Review: Brief communication: Estimating diffusion length from low resolution data by Shaw et al.

I. Overview – Significance of the work

The manuscript by Shaw et al. makes a relevant and significant contribution to the study of signal loss and reconstruction in ice-core water-isotope records. It is well suited to *The Cryosphere* and the Oldest Ice special collection. The mathematical approach is clearly presented, and the modelling experiments are informative and well designed. The manuscript is also well written and clearly organised.

It would be worthwhile, however, for the authors to consider that their target audience consists mainly of ice-core researchers dealing with the interpretation of ice-core data. Despite its clean structure and presentation, the manuscript is not an easy read for the average reader in this audience, who will typically not have a very strong mathematical background. It reads much more as a mathematical and theoretical paper, with too little ice-core context.

Additionally, the practical applicability of the enhanced correction appears rather narrow and concerns a specific case that is not often encountered in ice cores: a small diffusion length combined with a relatively low-resolution sampling scheme.

At the same time, although the fitting method accounts for gamma-distributed errors in the spectral estimates and Figure 3 shows the spread among the simulations, the practical implications of uncertainty in the spectral estimation and model fitting are not discussed in sufficient depth. In my view, the paper would benefit enormously from

- presenting the mathematics in a more approachable way;
- better describing the evolution of the total diffusion length with depth in an ice core;
- providing quantitative estimates for combinations of diffusion length and sampling interval that are representative of real ice-core examples, and comparing them with the existing correction approaches; and
- using real ice-core data to demonstrate the practical significance of the correction.

My recommendation is that the authors consider resubmitting the paper as a regular article rather than as a Brief Communication. This work presents a relevant correction scheme and, in my view, requires a more elaborate presentation and testing, which may be difficult to accommodate within the scope of a Brief Communication.

II. Items to consider for revisions

A. Novelty and state-of-the-art statements

The manuscript needs to address more clearly the state of the art and the actual novelty of the correction. Already in the abstract, the authors state that 'established approaches do not account for additional smoothing and aliasing introduced by discrete sampling'. This is inaccurate with respect to the additional smoothing component, as the manuscript itself acknowledges later in the text. Gkinis et al. [1] already described the loss of resolution caused by discrete 5 cm sampling as an additional smoothing variance and also quantified the smoothing introduced by a CFA measurement system through its transfer function. Gkinis et al. [2] subsequently introduced an explicit correction in which the rectangular sampling filter is approximated by an equivalent Gaussian filter and its contribution is subtracted from the estimated diffusion length. The same sampling correction was later used and described in more detail by Holme et al. [3].

The novelty of the present manuscript therefore appears to lie in representing the rectangular filter by its exact sinc response within the spectral model and, importantly, in explicitly accounting for aliasing. This is a useful advance over the Gaussian approximation, but it should be distinguished from the more general idea of accounting for finite-sample smoothing, which is already established in the literature.

a) Suggestion: Consider revising the abstract and introduction to acknowledge the existing finite-sample smoothing corrections and define the novelty specifically as the exact boxcar response and explicit aliasing term.

B. Clarification of the parameter a

The parameter a describes both the sample width and the spacing between the centres of the samples. It is central to the mathematical treatment. It would probably be clearer to express these two quantities using different variables and explain that, for contiguous discrete ice-core samples, they are equal because the samples are adjacent to each other.

a) Question: Consider introducing separate variables for the sample width and centre-to-centre spacing, stating explicitly where these quantities are assumed to be equal, and clarifying whether the method also applies when they differ?

C. Ice-flow thinning and solid-ice diffusion

What I miss as a reader is a clearer explanation of how the diffusion length evolves as a function of depth. Ice-flow thinning S and solid-ice diffusion are major controls on the value to be estimated. I think the following equation should be included, possibly together with a theoretical profile from a core such as Dome C:

$$\sigma_i^2(z) = S^2(z)\sigma_{\mathrm{firn}}^2 + \sigma_{\mathrm{ice}}^2(z). \tag{1}$$

For the particular case of paleotemperature estimation, the parameter of interest is σ_{firn}. This means that uncertainties in thinning and ice diffusion will play a significant role in the overall uncertainty of σ_{firn}.

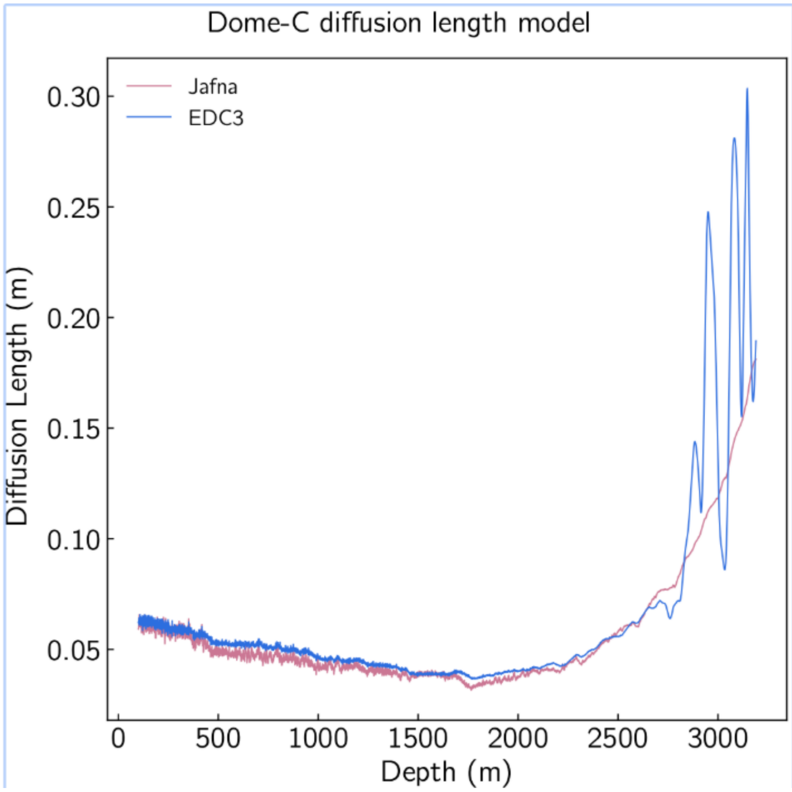

Fig. 1. Modelled evolution of the diffusion length with depth at Dome C.

