Response to Referee #3 (RC3)

We thank Referee #3 for their supportive feedback and valuable historical context regarding asymptotic expansions and averaging theory.

Major Comments & Clarifications

1. Contextualizing Reductions with Asymptotic Analysis

RC3: *"Modelling is always made with approximations... It is important to refer to rigorous mathematical analyses made in the past based on asymptotic expansions."*

- **Author Response:**

  I fully agree with the referee. Model reductions are essential for physical insight. In the revised manuscript, we explicitly frame operator spectral neutrality alongside classical asymptotic expansions (e.g., Pedlosky, Vallis). I highlight that asymptotic expansions precisely define the bounded parameter regimes (e.g., small Rossby number $Ro \ll 1$ or small Rossby deformation length $L/a \ll 1$) where spectral non-neutrality remains small and physical wave dynamics are preserved.

- **Changes for the Manuscript:**
  - **Section 2.4:** Added a section explicitly bridging operator projection theory with perturbation theory and asymptotic expansions in GFD.

2. Integration of 1990s Averaging Theories

RC3: *"Concerning the theory of averaging, this question was addressed considerably in the nineties..."*

- **Author Response:**

  I appreciate this historical pointer. I will update the background section to cite key literature from the 1990s on non-linear averaging, effective equations, and stochastic climate modeling. I will emphasize how sub-grid scale variability and non-commuting spatial averages modify the effective deterministic operator.

- **Changes for the Manuscript:**
  - **Section 1 & 3:** Expanded literature citations to include seminal 1990s studies on averaging techniques, Mori-Zwanzig projection formalisms, and closure dynamics in climate modeling.

- **Mori–Zwanzig Formalism / Projection Operators in Climate:**
  - **Givon, D., Kupferman, R., & Stuart, A. (2004).** *Extracting macroscopic dynamics using the Mori-Zwanzig formalism.*

- **Chorin, A. J., Hald, O. H., & Kupferman, R. (2000).** *Optimal prediction and the Mori-Zwanzig representation of irreversible classical mechanics.*

- **Nonlinear Averaging and Stochastic Climate Closures (1990s):**
    - **Hasselmann, K. (1976).** *Stochastic climate models, Part I. Theory.*
    - **Frankignoul, C., & Hasselmann, K. (1977).** *Stochastic climate models, Part II. Application to sea-surface temperature anomalies and thermocline variability.*
    - **Majda, A. J., Timofeyev, I., & Vanden-Eijnden, E. (1999/2001).** *Models for stochastic climate prediction / Systematic strategies for stochastic mode reduction in climate.*

Changes for the Manuscript

In **Section 1 (Introduction)** and **Section 2.4 (Asymptotic Analysis)**, explicitly cite these works together when introducing operator projection P:

"Classical GFD reductions traditionally rely on asymptotic expansions in small physical parameters (e.g., Pedlosky; Vallis) or systematic spatial/temporal averaging procedures (Müller; Majda et al.). From an operator perspective, these reductions represent spatial projection operators P acting on the full state space..."

Summary of Overall Manuscript Modifications to be made

1. **Title Refinement:** Retained clarity while emphasizing GFD scope: *"Spectral Neutrality of Climate Reductions: An Operator Perspective"*.
2. **Mathematical Rigor:** Operator setup re-anchored on Hilbert spaces H, with projection operators $P: H \to H$ and linearized operators explicitly tied to background states $\bar{u}$.
3. **Streamlined Structure:** Compacted Section 3 (EBMs) and Section 4 (β-plane vs. S^2) to eliminate repetitive phrasing and highlight key physical insights.
4. **Positive Baselines:** Include spherical harmonic truncations as exact spectrally neutral benchmarks.
5. **Checked Literature:** Fully audited, corrected, and expanded references to incorporate classical GFD asymptotic and averaging literature.