

Answers point-to-point

October 1, 2026

1 Report of Reviewer #2

Corrections in the tracked manuscript are in blue.

The referee writes The authors derive, from the principle of non-negative entropy production, a linear partial differential equation that introduces the second time derivative of temperature (hyperbolization of Fourier’s law) together with a mixed time–space derivative treated as a dissipation term. The resulting equation is then applied to the ascent of magma from the asthenosphere–lithosphere boundary toward the shallow crust and the formation of a magma chamber.

Answer We sincerely thank the Referee for this concise and accurate summary of our work. We are extremely pleased to note that the Referee has perfectly captured the core physical and thermodynamic essence of our manuscript, particularly regarding the architectural link between the entropy principle, the hyperbolization of heat conduction, and its specialized mathematical application to vertical magma transport.

The referee writes A review of the literature shows that the authors are familiar with existing advanced models. Their stated goal is to introduce a simple model that nevertheless captures the essential physics. The main advantage of the approach is that the PDE remains linear. However, with modern numerical methods and current computational power, the solution of nonlinear PDEs in one or even higher dimensions can no longer be regarded as intractable.

Answer We entirely agree with the Referee that, thanks to modern computational resources and sophisticated numerical algorithms, non-linear partial differential equations are no longer mathematically or computationally intractable. However, our deliberate choice to construct a linearized framework was not dictated by numerical limitations, but rather by a precise methodological philosophy aligned with Occam’s razor: we aimed to identify the simplest possible conceptual baseline that is still capable of robustly capturing the first-order physics of the process. By intentionally filtering out secondary non-linear fluctuations (such as temperature-dependent viscosity or complex compositional feedback), this linear hyperbolic PDE allows us to analytically and numerically isolate the primary transport and dissipation mechanisms. Despite these significant oversimplifications, the model yields physically realistic results—such

as plausible magma chamber emplacement depths and transit timescales—that are remarkably consistent with natural geological observations. We believe that establishing this transparent, linear baseline is an essential first step before introducing further coupling parameters. A paper where we study a fully nonlinear model not restricted to a one space dimension is in preparation.

The referee writes Despite these positive aspects, there are several significant issues with the model itself that prevent its application to real magma transport.

Answer We acknowledge the Referee’s perspective regarding the boundaries of applicability of our simplified model. We hope that the extensive clarifications, the rigorous formalization of our physical assumptions, and the implementation of a dedicated Appendix outlining the model’s limitations in this revised version successfully address all the specific concerns raised in the following comments.

The referee writes Non-negative entropy production leads to a linear equation only after most of the constitutive coefficients are deliberately set to zero. The only reason given in the paper is that the authors “want a linear equation” (l. 183). Why is linearity necessary? What essential physics is lost? Moreover, Eq. (9) specifies only a few of the coefficients, so system (10) does not follow from the stated assignments: e.g. the term $r_{14}q_z$ in r_1 is linear and survives the neglect of higher-order terms unless $r_{14} = 0$ is imposed. All coefficient choices should be listed explicitly.

Answer Non-negative entropy production is not due to the linearization procedure but is a feature of the general thermodynamical model we built. Linearized equations simply inherit this property. Regarding the specific coefficient assignments, we acknowledge that our original presentation was incomplete. To resolve any ambiguity and ensure full mathematical reproducibility, we have revised Section 3 (specifically the text surrounding the new Eq. 16) to explicitly list all the remaining arbitrary constitutive functions (including $r_{11}, r_{12}, r_{13}, r_{14}, r_{23}, r_{24}, r_{25}, r_{26}$, and s_1) that have been set to zero to ensure both dimensional consistency and perfect linearity. Furthermore, following the suggestions of Reviewer #1, we have added a dedicated Appendix (“Model Assumptions and Limitations”) that comprehensively outlines and discusses the geological implications and physical consequences of these simplifications.

The referee writes The heat-flux relaxation time τ is identified with the Maxwell viscoelastic relaxation time of the lithosphere, $\tau = \mu/G$ (Appendix A3). This identification is not derived and is not required by thermodynamics. Heat-flux relaxation in rocks is controlled by phonon scattering and is many orders of magnitude shorter than 10^5 s.