Looking at the profile for Dome C and assuming a typical 2.5 cm sampling resolution, we can see that the regime in which the aliasing correction becomes important is quite restricted. The minimum diffusion length only marginally reaches approximately 2.5–3 cm, after which ice diffusion becomes dominant.

a) Suggestion: Consider expanding the physical discussion with the relation above showing how thinning and solid-ice diffusion control the total diffusion length and its minimum, and how their uncertainties affect the recovery of σ_{firn} for paleotemperature applications.

D. A contour plot could be useful

A contour plot could provide a more nuanced picture. The x-axis could show diffusion length and the y-axis the sampling width. The plotted quantity could be the normalised power at the Nyquist frequency, $P_{\mathrm{meas}}(f_N)/P_0$ or only tha Alias term. One could assume a white-noise signal ($P_{X_0}(f) = P_0$), zero analytical noise ($P_n = 0$), and contiguous samples.

At the Nyquist frequency, this gives

$$
\begin{aligned}
P_{\mathrm{meas}}(f_N) &= P_{\mathrm{direct}}(f_N) + P_{\mathrm{alias}}(f_N) \\
&= P_0 \frac{4}{\pi^2} \exp\left[-\left(\frac{\pi\sigma}{a}\right)^2\right] + P_0 \frac{4}{\pi^2} \exp\left[-\left(\frac{\pi\sigma}{a}\right)^2\right] \\
&= P_0 \frac{8}{\pi^2} \exp\left[-\left(\frac{\pi\sigma}{a}\right)^2\right].
\end{aligned} \tag{2}
$$

This illustrates the contribution of the direct and alias terms at the Nyquist frequency. If we take the example of $\sigma = 5$ cm and $a = 2.5$ cm, the corresponding ratio, $P_{\mathrm{meas}}(f_N)/P_0$ is of the order of 10^{-18}. Thus, although the direct and alias terms are equal at the Nyquist frequency in this simplified case, their absolute contribution is extremely small. The aliasing term can therefore be neglected at the Nyquist frequency in this example.

Such a contour plot could provide an intuitive picture of the importance of the sampling and aliasing effects and could also serve as a useful guideline for sampling strategies in ice-core projects.

a) Suggestion: Include a contour plot of $P_{\mathrm{meas}}(f_N)/P_0$ (or only the Alias term) as a function of diffusion length and sampling width.

E. Real-data application

A real-data application would strengthen the manuscript considerably and test the methods better. Uncertainties in measurements noise, spectral estimation, sample cutting variability can be assessed as a whole by applying the corrections on real ice core data.

I think the early-Holocene NEEM data presented by Holme et al. [3] could be a useful benchmark. These data include overlapping discrete and CFA records from approximately the same depth interval, sampled at 5.0 and 0.5 cm, respectively.

Holme et al. showed that the noise part of the spectrum is much better resolved in the CFA record. With the 5 cm discrete data, the resolution was not sufficient to describe the noise clearly. At the same time, after correcting for sampling, ice diffusion, and thinning, the resulting firn diffusion lengths agree very well. For δ^{18}O, they are 10.33 ± 0.19 cm for the discrete record and 10.27 ± 0.19 cm for the CFA record. For δD, the corresponding values are 9.72 ± 0.20 and 9.65 ± 0.18 cm.

The early-Holocene Dome C section in Holme et al. [3] could be a second example. It covers 308–318 m depth at approximately 9.9 ka and was sampled at 2.5 cm resolution. The corrected firn diffusion lengths are 6.97 ± 0.15 cm for δ^{18}O and 6.34 ± 0.08 cm for δD.

The conventional fit, the Gaussian approximation used by Gkinis et al. [2] and Holme et al. [3], and the new method could be applied to exactly the same isotope signal (δ^{18}O or δD). Even better, estimates on the firn diffusion length could be obtained with all three methods and by applying Eq. 1

a) Suggestion: Consider adding a real-data benchmark based on the overlapping NEEM records and the early-Holocene Dome C section of Holme et al. [3], using Dome C as a contrasting case in which the new correction should be small.

III. Specific comments

I have no specific comments at this point - the more general revision i think is essential for the manuscript. I would be happy to provide my feedback to a revised version.

References

[1] V. Gkinis, T. J. Popp, T. Blunier, M. Bigler, S. Schupbach, E. Kettner, and S. J. Johnsen, "Water isotopic ratios from a continuously melted ice core sample," *Atmos. Meas. Tech.*, vol. 4, no. 11, pp. 2531–2542, 2011.

[2] V. Gkinis, S. B. Simonsen, S. L. Buchardt, J. W. C. White, and B. M. Vinther, "Water isotope diffusion rates from the NorthGRIP ice core for the last 16,000 years - Glaciological and paleoclimatic implications," *Earth and Planetary Science Letters*, vol. 405, no. 0, pp. 132–141, 2014.

[3] C. Holme, V. Gkinis, and B. M. Vinther, "Molecular diffusion of stable water isotopes in polar firn as a proxy for past temperatures," *Geochim. Cosmochim. Acta*, vol. 225, pp. 128–145, 2018.