To tip or not to tip

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Abstract. Tipping points have become a buzzword in Earth sciences, but ambiguous or overly narrow definitions of tipping are causing confusion around the concept. Agreeing on what tipping means, and whether a system tips or not, is important for robust science and communicating tipping risk. Based on a critical evaluation of existing tipping definitions, we propose a revised, general definition that characterizes a tipping event as a persistent nonlinear transition in forced systems. Our definition emphasizes both the phenomenology (observed time series) and cause (feedback mechanism) of a tipping event. Inspired by response theory, our proposition is compatible with more specific mathematical formulations while avoiding challenging notions such as bifurcations, equilibrium states, abruptness and irreversibility – making the definition testable also on transient dynamics in diverse complex systems under time-varying forcing. We showcase its practical use and limitations in a toy model and a case study of Earth system model data.

1 Introduction

Fifty years ago, René Thom's catastrophe theory was popularized as a mathematical framework capable of explaining sudden, abrupt changes in a wide range of real-world contexts (Thom, 1974; Zeeman, 1976). The theory describes such changes, or "catastrophes", in terms of singularities where a control parameter reaches a critical point. Seemingly a universal and powerful tool, the theory received great attention in science and beyond, finding application in diverse fields from biology (Thom, 1969) to economics (Zeeman, 1974), psychology (Stewart and Peregoy, 1983) and even linguistics (Wildgen, 1981).

However, soon after the initial rise, criticism grew regarding the theory's lack of justification and predictive power in applications (Zahler and Sussmann, 1977; Kolata, 1977). Dissonance emerged between the rather narrow mathematical assumptions underlying catastrophe theory and the complexity of systems under study. Arguably, loose applications of the theory and the difficulty of testing its hypotheses with empirical evidence eventually led to its demise (Rosser, 2007).

In recent years, a theory of tipping ~~behavior~~ dynamics has been developed to describe and understand critical change, centered around the concept of tipping points. Much of the mathematical formalism is rooted in bifurcation theory of dynamical systems (Ashwin et al., 2012; Kuehn, 2011), which is closely related to concepts of catastrophe theory (Arnol'd and Gamkrelidze, 1994). Popularized by Gladwell (2000), the term 'tipping point' has been adopted in various scientific disciplines – sometimes referring to a well-defined mathematical scenario while at other times invoked rather as a metaphor for large, rapid change (Russill and Nyssa, 2009; Lohmann, 2026a). Also here, concerns about overuse, ambiguity and limited applicability of the concept are increasingly being raised ~~(Kopp et al., 2025a, b; Milkoreit, 2023; Shaw and Stevens, 2025; Russill, 2015)~~

(Russill, 2015; van Nes et al., 2016; Milkoreit, 2023; Kopp et al., 2025a, b; Shaw and Stevens, 2025; Hansson et al., 2026). If not sufficiently clarified, there is a danger that tipping theory will suffer the same fate as catastrophe theory.

Since their introduction to Earth system science (Lenton et al., 2008), 'climate tipping points' have ~~been a fast growing~~ become a fast-growing research topic as well as a buzzword in climate change communication to the public ~~(Armstrong McKay et al., 2022; . The Global Tipping Points (GTP) Report 2025~~ (van der Hel et al., 2018; Armstrong McKay et al., 2022; Abrams et al., 2026; Hegerl et al. . The Global Tipping Points (2025) report discusses tipping points in the physical climate system and in society, extending the scope to both natural and socio-economic or technological systems~~(Global Tipping Points, 2025)~~. However, also here a lack of consensus persists about a general definition of tipping points and tipping behavior ~~(Abrams et al., 2026)~~(Abrams et al., 2026; Boers et al., . In light of the ongoing seventh assessment cycle of the Intergovernmental Panel on Climate Change (IPCC), we see a need and opportunity to revisit current tipping definitions, with the aim of establishing a terminology that accommodates both the mathematical formalism and the real-world complexity of tipping phenomena~~(Smith et al., 2025)~~. Agreeing on what tipping means, and whether a system tips or not, is important both for robust science as well as for communicating about nonlinear change.

This paper seeks to provide the discussion of tipping behavior with a clear, practical foundation. While we focus here on the domain of Earth system science, our proposition is more general and intended to be meaningful also in other contexts. Our goal is to define a tipping event – rather than a tipping point – in a way that is empirically testable and avoids overly restrictive assumptions, yet remains compatible with the existing mathematics and intuition of tipping phenomena. In section 2, we critically evaluate current tipping definitions, highlighting key issues. Based on this, we propose a revised definition of tipping and discuss its implications ~~in section 3~~(section 3). We illustrate its connection with bifurcation-, noise-, ~~and~~ rate-induced and overshoot tipping and apply the definition to earth system model simulations of ~~a~~ an example climate tipping system (section 4). ~~A summary and discussion conclude the paper, where we also reflect on~~ Finally, we reflect on the wider applicability and implications of our tipping definition.

2 Problems with existing definitions

The ~~latest Assessment Report of the IPCC~~ glossary of the latest IPCC Assessment Report defines a tipping point as "a critical threshold beyond which a system reorganizes, often abruptly and/or irreversibly" (IPCC, 2023). Other versions have been proposed by Lenton et al. (2008) and updated in the ~~GTP Reports (Global Tipping Points, 2023, 2025)~~ Global Tipping Points (2023, 2025) reports (see also van Nes et al. (2016); Milkoreit et al. (2018); Brovkin et al. (2021); Armstrong McKay et al. (2022); Winkelmann et al. (2025); Abrams et al. (2026)), but we argue in the following that they share common challenges causing confusion and misperceptions. These are mainly a lack of specificity, uniqueness and testability.

2.1 Threshold or event?

Originally, the term 'tipping point' was predominantly used to refer to a threshold, either in an external forcing parameter (control parameter) or in the system's state (Lenton et al., 2008; van Nes et al., 2016). A single, tiny perturbation past the

threshold would cause a drastic change in the long-term behavior of the system. In the case of a forcing threshold, this can be mathematically described as a bifurcation where the stability properties of the system change instantly and qualitatively. In particular, the fold (or saddle-node) bifurcation has become a paradigm of a tipping point, and bifurcation-induced tipping (Ashwin et al., 2012) shapes much of the thinking about tipping dynamics.

The problem is that such bifurcations characterize the equilibrium properties of the system. If the forcing changes slowly compared to internal system timescales (referred to as quasi-equilibrium conditions), the equilibrium description usually yields an adequate approximation of the dynamics: the tipping transition occurs ~~immediately after crossing the tipping point~~at the critical forcing level, and the system change is abrupt and irreversible relative to the forcing change. However, in most situations of interest (including anthropogenic climate change), the system and forcing timescales are not cleanly separated, meaning that the system can be driven far from equilibrium and transient dynamics become important ~~(Caraballo and Han, 2016; Hastings et al., 2018)~~ (Lai and Tél, 2011; Caraballo and Han, 2016; Hastings et al., 2018; Lohmann, 2026b). A common misunderstanding is that crossing a tipping threshold always coincides with observing a tipping event. On the contrary, owing to multiscale dynamics, inertia and chaos, tipping can be significantly delayed after crossing a threshold, avoided under temporary threshold ~~exceedance~~overshoot, or happen even without crossing a forcing threshold ~~(Bastiaansen et al., 2023; Ritchie et al., 2021)~~(Bastiaansen et al., 2023; Ritchie et al., 2

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The concepts of rate-induced (Wieczorek et al., 2023; Ritchie et al., 2023) and noise-induced ~~(Freidlin and Wentzell, 1998)~~ (Freidlin and Wentzell, 1998; Horsthemke and Lefever, 1984) tipping deal with scenarios where the system is pushed out of equilibrium by time-dependent forcing or random variability (Ashwin et al., 2012). Here, the relevant threshold (in multistable systems (Feudel, 2008)) is a basin boundary that separates competing basins of attraction (Feudel, 2023). The basin boundary can be a complicated geometrical object partitioning the state space – a tipping surface[1] rather than a tipping point in systems with more than one dimension. This tipping boundary generally moves in state space as a function of system parameters. Tipping occurs when the basin boundary is crossed without returning. Also bifurcation-induced tipping may be viewed as a special case of ~~crossing a basin boundary, since at a (fold) bifurcation~~ exiting a basin of attraction, provided that it corresponds to a situation where the stable equilibrium itself collides with its basin boundary and thereby disappears (e.g. a fold bifurcation).