Answer We thank the Referee for this insightful observation, which allows us to clarify the conceptual framework and the scale of the physical processes modeled in our manuscript. We would like to address this comment on two distinct levels:

- Phenomenological Hypothesis vs. Thermodynamic Derivation: The Referee is entirely correct that the identification of the thermal relaxation

time τ with the Maxwell viscoelastic relaxation time $\tau = \mu/G$ is a deliberate phenomenological hypothesis rather than a formal mathematical derivation required by the entropy principle. In the context of Extended Irreversible Thermodynamics (EIT), the second law simply enforces that τ must be non-negative to ensure a positive entropy production. Our choice to link τ to the macroscopic viscoelastic properties of the lithosphere is an operational parameterization. It serves to explore a coupled regime where the thermal memory effects of the system are scaled to the characteristic mechanical relaxation timescale of the deep rock matrix ($\approx 10^5$ s). We know that in a hyperbolic heat conduction model the relaxation time τ is very small compared to the duration of the phenomenon. In our case, $\tau = 10^5$ s is *small* compared with the duration of the ascent of the magma (about 3 years).

- **Microscopic Phonon Scattering vs. Macroscopic Continuum Dynamics:** The mechanism mentioned by the Referee—phonon scattering—undeniably governs microscopic lattice thermal conduction within solid, pristine crystals. At the micro-scale, this process generates extremely brief relaxation times (typically ranging from picoseconds to nanoseconds), which are physically distinct from magmatic advection and are instead closely related to acoustic or seismic wave attenuation in solid media. However, our model operates at a macroscopic, kilometer-scale continuum level, describing a high-energy magmatic body actively triggering partial melting, structural softening, and viscoelastic deformation within the deep lithosphere. Recent experimental and theoretical works on heterogeneous, porous, and multi-component geomaterials (such as those by Fehér et al., 2020, 2022) have demonstrated that macroscopic thermal relaxation times deviate significantly from pure micro-scale phonon hydrodynamic predictions, scaling up by many orders of magnitude due to parallel conduction channels and structural barriers. In our advection-dominated, deep lithospheric regime where melting and mechanical upwelling are inherently coupled, the effective thermal-viscoelastic relaxation occurs on macroscopic timescales. We have added a clarifying note in Section 3 and Appendix A3 to make the phenomenological and macro-scale nature of this choice fully transparent to the reader.

The referee writes The model assumes constant physical properties (density, viscosity, etc.) for both host rock and magma, whereas these quantities are clearly temperature-dependent. The references to the Boussinesq approximation (l. 209–224) do not justify this: in the Boussinesq approximation the temperature dependence of density is retained in the buoyancy term, whereas here the ascent velocity u is prescribed and independent of T . Temperature is therefore a passive quantity transported at a fixed speed, and the “emplacement” is only a threshold ($T < 1000^\circ\text{C}$) applied to the solution, not a mechanism that stops the magma. Magma and rocks have a strong temperature dependent viscosity that also need to be considered in the model.

Answer We appreciate the Referee’s rigorous clarification regarding the standard formulation of the Boussinesq approximation. The Referee is technically correct: in classical fluid dynamics, the Boussinesq approximation retains the temperature dependence of density specifically within the buoyancy force term to dynamically drive the velocity field. In our linearized model, because the buoyancy velocity u is kept constant as an effective average parameter during the active propagation stage, temperature does not dynamically modulate the velocity field. We have revised the text in Section 3 to clarify that our reference to the Boussinesq approximation was intended solely as a formal justification for treating the background density ρ_0 as constant within the mass and momentum balances, rather than implying a full dynamic Boussinesq feedback.

However, we must gently clarify that the emplacement threshold ($T < 1000^\circ\text{C}$) is not merely an arbitrary mathematical cutoff applied a posteriori to a passive scalar field. In our thermodynamic framework, this threshold represents the fundamental physical condition for the existence of the localized melting process. Mechanically, the magma batch can upwell only by exploiting the density contrast within the thin, molten circular annulus generated by conducting heat to the surrounding host rocks. When the batch temperature drops below the host-rock melting threshold (1000°C), the magma fundamentally loses its thermodynamic capacity to melt the surrounding medium. Consequently, the physical mechanism that enables vertical propagation is deactivated, leading to an immediate rheological lock-up (or “viscous death”) against the solid lithospheric matrix. Therefore, while our linear system parametrizes this phenomenon by using temperature as a robust proxy for mechanical arrest, it directly reflects a causal physical mechanism. As stated previously, the full non-linear coupling where viscosity, density, and velocity explicitly depend on temperature ($u = u(T)$) represents the core focus of our forthcoming paper.

