 and 6 of the main text, and in Figure S3, which looks very similar to the example shown here. Consistent with these known issues in non-identifiable inverse problems, the posterior distributions in Figure 4 span much of the prior space, yet still show maxima that the authors interpret as meaningful. Based on the inversion framework presented, however, I do not see sufficient justification for treating these maxima as physically interpretable regions of higher support. Even if the sampler concentrates in some regions of parameter space, this concentration should not automatically be interpreted as evidence for preferred solutions when the problem remains practically non-identifiable.

My concern, therefore, is that several of the interpretations derived from the posterior distributions may reflect inversion artefacts rather than features uniquely supported by the data. I therefore encourage the authors either to justify more explicitly why the MCMC framework provides additional insight beyond direct exploration of the analytical solution space, or to change the interpretation of posterior maxima and frame the results primarily in terms of admissible parameter trade-offs rather than preferred parameter values.

References:

- Raue, A., Kreutz, C., Maiwald, T., Bachmann, J., Schilling, M., Klingmüller, U., & Timmer, J. (2009). Structural and practical identifiability analysis of partially observed dynamical models by exploiting the profile likelihood. *Bioinformatics*, 25(15), 1923–1929. <https://doi.org/10.1093/bioinformatics/btp358>
- Raue, A., Kreutz, C., Theis, F. J., & Timmer, J. (2013). Joining forces of Bayesian and frequentist methodology: A study for inference in the presence of non-identifiability. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 371(1984), 20110544. <https://doi.org/10.1098/rsta.2011.0544>