Thus, from a dynamical systems perspective, it seems reasonable to generally define tipping as the persistent crossing of a state space ~~threshold~~ boundary partitioning different domains of stability. However, applying this definition in practice requires knowledge of the global stability landscape of the system. This is often not known and the location of basin boundaries cannot simply be derived from observed trajectories of the system.

Given the challenges of identifying critical thresholds in complex systems (Hillebrand et al., 2020; van Westen et al., 2026) , the term 'tipping point' is increasingly used to refer to an event, not a threshold. For example, the ~~GTP Report 2025~~ Global Tipping Points (2025) report defines: a tipping point "occurs when change in part of the system becomes self-perpetuating beyond some threshold [...]"~~(Global Tipping Points, 2025)~~. This shift from threshold to occurrence is widely reflected in current tipping point language, where many use 'tipping point' as a synonym for transition. However, this blurs the important

[1]Note that the tipping surface in *state* space discussed here is different from a tipping surface in *parameter* space (or forcing space) discussed by Abrams et al. (2026).

distinction between the stability properties of the system and its transient dynamics, causing misunderstanding about the timing and predictability of tipping. For example, a change point detected in a timeseries generally does not indicate the location of a critical threshold but is often labeled as the tipping point.

2.2 Abruptness and/or irreversibility

Tipping behavior is considered important because it is associated with large, ~~abrupt, nonlinear~~ rapid and possibly unexpected change that cannot easily be reversed. ~~The~~ Emphasizing this high-impact potential, the notions of abruptness and irreversibility are ~~thus~~ central to many tipping definitions ~~(IPCC, 2023; Global Tipping Points, 2025)~~(IPCC, 2023; Global Tipping Points, 2025; Winkeln . However, ~~it is often not useful to define a concept via its relevance~~ defining a type of dynamics based on relevance or impact bears the risk of being subjective, poorly distinctive and inconsistent with mathematical descriptions. In fact, ~~the two attributes~~ abruptness and irreversibility are neither necessary nor sufficient criteria for tipping, as is evident in the IPCC formulation, "often abrupt and/or irreversible" (IPCC, 2023).

Abruptness implies a rapid change relative to a given timescale, but the choice of reference timescale is unclear. Is it the forcing timescale (Global Tipping Points, 2025), the typical rate of change in the system's history (IPCC, 2023), or a timescale of policy relevance (Lenton et al., 2008)? For example, present-day mass loss of ice sheets may be considered fast compared to glaciological (multi-millennial) timescales but not compared to the current forcing timescale, and is arguably not perceived as abrupt in society. Some tipping events may thus appear slow and gradual, especially relative to changing external influences. Nonetheless, it is possible to distinguish a slow tipping event from other gradual changes ~~, as we will discuss in section 3~~(see section 3). On the other hand, arguably not all abrupt changes should classify as tipping events~~;~~: they could be entirely driven by an abrupt forcing change or represent an extreme event without a 'reorganization' (IPCC, 2023) of the system. Various dynamical mechanisms can produce an abrupt change in an observable despite the absence of critical thresholds or alternative stable states.

Defining irreversibility is an open problem in itself ~~. While some~~ and, like abruptness, requires a clarification of timescales (Buhr et al., 2025). Some call a change irreversible if it is not reversed under reversal of the forcing that caused it (Global Tipping Points, 2025)~~, other definitions are related to the recovery rate of a system after a perturbation (IPCC, 2023). These are two very different aspects in dynamical systems theory, and like abruptness require a clarification of timescales~~. Under quasi-equilibrium conditions, this is known as (static) hysteresis and indicates a forcing range, bounded by a bifurcation, in which the system features multistability. One major problem with linking tipping to irreversibility is that even a linear system (which cannot have tipping points) can be irreversible under forcing reversal if it has inertia. ~~This~~ On a finite time horizon, transient forcing that interferes with the system's internal damping timescales can result in spurious hysteresis effects ~~under forcing rates faster than or similar to the internal damping timescale. Conversely, there is~~ (Broner et al., 1997; Hankel and Tziperman, 2023) . In the glossary of the IPCC (2023), irreversibility relates to an entirely different property in dynamical systems theory, the recovery rate of a system after a perturbation. There, irreversibility is attributed to a system state rather than a change process, which neglects the importance of path dependence.

Irreversibility on finite timescales does not imply tipping, but does tipping imply irreversibility? There is controversy about whether nonlinear but reversible shifts should be considered tipping events. This question relates to the debate about including Arctic sea ice and permafrost in the list of Earth system tipping elements ~~(Notz, 2009; Li et al., 2013; Levermann et al., 2012; Brovkin et al.~~ (Notz, 2009; Li et al., 2013; Wagner and Eisenman, 2015; Levermann et al., 2012; Brovkin et al., 2025; Armstrong McKay et al., 2022) . As demonstrated in ~~simple, monostable dynamical systems~~ monostable toy models without bifurcations, a nonlinear sensitivity to a control parameter can give rise to ~~tipping-like behavior in the absence of multistability and hysteresis (Scheffer et al., 2001) . Lastly, irreversibility with respect to forcing reversal can only be empirically verified if the forcing is actually reversed, making it difficult to test in practice.~~ relatively abrupt, lasting shifts in the system state that match the classical phenomenology of tipping behavior even though no multistability or hysteresis exists (Scheffer et al., 2001).

In summary, while abruptness and irreversibility can be important features of tipping behavior, they are poorly suited to define tipping dynamics and to distinguish tipping from other forms of change. Moreover, irreversibility with respect to forcing reversal can only be empirically verified if the forcing is actually reversed, making it difficult to test in practice.

2.3 Metastability

The coexistence of multiple stable states (multistability) is central to many tipping definitions ~~(van Nes et al., 2016; Milkoreit et al., 2018)~~ (van Nes et al., 2016; Milkoreit et al., 2018; Boers et al., 2026). The mathematical concepts of bifurcation-, rate-, and noise-induced tipping ~~all require multistability~~ are commonly illustrated using bistable systems (Ashwin et al., 2012). At the same time, the IPCC and ~~GTP~~ Global Tipping Points definitions are more vague, requiring merely a "reorganization" (IPCC, 2023) of the system or "substantial and widespread impacts" (Global Tipping Points, 2025), without mention of an alternative stable state.