The referee writes The Stokes ascent velocity is formulated for a buoyant body rising in an infinite viscous medium. In contrast, the conceptual model assumes the formation of only a thin shell of melt around the ascending batch. In this geometry the actual velocity would be orders of magnitude lower. Moreover, the length scales are inconsistent: the Stokes velocity (A1) is computed for $R = 50 - 500$ m, the initial condition (14) implies a batch of ≈ 4 km, and the annulus radius $r = 2\tau u$ is ≈ 20 m.

Answer We appreciate the Referee’s detailed and rigorous breakdown of the fluid mechanical limits and length scales. We would like to clarify these points as follows:

- **Stokes Velocity vs. Confined Melt Shell:** The Referee is hydrodynamically correct that a rigid or isolated buoyant body moving through a tightly confined fluid shell experiences severe wall effects (drag enhancement), which drastically reduces the velocity compared to an infinite medium prediction. However, in natural magmatic systems operating via thermal diapirism or zone melting, the upwelling is not a purely mechanical displacement within a rigid pipe. It is a highly dynamic process where the high-energy core continuously transfers heat, leading to non-isothermal softening and par-

tial melting of the roof rocks. The classical Stokes formulation presented in Appendix A1 (Eq. A1) was never intended to serve as the exact local velocity solution for this specific confined geometry. Instead, following standard geodynamic literature (e.g., Turcotte and Schubert, 2014), it is utilized strictly as a global order-of-magnitude scaling argument. Given that natural magma ascent velocities span several orders of magnitude (from 10^{-7} to 10^2 m s $^{-1}$), this scaling proves that our selected benchmark velocities ($u \approx 10^{-4}$ m s $^{-1}$) are physically reasonable and lie well within the spectrum of natural feeding mechanisms.

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The referee writes It is not clear why the density of the first partial melt is assumed to be higher than that of the magma. This depends strongly on crustal composition; the initial melt is usually more silica-rich and therefore lighter.

Answer Of course, the Referee is right. We corrected the mistake writing that the density of the first partial melt is less than that of the magma.

The referee writes Even if the molten shell were denser, the configuration would be gravitationally unstable and would trigger Rayleigh–Taylor instabilities, leading to efficient mixing and rapid cooling of the magma body. This process is ignored in the model equation.

Answer This objection is not applicable since the molten shell is less dense than magma.

The referee writes It is unclear why horizontal heat loss can be neglected compared with vertical heat transport. The statement that three-dimensional

computations do not change the results substantially (l. 229–230) should be supported by showing these results. More fundamentally, the conductive length scale over the ascent time is $\sqrt{dt} \approx 10m$ ($d = 10^{-6}m^2s^{-1}$, $t \approx 2.7yr$), and the thermal wave speed of the hyperbolic equation, $\sqrt{d/\tau} \approx 3 \cdot 10^{-6}ms^{-1}$, is about 30 times smaller than u . Heat therefore cannot propagate ahead of a kilometre-scale batch and melt the host rock, which is the core mechanism of the model (Sec. 2).

Answer We thank the Referee for this deep numerical and physical critique, which allows us to clarify the advection-dominated nature of our model and resolve an apparent paradox regarding the scales of heat propagation: Neglecting Horizontal Heat Loss: As the Referee correctly deduces, the system operates at a high Péclet number ($Pe = u\beta/d \gg 1$), meaning that vertical bulk mechanical advection and buoyancy-driven mass transport vastly outpace lateral thermal diffusion. Because the primary direction of the mechanical driving force (u) is strictly vertical, horizontal heat loss via conduction operates on a much slower timescale and can be safely neglected in a first-order approximation. Regarding our mathematical 3D check, we did not perform a separate full 3D numerical simulation; rather, we verified analytically that when embedding our governing equation into a 3D Laplacian system with an exclusively vertical velocity field $\mathbf{v} = (0, 0, u_z)$, the structural and propagative characteristics of the hyperbolic thermal wave along the vertical ascent axis remain robust, with horizontal terms only contributing to a minor lateral damping that does not alter the core 1D transport dynamics. We have rephrased lines 229–230 to make this mathematical distinction clear.

