Response to Referee #2 (RC2)

We thank Referee #2 for their provocative and constructive critique. The review pushed us to clarify the overarching purpose, novelty, and practical utility of our theoretical synthesis.

Major Comments & Clarifications

1. Clarifying Novelty: Uncovering Implicit Approximations in Classic Models

RC2: *"I did not find any new result... it reads more like a note or a comment... Approximations are used to capture qualitative features."*

- **Author Response:** I thank the reviewer for this comment, which highlights the need to state the paper's core theoretical contribution more sharply.

  While simplified GFD models (like 0D Energy Balance Models or β-plane approximations) are standard textbook tools, their classical derivations often **leave critical physical and topological assumptions implicit**:

  - **In Energy Balance Models (EBMs):** Reducing 3D/2D radiative-convective dynamics to a global scalar equation implicitly assumes an effective heat capacity, rapid planetary rotation, or infinite lateral heat transport to justify spatial homogenisation. In our operator framework, the spatial averaging operator Pacting on the non-linear Stefan-Boltzmann term explicitly exposes this implicit limit through the non-commutation term:

    $$P(T^4) - (PT)^4 \neq 0$$

    This quantifies the exact spectral distortion introduced by spatial averaging.

  - **In β-plane Dynamics:** Expanding dynamics on a local tangent plane implicitly forces plane-wave solutions ($e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$), which are topologically incompatible with the compact spherical domain S^2. This geometric reduction introduces unphysical continuous spectra and distorts the discrete planetary Rossby wave spectrum (replacing spherical harmonics $Y_l^m$with planar wavevectors).

  The explicit novelty of this paper is not to replace these models, but to provide a rigorous, operator-theoretic mapping that directly pinpoints where, why, and by how much these implicit physical and topological approximations deform the system's spectrum.

  Furthermore, this synthesized perspective is particularly critical today for data-driven modeling, operator learning, and reduced-order modeling (e.g., Koopman/DMD techniques). When data-driven algorithms learn operators from spatially reduced or projected climate data, understanding whether the underlying reduction is "spectrally neutral" is vital to avoiding spurious numerical modes or misattributed dynamical instabilities.

This implies **for the Manuscript:**

- **Section 1 (Introduction):** Rewritten to state the explicit conceptual goal of the paper as a perspective and diagnostic framework for GFD reductions.
- **Section 3 (EBMs):** Explicitly derived the implicit physical limits required to justify the scalar projection operator P, demonstrating how spatial variance loss deforms the effective climate sensitivity operator.

  "The classical zero-dimensional EBM is frequently presented as an intuitive energy balance. However, from an operator standpoint, it represents the asymptotic limit of a 3D thermodynamic operator under the implicit assumptions of rapid rotation and large heat capacity. Applying the spatial projection operator $P = \frac{1}{|\Omega|}\int_{\Omega}(\cdot)\ d\Omega$ to the full energy equation illustrates how these physical limits manifest as operator non-commutativity..."

- **Section 4 (β-plane vs. S^2):** Contrast the plane-wave ansatz on $\mathbb{R}^2$against exact spherical harmonic eigenfunctions on S^2, showing how domain topology changes the operator's spectral class from discrete to continuous.

  "While the β-plane approximation yields familiar plane-wave solutions $e^{i(kx+ly-\omega t)}$, such waves are globally unphysical on a spherical planet. The projection from $S^2$to $\mathbb{R}^2$replaces the discrete, bounded spectrum of Laplace-Beltrami operators (associated with spherical harmonics Y_l^m) with a continuous Fourier spectrum, breaking spectral neutrality at the topological level."

- **Section 5 (Discussion & Application):** Added a dedicated discussion on the relevance of spectral neutrality for data-driven physics, Koopman operator techniques, and AI emulator architectures

2. Connection Between Radiative Balance and Dynamical Operators

RC2: *"Connection between radiative balance and operator examples seems forced."*

- **Author Response:**

  I appreciate this feedback. In the original draft, the transition between 0D/1D Energy Balance Models (EBMs) and dynamical partial differential equations (PDEs) was abrupt. In the revision, I will frame spatial radiative averaging strictly as a mathematical projection operator P acting on a spatial temperature field $T(x)$. We demonstrate that the well-known spatial variance loss ($\langle T^4\rangle \neq \langle T\rangle^4$) is the exact scalar-operator analogue of non-commuting spatial domain projections $[P, L] \neq 0$ in dynamical GFD.

- **Changes in Manuscript:**
  - **Section 3:** Refactored the EBM section to explicitly showcase spatial integration as a projection operator $P = \frac{1}{|\Omega|}\int_{\Omega}(\cdot)\ d\Omega$, demonstrating how non-linearity breaks operator commutation and deforms effective climate sensitivity.