Multistability is an important concept for understanding tipping dynamics, but it describes the equilibrium properties of the system – like bifurcations. The term 'metastability' has been introduced to account also for transient dynamics, such as ~~transient chaos or nonautonomous dynamics (Rossi et al., 2025)~~long transients (Morozov et al., 2020), transient chaos (Grebogi et al., 1983; Lai and , excitability (Crucifix, 2012; Pierini and Ghil, 2021) and nonautonomous dynamics (Wieczorek et al., 2011). Metastability does not rely on equilibrium states but more generally refers to distinct, relatively long-lived dynamical regimes and shifts between them ~~.~~ (Rossi et al., 2025; Koch et al., 2024a). An example are ghost states (Koch et al., 2024b), which can lead to climate tipping after a long transient (Börner et al., 2026). Metastability allows to generalize the concept of basins of attraction to domains of state space with attracting or "sticky" dynamics on a finite timescale.

We argue that defining tipping as a transition between different stable equilibrium states makes it difficult to apply the definition to the full variety of tipping phenomena observed in complex systems. As mentioned above, the global stability properties are often not known. In adaptive systems and systems exposed to continuous time-dependent forcing, the notion of equilibrium can become less clear since the system may never settle in an equilibrium. A general tipping definition should accommodate aspects of metastable, transient dynamics and clarify what form of 'reorganization' occurs.

3 ~~Tipping~~ Revised tipping definition

Besides threshold behavior, abruptness, and irreversibility, a common aspect of existing tipping definitions is the self-amplifying, self-sustaining or accelerating nature of the change ~~(Abrams et al., 2026; Armstrong McKay et al., 2022)~~(Abrams et al., 2026; Armstrong M
. This is typically linked to a positive (i.e. destabilizing) feedback mechanism becoming dominant in the system, leading to ~~a nonlinear self-sustained response. We propose to focus on this aspect, taking both~~ an internally driven nonlinear response. Feedback is a widespread concept in the study of natural, social and engineered complex systems (Åström and Murray, 2008; DeAngelis et , central for building conceptual models of tipping dynamics (Dijkstra, 2024; DeAngelis and Waterhouse, 1987; Thomas and d'Ari, 1990) . Importantly, understanding of feedbacks helps link the observed behavior (phenomenology) ~~and knowledge of~~ to the underlying mechanisms (cause)~~into account.~~ . We propose that a robust tipping definition combines both aspects. Phenomenologically, a tipping event is a transition process in the system's state. The cause aspect of tipping requires system-specific knowledge about internal feedback processes and the mechanistic connection between forcing and response.

3.1 Definition

In short, we define a tipping event as a persistent nonlinear transition driven by a self-amplifying feedback process. Persistence means that the system settles in an alternative dynamical regime following the transition. By nonlinear we mean that the system observable responds disproportionately to a driver (nonlinear response), which could be an external forcing or internal fluctuations. The change of the observable is thus not merely a linear ~~function of~~ response to a nonlinear forcing. Self-amplification means that a change in the observable ~~leads to~~ causes further change of the observable in the same direction.

To make this more precise, let $\tau = [t_0, t_1]$ denote a time horizon over which the time series of the system observable $\psi(t)$ and the relevant forcing $F(t)$ are known. This time horizon sets the reference timescale on which tipping is assessed. If tipping occurs (once) during τ, we can conceptually divide τ into a pre-tipping interval $\tau_A = [t_0, t_s]$, a transition phase $\tau_{\text{tr}} = [t_s, t_e]$, and a post-tipping phase $\tau_B = [t_e, t_1]$, where $t_0 < t_s < t_e < t_1$ ~~.~~(Fig. 1). The exact choice of t_s and t_e is not important but τ_{tr} should roughly reflect the transition suspected in the time series of ψ. Furthermore, we require that $t_1 - t_e > P(t_e - t_s)$ for some persistence $P \geq 1$, which means that τ_B should last at least as long as τ_{tr}.

We define that tipping occurs during τ if and only if there exists t_s, t_e such that

1. A persistent nonlinear transition occurs, i.e. the observable ψ
   (a) evolves in a regime of approximately linear response during τ_A;
   (b) increasingly and significantly deviates from linear response during τ_{tr};
   (c) ~~evolves~~ persists in a dynamically distinct (compared to τ_A) regime of approximately linear response during τ_B~~,~~ ~~where τ_B lasts at least as long as τ_{tr} (i.e. $t_1 - t_e > P(t_e - t_s)$ for some persistence $P \geq 1$)~~.
2. The deviation from linear response during τ_{tr} is attributed to a positive feedback process that operates on the timescale τ_{tr}.

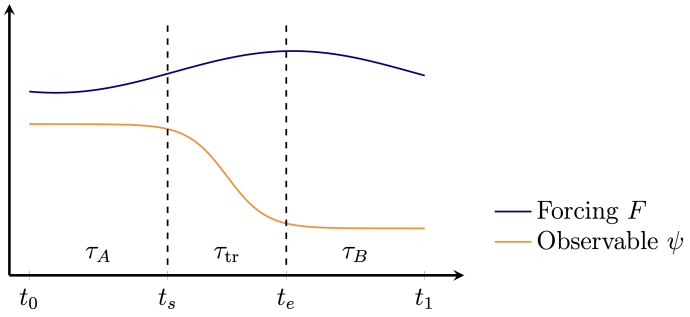


Figure 1. Schematic of the problem setting. During the time interval $\tau = [t_0, t_1]$, is the system tipping in the observable ψ under the forcing F?

Criterion 1 (phenomenology) and criterion 2 (cause) are necessary conditions for tipping: phenomenology and system understanding must come together. In combination, these criteria are sufficient. How to identify linear response regimes and significant deviations from them is outlined in section 3.2 and illustrated in section 4. The attribution to a positive feedback is discussed in section 4.

A *tipping system* is a dynamical system in which tipping can ~~be shown to~~ occur. The *tipping duration*, i.e. the duration of the tipping event, can be inferred from the duration of the amplifying nonlinear response. The *tipping onset* marks the point in time where the ~~identified destabilizing feedback becomes dominant relative to stabilizing processes~~balance of internal feedbacks begins to drive a self-amplifying response. This point in time does not necessarily coincide with an abrupt change point in the timeseries of the observable.

On a given forcing timescale, a tipping system is a fast (slow) tipping system if τ_{tr} is short (long) compared to the timescale on which the forcing varies. Tipping events can be bifurcation-induced, rate-induced or noise-induced as defined ~~in Refs. (Ashwin et al., 2012; Wieczorek et al., 2023)~~by Ashwin et al. (2012); Wieczorek et al. (2023), or a combination of these. Furthermore, tipping can be induced by a singular shock ~~(Halekotte and Feudel, 2020)~~ (Klinshov et al., 2015; Halekotte and Feudel, 2020) or occur spontaneously following a long transient ~~(Lai and Tél, 2011; Börner et al., 2026)~~(Hastings et al., 2026; Börner et al., 2026)
.

3.1.1 Remarks

– ~~Constant forcing , $\mathrm{d}F/\mathrm{d}t = 0$ for all $t \in \tau$, represents a special case where any nonlinearity in the observable is necessarily a deviation from linear response~~ Our definition assumes that tipping can be determined based on the evolution of the forcing and observable during the finite time horizon τ, without knowledge of the system's history before τ. This is only the case if the system has sufficiently lost its memory at the beginning of τ. Furthermore, without knowledge of the future after τ, persistence (criterion 1c) is assessed on the timescale τ_B. The value of P should be large enough to ascertain whether a dynamically distinct linear response regime has been reached. Setting τ or P too small for the given context

can result in a misclassification of tipping (Bastiaansen et al., 2023). ~~Transitions are then a result of internal variability, e.g. a noise-induced transition between metastable states or a chaotic transient.~~

- Our definition permits both irreversible and reversible tipping ~~. For example,~~ in the quasi-equilibrium sense. Bifurcations are not necessary; also a monostable system with a nonlinear sensitivity to a forcing parameter could produce tipping ~~dynamics~~ events in the sense of our definition. We justify this from a phenomenological or impact perspective: different mathematical structures can lead to observable timeseries that are qualitatively equivalent and mechanistically relate to a positive feedback.