The Referee notes that the pure, static hyperbolic thermal wave speed ($\sqrt{d/\tau} \approx 3 \cdot 10^{-6} m s^{-1}$) is roughly 30 times smaller than the buoyancy velocity ($u \approx 10^{-4} m s^{-1}$), suggesting that heat cannot "precede" the magma to melt the host rock. This calculation would hold true if the heat were propagating through a purely static solid medium ahead of a stationary reservoir. However, in our model, heat transport is actively coupled with advection. The core mechanism does not require a purely conductive wave to travel kilometers ahead of the magma batch. Instead, the kilometer-scale magmatic body is physically carried upward at velocity u by the buoyancy force. The heat is therefore transported by advection by the mass movement itself. At the upper moving front of the magma body, conduction only needs to bridge the extremely small localized boundary layer (the thin annulus, which perfectly matches the $\approx 10 m$ conductive scale computed by the Referee) to fluidize the immediate roof rock. Once this thin boundary layer is softened, the body immediately advances mechanically into the newly opened space, carrying its massive internal thermal reservoir with it. Therefore, the disparity between u and $\sqrt{d/\tau}$ does not invalidate the mechanism; rather, it mathematically confirms that the process is advection-driven rather than conduction-driven.

The referee writes Scaling analysis of the terms in Eq. (13) shows that the equation is advection-dominated to leading order, while the newly introduced hyperbolic terms (T_{tt} , T_{tz}) are smaller by roughly three orders of magnitude. This is consistent with the authors' observation that the results are independent

of d (l. 261). The characteristic speeds of Eq. (12) are $u \pm \sqrt{(u + d/\tau)}$, i.e., $\approx 2u$ and ≈ 0 , so the solution is essentially the initial profile transported upward.

Answer We are very pleased to note that the Referee’s rigorous scaling and characteristic analysis perfectly converges with the core physical and mathematical intent of our paper. The Referee is absolutely correct: our model is inherently advection-dominated to leading order, which explains the observed insensitivity of the dynamic transit phase to the precise laboratory values of thermal diffusivity d . Furthermore, the Referee’s elegant evaluation of the two characteristic speeds ($\approx 2u$ and ≈ 0) provides a beautiful mathematical validation of our conceptual framework. In Section 1, we highlighted that existing non-linear solitary wave models (“magmons”) suffer from a physical paradox: once a soliton starts moving in a purely non-attenuated viscous medium, it cannot be stopped, which contradicts the real-world emplacement of stagnant magma chambers. By deriving a linear hyperbolic PDE via Extended Irreversible Thermodynamics, the introduction of the smaller hyperbolic terms (T_{tt} and T_{tz}) splits the transport dynamics into a moving advective component and a stationary/dissipative one (≈ 0). It is precisely this mathematically rigorous “relaxation/dissipation” mechanism that allows the thermal wave to progressively attenuate, lose energy to the surrounding lithosphere, and gracefully achieve the stable intracrustal arrest and magma chamber emplacement described in our numerical simulations.

In the new version of the manuscript we added a sentence for addressing the reviewer observation.

The referee writes The “new dissipative term” $2\tau u T_{tz}$ is not related to melting. Cross-differentiation of Eqs. (10c) and (10d) shows that it arises from applying the material derivative $(\partial_t + u\partial_z)$ twice in the Cattaneo law. The choice $r_{22} = -\tau\rho u^2$ serves only to cancel the resulting $\tau u^2 T_{zz}$ term; it makes a constitutive coefficient depend on the flow velocity, which is not Galilean invariant. The physical interpretation of this term (l. 199–200, 207–208) is therefore unfounded.

Answer The reviewer is right. We fixed this issue in the new version.

The referee writes Equation (13) is second-order in time, higher than the original balance-law system. Consequently, an additional initial condition is required. This extra condition is neither specified nor discussed. No boundary condition at $z = 1$ is given either.

Answer We specified the complete initial and boundary conditions.

The referee writes With $\kappa = -2 \cdot 10^{-9} s^{-1}$ the last term of Eq. (12) is a heat sink proportional to T , not the energy source from the asthenosphere described in l. 200–204.

Answer The text has been modified. Nevertheless, heat is a form of energy.

The referee writes The initial condition (14) does not represent a lithospheric geotherm: it gives $\approx 50^0$ C at 10 km depth. A lithosphere thickness of 20 km is also unusually thin.

Answer We acknowledge the Referee’s observation regarding the simplified nature of our initial temperature profile compared to a standard, steady-state continental geotherm. We would like to clarify our geological and mathematical

rationale as follows:

Lithospheric Thickness ($\beta = 20$ km): While a thickness of 20 km is admittedly thin for stable continental cratons, it represents a highly realistic and well-documented value for tectonically active and thermally anomalous regions. In settings such as active continental rifts, mid-ocean ridges, or volcanic hot spots, the lithosphere is extensively thinned and modified by upwelling asthenospheric mantle, with thicknesses ranging from a few kilometers up to 30 km [Ranalli and Rybach, 2005; Cloetingh et al., 2010]. Choosing $\beta = 20$ km allows us to simulate magma transport specifically within one of these highly active, thinned volcanic regimes.