- We strongly recommend to distinguish between abrupt ~~shifts~~ change and tipping. A tipping event can occur as an abrupt shift but may also appear more gradual in slow tipping systems. Conversely, not every abrupt shift is a tipping event: it can be a linear response to an abrupt change in the forcing, for example, or a non-persistent deviation from linear response. The persistence of a distinct dynamical regime after τ_{tr} also clarifies the distinction between a tipping event (persistent change) and an extreme event (non-persistent change).

- ~~Our tipping definition assumes that tipping can be asserted based on the evolution of the forcingand observable during the interval τ, without knowledge of the system's history before τ. This is only the case if the system has sufficiently lost its memory at the beginning of τ.~~ Constant forcing, $\mathrm{d}F/\mathrm{d}t = 0$ for all $t \in \tau$, represents a special case where any transition is a result of internal variability. Noise-induced transitions between metastable states will typically fulfill the phenomenology criteria of our tipping definition, as well as chaotic transients between a long-lived unstable chaotic set and an attractor. We believe that classifying such events as tipping events is desired, except in cases described in the following remark.

- In multiscale systems with a slow timescale relative to τ, fast-slow ~~dynamics such as relaxation oscillations could be classified~~ oscillations are another form of internal variability that could be labeled as tipping events ~~when considering only the dynamics during~~ by our definition. Specifically, if τ ~~,~~ covers a fraction of one oscillation period in which a fast switch occurs between slow regimes, that transition might classify as tipping on the timescale τ – whereas on longer timescales ~~they~~ it would be interpreted as a mode of internal variability associated with a single dynamical regime. From a dynamical systems perspective, this may be considered a misclassification caused by an incomplete estimation of the linear response regime and inappropriate choice of τ. From an impact perspective on the time horizon τ, we may argue again that such dynamics have characteristics of a tipping event. This question applies, for example, to Dansgaard-Oeschger events (Crucifix, 2012; Dijkstra et al., 2026).

3.2 Tipping test

To be useful, a tipping definition should be testable in the sense that applying the definition to empirical data yields a consistent ~~assertion~~ detection of whether tipping has occurred. In the following, we present a generic tipping test inspired by response theory (Ruelle, 2009; Ghil and Lucarini, 2020) that aligns with our tipping definition in section 3.

Given a timeseries pair of an observable $\psi(t)$ and forcing $F(t)$, the general idea is to test the null hypothesis: the evolution of $\psi(t)$ can be explained by a linear response model (null model) subjected to $F(t)$. Rejection of this null hypothesis is a necessary condition for tipping. If the deviation from the null model is then shown to be persistent and attributed to a plausible feedback mechanism, this signifies tipping.

3.2.1 Deviation from linear response

Consider a nonlinear dynamical system described by[2] ~~$\dot{u} = f(u, F(t))$~~ $\dot{\boldsymbol{u}} = f(\boldsymbol{u}, F(t))$. The state ~~$u$~~ vector $\boldsymbol{u}$ maps to the observable ψ via the projection ~~$\psi = Mu$. Linearizing around an unperturbed reference state u_* for reference forcing $F = F_*$~~ $\psi(\boldsymbol{u})$. For simplicity, we assume here that the unperturbed system has an equilibrium $x_* = \langle \psi(\boldsymbol{u}) \rangle_*$ at reference forcing F_* (let $F_* = 0$ without loss of generality), ~~we obtain a counterfactual linear system, $\dot{y} = Ay + BF(t)$ with $y(t) = u(t) - u_*$. The~~ where $\langle \cdot \rangle_*$ denotes the expectation value with respect to the reference state. The idea sketched below can be extended to more general cases and multiple forcings (see Appendix A).

The null model predicts the solution

$$x(t) = x_* + e^{-t/m - (t-t_0)/m}(x_0 - x_*) + \int_{t_0}^{t} G(t,t')F(t')\mathrm{d}t' \,, \quad t \geq t_0 \,, \quad x_0 = x(t_0) \,, \tag{1}$$

for the ~~observable $x = My$ of the linear system. Here~~ expectation value $x(t)$ of a counterfactual linear system, given the initial condition $x_0 = \psi(t_0)$. Here the exponential decay term describes the unforced relaxation towards equilibrium with damping timescale m ~~is a damping timescale of the unperturbed system (with projected reference state $x_* = Mu_*$)~~ related to the system's inertia, more specifically the leading relaxation mode of the linearized dynamics. The linear response to the forcing F, described by the integral term, is encoded in the (first-order) Green's function G (Kubo, 1966; Ghil and Lucarini, 2020; Ragone et al., 2016; Lucarini and Chekroun, 2024). Although the approximation (1) ~~is exact only if the system is linear, it often remains~~ assumes a single damping timescale and linearity of the response and projection, such linear response formulas often remain valid also in complex nonlinear systems for a relatively wide range of forced perturbations ~~(Leith, 1975; Hairer and Majda, 2010; Ragone et al., 2016)~~(Leith, 1975; Bell, 1980; Majda et al., 2010; Hairer and Majda, 2010; Wormell . However, linear response breaks down near tipping thresholds (among other possible reasons, such as large perturbations) ~~(?)~~ (Lucarini et al., 2014; Lucarini and Chekroun, 2023).

We introduce the absolute residual,

$$\eta(t) = |\psi(t) - x(t)| \,, \quad t \in \tau \,, \tag{2}$$

which quantifies the deviation from linear response. The timeseries of η during τ can be analyzed to identify tipping. The null hypothesis is rejected if η diverges significantly from zero, i.e. $\eta(t) > \eta_c$ for some tolerance η_c and $t \in \tau$. In practice, tipping requires that:

- The null model explains the behavior during τ_A (i.e. $\eta(t) \leq \eta_c$ for almost all $t \in \tau_A$).

[2] Here $\dot{x}$ is typical short-hand notation for $\mathrm{d}x/\mathrm{d}t$.

– The same null model persistently fails to predict the behavior during τ_B (i.e. $\eta(t) > \eta_c$ during most of τ_B), but a recalibrated null model (e.g. with a different reference state) captures the behavior during τ_B.

– The increase of $\eta(t)$ during τ_{tr} is attributed ~~in magnitude and rate~~ to a positive feedback process.

This practical test highlights several implications of our tipping definition. First, if the system does not fulfill approximately linear response conditions before and after the transition, tipping cannot be asserted. This seems a reasonable restriction, since tipping should lead from one well-defined dynamical regime to another; it is less restrictive than considering transitions between equilibrium states. Essentially, this reflects that tipping is difficult to detect if the system is strongly forced or features strong nonlinear variability on the timescale τ. Secondly, tipping can only be determined retrospectively, after observing the system during the persistence ~~time~~ interval $[t_e, t_e + P(t_s - t_e)]$. Of course, it may be possible to monitor tipping indicators throughout τ, such as an increase in nonlinear response ($\mathrm{d}\eta/\mathrm{d}t > 0$), or to potentially predict tipping based on knowledge of the system's stability properties and feedback processes.

The purpose of constructing a null model is that it allows to separate linear from nonlinear response. With suitable null models, the evolution of the residual η provides an approximation of linear response regimes (criteria 1(a) and 1(c) in section 3) as well as the deviation from linear response (criterion 1(b)). The null model also provides a counterfactual system in which the positive feedback that drives tipping is absent.

3.2.2 Elementary null models

In applications where the Green's function G is not known for the specific system, a simple Green's function can be fitted to the empirical data. The simplest choice is an exponential memory kernel,

$$G(t,t') = \frac{\beta}{m} e^{-(t-t')/m}, \tag{3}$$

where β is the sensitivity to the forcing F and the timescale m represents the system's inertia. This Green's function corresponds to the null model,

$$m\dot{x} = -(x - x_*) + \beta F(t), \tag{4}$$

with free parameters m, β and x_*. This null model captures the inertial relaxation to a moving reference equilibrium $x_* + \beta F(t)$ (provided that the system has sufficiently lost memory of its history before t_0; otherwise the pullback attractor must be considered (Ghil and Lucarini, 2020)). However, it cannot describe internal oscillations of the system. For systems featuring oscillatory dynamics, a more appropriate null model is given by the damped, driven harmonic oscillator (see Appendix B). Equation (4) may be viewed as a harmonic oscillator in the overdamped limit.

4 Examples

To illustrate our tipping definition, we apply the tipping test to simulated tipping events in a simple toy model and in an Earth system model.

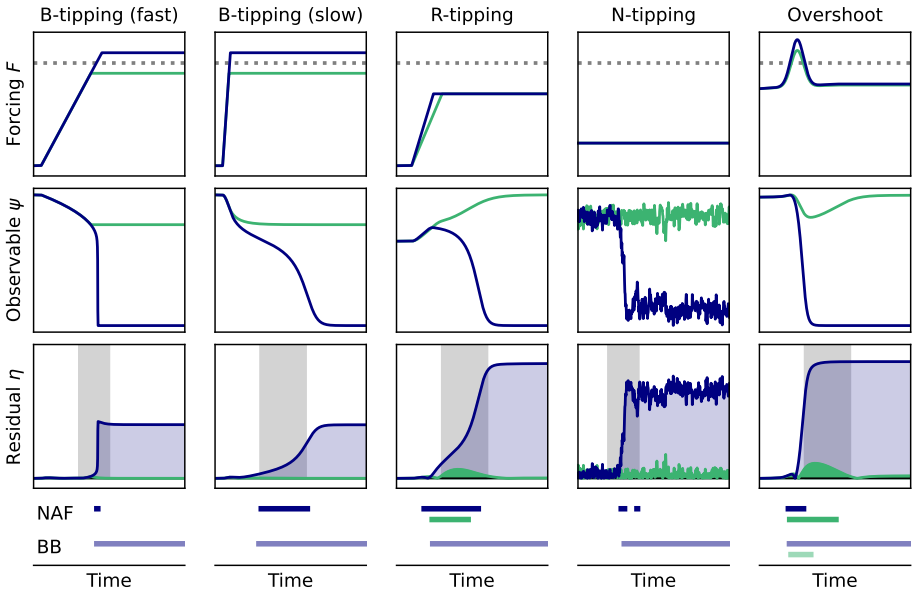


Figure 2. Different types of tipping demonstrated in the simple tipping system (5). Each scenario (column) shows a tipping case (blue) and a non-tipping case (green). Top row: Forcing protocols; the gray dotted line marks the fold bifurcation (i.e. the 'classical' tipping point). Row 2: Trajectories of the system determined under the respective forcing. Row 3: Nonlinear residual of the response η (see Eq. (2)). The null model (Eq. (4)) is fitted to each non-tipping case outside of the gray shaded area marking τ_{tr} (see Appendix C). Bottom: Horizontal bars mark the time periods during which the system evolves in a region of net accelerating feedback (NAF, defined by a downward concave potential landscape) or has crossed the basin boundary (BB) to the competing stable state.

4.1 Simple tipping system

Consider a simple dynamical system that features a bistable regime with basin instability (O'Keeffe and Wieczorek, 2020), bounded by a fold bifurcation ~~(based on Ritchie et al. (2023)):~~ (based on Ritchie et al., 2023):

$$m\,\mathrm{d}\dot{x} = -x\left[(x - A - s(F(t)-1))^2 + F(t) - 1\right]\mathrm{d}t + \sigma \mathrm{d}W_t\,, \tag{5}$$

where m denotes inertia, A and s are parameters and W_t represents a Wiener process (Gaussian white noise) scaled by the noise strength $\sigma \geq 0$. This system is susceptible to bifurcation-, rate-, and noise-induced tipping. We can thus test our definition on these different cases as well as an overshoot case where the bifurcation threshold is crossed temporarily (Ritchie et al., 2021).

As a null model, we use the linear system in Eq. (4), with x_* set to the upper stable fixed point at reference forcing $F_0 = F(t_0)$. For each tipping scenario, we generate one trajectory that tracks the initial fixed point (non-tipping) and one trajectory that tips to the alternative stable fixed point (Fig. 2; details in Appendix C). ~~In~~ We exclude noise ($\sigma = 0$) except in the noise-induced scenario, ~~we add Gaussian white noise to the system~~ where we set $\sigma > 0$ under otherwise constant forcing. We consider bifurcation-induced tipping both in a fast system (small inertia) and a slow system (large inertia). In the fast case, tipping is

abrupt relative to the forcing. In the slow case, both the tipping and non-tipping trajectories continue to change long after the forcing has stabilized, and the transition in the tipping case is delayed compared to the crossing time of the bifurcation.

In all non-tipping cases, the fitted null model accurately describes the evolution of the observable during τ_A and τ_B, even though the system is nonlinear (Fig. 2). During τ_{tr}, a transient nonlinear response signal is sometimes visible, for example under overshoot forcing. This is because the system temporarily crosses the fold bifurcation; the positive feedback amplifies the response but is subsequently reversed. The key point is that the residual η decays back to zero in the non-tipping cases, whereas it grows significantly larger and never recovers in the tipping cases. After the transition, the residual stabilizes in the tipping cases, showing a linear response again but now with the alternative stable fixed point as reference state.

~~The~~ Since the system is one-dimensional, the only possible feedback is the state variable y feeding back on itself. A positive feedback arises from the presence of an unstable equilibrium that generates a basin boundary and, locally, a downward concave stability landscape (see Appendix C). Within the concave state space region, the linearized dynamics are destabilizing and accelerating, which we propose to interpret as a net positive feedback in an out-of-equilibrium context. Indeed, major increases of the nonlinear residual coincide with periods in which the system evolves within the region of net positive feedback ~~.~~ (Fig. 2). Additionally, tipping occurs when the system crosses the unstable equilibrium (basin boundary) permanently. We can thus attribute this nonlinear response to the action of a positive feedback.

4.2 AMOC tipping in an Earth system model

We next consider simulations with the Community Earth System Model (CESM) version 1, a comprehensive climate model with around 10^7 degrees of freedom (Hurrell et al., 2013). This model has been used extensively to study the tipping behavior of the Atlantic Meridional Overturning Circulation (AMOC), a major ocean current system, under changes in atmospheric CO_2 concentrations ~~(van Westen et al., 2025, 2026)~~(van Westen et al., 2025, 2026; Oh et al., 2025). The AMOC is a widely studied climate tipping system and has been shown to feature bistability in CESM, where the two stable states correspond to a strong and a collapsed circulation, respectively (Dijkstra et al., 2026).