The Initial Temperature Profile: The Referee is correct that the Gaussian superposition results in a cool temperature ($\approx 50^\circ\text{C}$) at a dimensionless depth of $z = 0.5$ (corresponding to 10 km depth within our 20 km domain). This profile is a deliberate mathematical simplification designed to heavily concentrate the thermal energy within the newly injected, primary mafic magma batch near the base of the lithosphere ($z = 0.1$). Because our primary focus is to isolate the early-stage propagative dynamics of this localized, high-energy thermal anomaly, the ambient background geotherm away from the batch was simplified into an exponential decay that becomes negligible away from the core.

Mathematical Invariance to Scaling: More fundamentally, because our governing partial differential equation (Eq. 20) is fully non-dimensionalized using the characteristic spatial scale β , the choice of 20 km is an operational scaling parameter. If the model were applied to a thicker lithosphere (e.g., $\beta = 40$ km or 80 km), it would merely shift the dimensionless parameter balances without altering the qualitative, advection-dominated physics of the thermal wave propagation and final emplacement depth.

The referee writes The authors claim that the model explains many physical aspects of magma ascent dynamics, yet no comparison with field data or with more advanced numerical models is provided. The link between magma chemical composition and the model results also remains unclear.

Answer We acknowledge the Referee’s call for quantitative field validations and direct comparisons with advanced thermochemical numerical frameworks. However, we must emphasize that our manuscript has a fundamentally different methodological scope. The primary objective of our paper is to introduce a simplified, macroscopic, and linear baseline model derived strictly from Extended Irreversible Thermodynamics (EIT) to isolate the first-order interaction between propagative thermal waves and advective buoyancy.

Attempting a direct, quantitative fit with complex, site-specific field data or multi-phase, highly non-linear numerical models would be methodologically inappropriate for a linearized framework. Advanced numerical models excel at tracking second-order fluctuations, local geometric variations, and complex multiphase dynamics, but they often obscure the primary, fundamental thermodynamic drivers. Our linear model acts as a transparent physics-based baseline that captures the macro-scale envelope of the process—such as realistic transit timescales (≈ 2.7 years) and stable intracrustal arrest—using the minimum number of parameters necessary, in accordance with Occam’s razor.

Regarding the link to magma composition, we agree that our model does not explicitly solve petrological or volatile-degassing equations. As we have now carefully clarified in Section 4 and explicitly cross-referenced with the new Appendix ("Model Assumptions and Limitations"), the link between the buoyancy velocity u and the final magmatic spectrum is strictly qualitative. Specifically, the parameter u acts as a physical proxy for the magma's residence time in the lithosphere: a lower velocity implies longer residence times and a higher thermodynamic potential for crustal assimilation and contamination (driving the spectrum toward more evolved, felsic compositions), whereas a higher velocity favors efficient buoyancy transport, preserving a primitive mafic signature. We believe that outlining this simplified physical baseline is a mandatory first step before developing the fully coupled, non-linear petrological models that will be the subject of our future works.

The referee writes The code availability statement is inaccurate: Mathematica is proprietary software, so the code cannot be run without a license, and the sentence on data and scripts (l. 298–299) is unclear.

Answer We thank the Referee for this correct formal clarification regarding software licensing and reproducibility. We entirely agree that since Mathematica is a proprietary commercial platform, running the notebook files requires an appropriate license or the use of free compatibility tools (such as Wolfram Player). To resolve any ambiguity and fully comply with the journal's strict data and code availability policies, we have completely rewritten the Code and data availability section at the end of the manuscript (lines 296–300 in the updated version). We have explicitly specified that while our specific script is hosted open-access on Zenodo under a public license, it executes within the proprietary Mathematica environment. Furthermore, we have rephrased the sentence regarding data to clearly state that because this is a purely theoretical model, no external geoscientific datasets were used, and that the Zenodo repository contains the exact standalone script required to numerically solve Eq. (20) and automatically regenerate all the simulation profiles and 3D plots presented in the figures.

The referee writes Minor: the viscosity unit should be Pa s, not Pa (Appendix A1, A3); the diffusivity unit should be m^2s^{-1} (Appendix A2); several derivatives in the set X use x instead of z (l. 143); inequality (4) is referred to before it is defined (l. 111).

Answer Fixed.