The AMOC response to radiative forcing depends strongly on the forcing path, meaning that not only the magnitude but also the rate of radiative forcing change is crucial (Stocker and Schmittner, 1997; Hankel, 2025; van Westen et al., 2026) . In CESM, the AMOC tips under certain CO_2 forcing scenarios while ~~returning to the stable strong state~~ recovering in others, sometimes after a transient weakening and sometimes exhibiting ensemble splitting (Oh et al., 2025). Since radiative forcing is related to the logarithm of CO_2 concentration, we use ~~$\log[CO_2]$~~ $\log(CO_2/CO_2^{\text{pi}})$ as the forcing parameter F, where $CO_2^{\text{pi}} = 285$ ppm is the preindustrial reference. The AMOC state is also affected by the surface freshwater forcing $\bar{F}_H$, which is set to a constant background value in the simulations ~~(van Westen et al., 2026)~~(van Westen et al., 2025) . The observable ψ is the AMOC strength (defined in Appendix D), measured as volume transport in units of Sv ($1\,\text{Sv} = 10^6\,\text{m}^3\,\text{s}^{-1}$)~~(Johns et al., 2023; Dijkstra et al., 2026)~~, and x_* corresponds to the mean equilibrium AMOC strength under reference forcing.

One complication with the CESM simulations is that the AMOC strength can feature oscillatory dynamics in response to the forcing. The ~~appropriate elementary null model is thus the~~ damped, driven harmonic oscillator is thus a candidate elementary

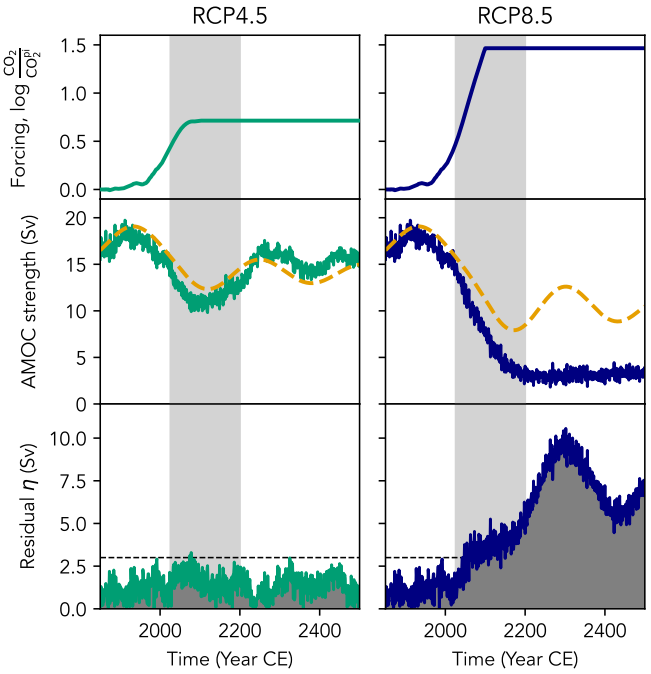


Figure 3. Tipping analysis of CESM simulations under the RCP4.5 (~~blue~~left) and RCP8.5 (~~orange~~right) scenarios for constant $\bar{F}_H = 0.18\,\mathrm{Sv}$~~(van Westen et al., 2025)~~. The light gray shaded interval indicates τ_{tr}. The null model (~~light blue~~ orange dashed, see Table D1 for parameter values and Appendix D for details) is fitted to the RCP4.5 case using data outside of τ_{tr}, with ~~log [CO_2]~~ logarithmic CO_2 concentration as forcing~~(see Appendix D)~~. The same null model is applied to the RCP8.5 case. In RCP4.5, the absolute residual η remains almost entirely below $\eta_c = 3\,\mathrm{Sv}$ (~~red~~ black dashed line), whereas in RCP8.5 η significantly increases during τ_{tr} and persists above η_c thereafter.

null model. With a background freshwater forcing of $\bar{F}_H = 0.18\,\mathrm{Sv}$, the AMOC tips under the high-emissions Representative Concentration Pathways ~~(RCP)~~ (RCP; van Vuuren et al., 2011) scenario RCP8.5 but recovers to a strong AMOC state in the intermediate-emissions scenario RCP4.5. A null model with a natural frequency of roughly $1/41$ ye~~a~~r $^{-1}$ and damping ratio of around 0.1 (see Appendix B) can approximately explain the AMOC evolution under RCP4.5 (Fig. 3), suggesting that no tipping occurs. If the ~~system~~ response would be linear, the same null model should also describe the evolution under RCP8.5; however, the null model persistently underestimates the magnitude of AMOC weakening, resulting in a nonlinear residual of more than 53 Sv. ~~This nonlinear response can be linked to the positive salt-advection feedback, which has been shown to govern an AMOC collapse in CESM (Vanderborght et al., 2025).~~ Beyond the year 2200, both the forcing and the AMOC strength are relatively constant under RCP8.5, giving a linear, non-oscillating response with a significantly weaker AMOC as the reference state.

A second complication arises from the timescale-dependent relationship between the AMOC and radiative forcing (van Westen et al., 2026), shown for a background freshwater forcing of $\bar{F}_H = 0.45\,\mathrm{Sv}$ (Fig. 4). Under rapid CO_2 increase (+2.5 ppm

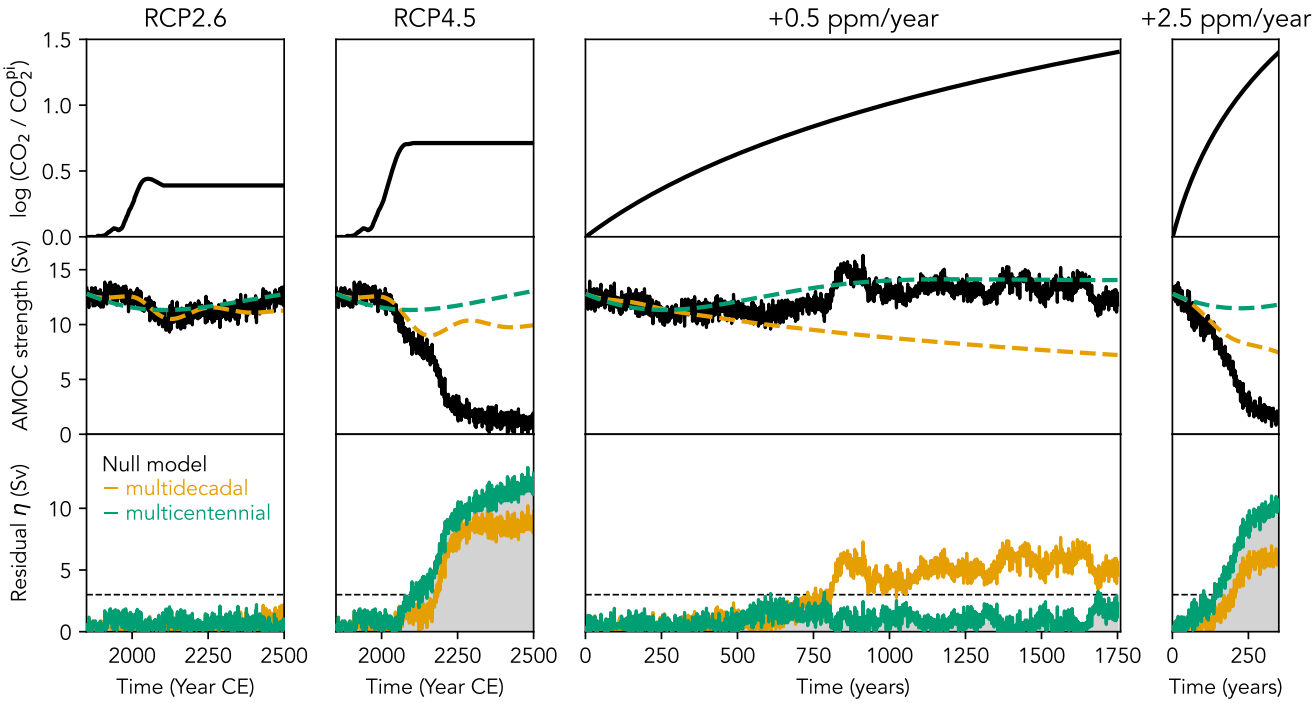


Figure 4. Tipping analysis of CESM simulations with constant $\bar{F}_H = 0.45\,\mathrm{Sv}$ for two RCP scenarios (from 1850 CE extended to 2500 CE, van Westen et al. (2025)) and two linear CO_2 ramp protocols with different rates ~~(van Westen et al., 2026)~~(+0.5 ppm yr^{-1} and +2.5 ppm yr^{-1}, van Westen et al. (2026)). All trajectories are initialized from the same statistical equilibrium state (preindustrial reference). Predictions of a null model with multidecadal (~~blue~~orange) and multicentennial (~~orange~~green) oscillations are shown for each case (see Appendix D and Table D1). Tipping cases (RCP4.5 and ~~+2.5~~+2.5 ppm~~/year~~ yr^{-1}) and non-tipping cases (RCP2.6 and ~~+0.5~~+0.5 ppm~~/year~~ yr^{-1}) are accurately classified by the multicentennial null model but not by the multidecadal null model, which would classify also the +0.5 ppm yr^{-1} simulation as tipping. The dashed horizontal line indicates $\eta_c = 3$ Sv.

yr^{-1}), the AMOC strength tends to decrease, implying a negative forcing sensitivity β in Eq. (B1). Conversely, the AMOC can strengthen due to a slow (+0.5 ppm yr^{-1}) multi-centennial increase in CO_2 concentration. An overdamped null model or multi-decadal oscillator model cannot simultaneously explain the linear response of the AMOC to slow and fast radiative forcing change. However, when allowing for centennial-scale ~~oscillations~~variability, we can determine a single null model that correctly distinguishes between tipping and non-tipping cases under various forcing scenarios (Fig. 4; details in Appendix D).

Determining the role of feedbacks is generally challenging in high-dimensional climate models, but in the case of the AMOC tipping there is strong evidence for the salt-advection feedback being the dominant positive feedback mechanism (Kuhlbrodt et al., 2007; Dijkstra, 2024). This process understanding is derived from a model hierarchy linking the dynamics of earth system models to conceptual models in which the feedbacks are explicitly represented. In CESM, a mechanistic analysis has shown that the salt-advection feedback indeed dominates the internal feedbacks that destabilize the AMOC

(Vanderborght et al., 2025). For scenarios where the AMOC undergoes a persistent nonlinear transition, the residual η is consistent in strength and timescale with this feedback, so we conclude that we can attribute the deviation from linear response to the salt-advection feedback.

This example demonstrates that our tipping test can be successfully applied to complex tipping phenomena in high-dimensional, multiscale dynamical systems. It also highlights the challenges of determining an appropriate null model based on limited empirical data. A poorly constrained choice of null model can inhibit the robustness of the tipping test and yield false positives or negatives.

5 Discussion and conclusion

Tipping points have become a highly popular concept in Earth sciences, but inconsistent language around tipping points is causing ~~growing confusion~~concern. In this paper, we critically evaluated existing tipping definitions, in particular the definition used by the Intergovernmental Panel on Climate Change (IPCC). We identified several key issues: confusion between tipping thresholds versus tipping events, a problematic focus on abruptness and irreversibility, and a reliance on equilibrium concepts that have limited applicability to transient dynamics.

We proposed a revised definition that characterizes tipping as a persistent nonlinear transition of a system observable in relation to ~~an external forcing. This~~ external drivers. In line with Boers et al. (2026), this effectively shifts the focus from tipping ~~points to tipping events~~*points* to tipping *transitions*, acknowledging the increasingly common and sometimes confusing use of the term 'tipping point' as a synonym for transition. Our goal was to define tipping in a general way that applies to diverse complex systems, minimizing restrictive assumptions while still formulating clear, testable, necessary and sufficient criteria for tipping. The definition avoids notions of abruptness, irreversibility, bifurcations and equilibria, instead emphasizing the nonlinearity and persistence of the change driven by a self-amplifying feedback process. ~~Our goal was to define tipping in a general way that applies to diverse complex systems, minimizing restrictive assumptions while still formulating clear, testable, necessary and sufficient criteria for tipping~~Attributing tipping to a mechanistic cause ensures that the use of tipping terminology is grounded in a robust understanding of system dynamics. Conceptual models and model hierarchies are important tools for developing this understanding.

Our tipping definition is inspired by the idea that linear response breaks down near tipping points. While we did not aim to make this rigorous here, the relationship between response theory, tipping behavior and climate prediction is an active research field ~~(Ghil and Lucarini, 2020; Santos Gutiérrez and Lucarini, 2022; **?**; Galatolo and Lucarini, 2026)~~(Santos Gutiérrez and Lucarini, 2022 . Moreover, there has been great interest recently in data-driven approaches for estimating Green's functions ~~(Aengenheyster et al., 2018; Lu~~ (Torres Mendonça et al., 2021; Zagli et al., 2026; Lucarini, 2026; Aengenheyster et al., 2018). This holds potential for expanding the mathematical basis of our tipping definition, and for developing more advanced null models. Conceptualizing tipping as a deviation from a counterfactual linear world frames the problem in terms of detection and attribution ~~(Lucarini and Chekroun, 2024)~~ (Hasselmann, 1993; Palmer, 1999; Lucarini and Chekroun, 2024), which can offer a useful complement to the prediction prob-

lem posed by early warning methods (Dakos et al., 2008; Rietkerk et al., 2026). How our definition applies to adaptive systems, such as socio-technical or biological systems, is an interesting question for future work.

Overly simplistic pictures of tipping dynamics have led to a common belief that tipping is restricted to simple systems (Shaw and Stevens, 2025; Kopp et al., 2025a). However, the mathematical theory of tipping goes far beyond fold bifurcations and quasi-equilibrium dynamics. Methods from global stability analysis, nonautonomous dynamics and chaos theory have been successfully applied to high-dimensional systems, and concepts such as rate-induced tipping enable a more nuanced view that incorporates transient, metastable dynamics. What has been missing is a practical and overarching tipping definition that bridges between specific mathematical scenarios and purely qualitative notions that often remain vague.

Carefully defining what tipping means is important because it shapes research questions and the public perception of tipping risk. A strong emphasis on abruptness has possibly caused a misunderstanding of climate tipping events occurring instantly once a threshold is crossed, conveying a misleadingly binary picture of "safe" versus "catastrophic" climate change. We argue that from an impact perspective, the essential aspect is the internally driven nonlinearity of the change because the change is then no longer externally controlled, making it difficult to stop or reverse. Prediction models based on linearity would then fail and lead to false risk assessments.

Code and data availability. The TippingTest.jl package (https://github.com/reykboerner/TippingTest.jl) provides a generic implementation of our tipping test in the Julia language. The CESM model simulation data used in this work are archived on Zenodo under the DOIs 10.5281/zenodo.15088540 (Fig. 3) and 10.5281/zenodo.18599844 (Fig. 4).

Appendix A: Linear response equation

The linear response equation (1) is based on the simplifying assumptions that (i) the unforced dynamics are well approximated by linear relaxation with a single damping timescale associated with a dominant leading eigenmode of the linearized dynamics, (ii) the reference state is a stationary equilibrium with a representative expectation value x_* of the observable ψ, and (iii) the relevant forcing is captured by a single scalar variable $F(t)$.

However, the linear response framework (Ruelle, 2009) can be extended to more general settings (Wormell and Gottwald, 2019; Lucarini . The unforced dynamics can be described by a spectrum of relaxation modes or, most generally, by the propagator $P(t_2,t_1)$ that evolves the expectation value $x(t_1)$ of the unforced system at time t_1 to time t_2. Recent work has also considered the case where the reference state is time-dependent, e.g. periodic in time (Branicki and Uda, 2021; Galatolo and Lucarini, 2026) . Moreover, it is straightforward to extend linear response to multiple forcings $\{F_i(t) : i = 1, \ldots, N\}$ acting simultaneously (Lucarini and Chekroun, 2024). In linear response, the forcings combine additively; higher-order terms accounting for the interacting effects of multiple forcings are neglected.

Considering these aspects, a more general version of Eq. (1) could take the form

$$x(t) = P(t,t_0)x(t_0) + \sum_{i=1}^{N} \int_{t_0}^{t} G_i(t,t')F_i(t')\,\mathrm{d}t' \,, \tag{A1}$$

where x is, as before, the expectation value of an observable of the system. Note that there is an individual Green's function G_i for each forcing F_i.

Appendix B: Damped oscillator null model

A simple linear null model for oscillatory dynamics is given by the damped, driven harmonic oscillator,

$$m^2\ddot{x} = -(x - x_* - \beta F(t)) - \gamma\dot{x} \tag{B1}$$

Here $\ddot{x} = \mathrm{d}^2x/\mathrm{d}t^2$ denotes the second time derivative (acceleration) of the state x, which relaxes to the time-dependent equilibrium $x_* + \beta F(t)$ with inertia m. Displacements out of equilibrium lead to damped oscillations with eigenfrequency $\omega_0 = 1/m$ and damping ratio $\xi = \gamma/(2m)$. The oscillator is overdamped (underdamped) if $\xi > 1$ ($\xi < 1$).

Eq. (B1) can equivalently be written as a pair of first-order ordinary differential equations,

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{1}{m^2}\left[(x - x_* - \beta F(t)) - \gamma v\right], \end{aligned} \tag{B2}$$

which can be solved given an initial condition (x_0, v_0) comprising the initial displacement $x_0 = x(t_0)$ and velocity $v_0 = v(t_0)$.

Appendix C: Simple tipping system

Trajectories of the simple tipping system (Eq. (5)) shown in Fig. 2 were generated using the parameter settings given in Tab. C1. The fold bifurcation occurs at $F = 1$. In the B-tipping cases, the forcing F is linearly increased up to a final constant value of $F = 0.9$ (no tipping) or $F = 1.1$ (tipping). In the R-tipping case, the linear increase is stopped at $F = 0.7$; the durations of increase are $T/5$ (no tipping) and $T/7$ (tipping), where $T = t_1 - t_0$. In the N-tipping case, $F = 0.22$ is held constant. In the overshoot case, the forcing profile is given by $F(t) = h[\exp(-70(t - (t_0 + t_1)/2)^2/(t_1 - t_0)) + 0.1t]$ with $h = 0.355$ (no tipping) and $h = 0.455$ (tipping).

The location of the saddle point x_S is given by $x_S(F) = A + (F-1)s - \sqrt{1-F}$ for $F < 1$. The potential V of the system is defined as $f(x) = -\mathrm{d}V/\mathrm{d}x$, where $f(x) = -x[(x - A - s(F-1))^2 + F - 1]$. The inflection points of the potential (where its curvature changes sign) are located at $x_\pm = [2(A - s + Fs) \pm \sqrt{A^2 + 2As(F-1) + (F-1)(s^2(F-1) - 3)}]/3$.

Table C1. Parameter settings for the toy model trajectories shown in Fig. 2. Here m is the inertia, s is the tilt parameter, $F(t_0)$ is the initial forcing value at t_0, σ is the noise strength and $T = t_1 - t_0$ (in system time units) is the time intercal shown in each panel of Fig. 2.

	m	A	s	$F(t_0)$	σ	T
B-tipping (fast)	1	2	0	0	0	1000
B-tipping (slow)	120	2	0	0	0	1000
R-tipping	400	5	3	0	0	1000
N-tipping	1	5	3	0.22	0.75	50
Overshoot	300	5	3	0.75	0	1000

Appendix D: CESM simulations

In all CESM simulations, the atmospheric CO_2 concentration is prescribed as a function of time. The AMOC strength ψ (Johns et al., 2023) is defined as the maximum of the Atlantic meridional overturning streamfunction at the latitude $y_0 = 26°$N and below $z_0 = 1000$ m depth, $\psi(t) = \max_{z \geq z_0} \Psi(y = y_0, z, t)$, where

$$\Psi(y,z,t) = \int_z^0 \int_{x_W}^{x_E} v(x,y,z',t)\,\mathrm{d}x\,\mathrm{d}z' \,. \tag{D1}$$

Here v is the meridional velocity field, which is a function of longitude x, latitude y, depth z and time t. The inner integral is over the Atlantic basin from its western (x_W) to eastern (x_E) boundary. The observable ψ is thus a nonlinear one-dimensional mapping of the full state vector including ocean variables (velocity, temperature, salinity, ...), atmospheric variables (air temperature, pressure, humidity, ...) and all other model variables such as sea ice, soil moisture etc. at all model grid points.

The null model trajectories shown in Figs. 3 and 4 are generated with the harmonic oscillator null model (Eq. (B1)) using the parameter values and initial conditions listed in Tab. D1. In the case $\bar{F}_H = 0.18$ Sv (Fig. 3), we fitted the null model to the RCP4.5 scenario using the data in $\tau_A = (1850, 2026)$ and $\tau_B = (2200, 2500)$, where the years refer to the Common Era (CE).

Table D1. Parameter settings of the harmonic oscillator null models (Eq. (B1)) shown in Figs. 3 and 4.

Null model	x_* (Sv)	x_0 (Sv)	v_0 (Sv yr^{-1})	β (Sv)	m (yr)	γ (yr)
Decadal oscillator $\bar{F}_H = 0.18$ Sv (Fig. 3, orange)	17.5	16.5	0.04	−4.76	41.3	5.9
~~Centennial~~ Decadal oscillator $\bar{F}_H = 0.45$ Sv (Fig. 4, orange)	12.8	12.8	−0.012	~~1.0~~ −4.0	~~223.6~~ 41.3	~~200.0~~ 20.0
~~Decadal~~ Centennial oscillator $\bar{F}_H = 0.45$ Sv (Fig. 4, ~~blue~~green)	12.8	12.8	−0.012	~~−4.0~~ 1.0	~~41.3~~ 223.6	~~20.0~~ 200.0

Author contributions. R.B. and H.A.D. conceived the study and developed the tipping definition. R.B. performed the analysis and created the figures. R.B. wrote the manuscript with input from H.A.D.

Competing interests. The authors declare no competing interests.

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