Response to Referee #1 (RC1)

We thank Referee #1 for their thorough reading and insightful comments on our manuscript. The constructive feedback regarding functional analysis setup, notation, literature integration, and the definition of spectral neutrality has significantly strengthened the clarity and mathematical rigor of the manuscript.

Major Comments & Clarifications

1. Scope & Definition of Spectral Neutrality

RC1: *"The concept of spectral neutrality... is too restrictive... makes all plane/channel models non-neutral... The article does not provide any examples of spectrally neutral reduced models."*

- **Author Response:** I agree with the referee's assessment. Spectral neutrality is not an all-or-nothing requirement for model validity, but rather an idealized theoretical benchmark—analogous to exact energy or enstrophy conservation in numerical discretization schemes. Plane and channel approximations remain invaluable physical tools; the purpose of our analysis is simply to provide a precise operator-theoretic diagnostic for *where* and *how* spectral shifts (such as topology-induced coordinate singularities or boundary layer localization) occur.

  To address the reviewer's concern directly, I will expand the framework to explicitly present **spectrally neutral (and near-neutral) positive baseline examples**, specifically Galerkin projections onto spherical harmonics and normal-mode truncations on S^2, where self-adjointness and domain topology are preserved by construction.

- **This means for the Manuscript:**
  - **Section 2.2:** Added a dedicated subsection defining *"Spectrally Neutral Reductions: Spherical Normal-Mode Truncations"* showcasing exact spectral preservation under orthogonal projection on S^2.
  - **Sections 1 & 5:** Clarified that "spectral non-neutrality" is a diagnostic metric for operator deformation rather than a fundamental flaw of idealized modeling.

2. Linearization, Leading Eigenspaces, and Background States

RC1: *"Line 144 mentions 'leading eigenspaces', but no precise definition is given... Does it apply to linearized dynamics only? How is the background state accounted for?"*

- **Author Response:** The reviewer raises a crucial point regarding the domain of applicability. The spectral definitions in the original draft were overly broad. I will now strictly define the operator framework around **linearized dynamics $L_{\bar{u}}$ evaluated at a specified background state $\bar{u}$** (e.g., state of rest vs. non-trivial baroclinic zonal flow).
- **Changes for the Manuscript:**

- **Section 2.1:** Formally reformulated the operator $L \equiv L_{\bar{u}} = \nabla_{\phi} F(\bar{u})$, explicitly denoting the background field $\bar{u}$.
- **Section 2.3:** Defined "leading eigenspaces" strictly via spectral projection operators associated with the discrete eigenvalues of highest real part Re(λ) or key physical wave modes (e.g., planetary Rossby modes).

3. Mathematical Precision & Operator Spaces (Section 2)

RC1: *"Equation (3) and (4) mathematical notation issues... $P: H \rightarrow H_r$ means LP is not well-defined."*

- **Author Response:** I apologize for the imprecise functional analysis notation in the original manuscript. The reviewer is correct: defining P as mapping to a distinct space H_r breaks the composition $[P, L]$. I will refactor Section 2 within a unified Hilbert space setting $H = L^2(\Omega)$. P is now defined strictly as an orthogonal projection operator $P: H \rightarrow H$ whose range is the reduced subspace $H_r = P(H) \subset H$. Consequently, $P^* = P = P^2$, and the reduced operator $L_r = PLP$ as well as the commutator $[P, L] = PL - LP$ are rigorously defined on the domain $\mathcal{D}(L) \subset H$.
- **Changes in Manuscript:**
  - **Section 2.1:** Rewritten operator setup:

$$P: H \longrightarrow H, \quad P^2 = P = P^*$$

$$L_r = PLP: \mathcal{D}(L) \cap H_r \longrightarrow H_r$$

$$\text{Commutator: } [P, L] = PL - LP$$

4. Literature Integration & Reference Errors

RC1: *"Citation errors (e.g., Müller et al.) and missing references to classical asymptotic expansions and averaging theories."*

- **Author Response:** I sincerely regret the errors in the bibliography of the initial draft. All references have been rigorously audited and corrected against primary scientific databases. Furthermore, I will thoroughly integrate classical asymptotic expansion theory (Pedlosky, Vallis) and 1990s GFD averaging/projection formalisms into the introduction and theoretical context.
- **Changes for the Manuscript:**
  - **References:** Meticulously revised all citation entries. Corrected primary historical citations for climate stochastic models, ocean closures, and operator decompositions.
  - **Section 1:** Added extensive contextual references connecting operator projections to asymptotic scale analysis and Mori–Zwanzig projection formalisms.

Minor Technical Comments & Figures

5. Compactness & Operator Properties (Lines 18–19)

RC1: *"The spherical Laplace tidal operator defines a compact global eigenvalue problem... please either reference the relevant result, or provide an argument for compactness."*

- **Author Response:** I appreciate this correction. I will clarify that compactness applies to the resolvent operator $(L_{\bar{u}} - \lambda I)^{-1}$ on bounded domains for elliptic components, rather than asserting self-evident compactness for the unbounded differential operator itself. Appropriate functional analytic references have been added.
- **Changes in Manuscript:**
    - **Section 1:** Added rigorous framing regarding resolvent compactness and cited standard operator references.

6. Parameter Definitions and Equations (Eq. 5, 7, 10, 11, 13, 15)

RC1: *"Eq. (5): all parameters are not introduced... Eq. (10): left hand side should involve averaged temperature... Eq. (13), (15): ψ, L_D, L_M not defined."*

- **Author Response:** All symbols across Equations (5)–(15) have been systematically defined upon first appearance. Specifically, the spatial averaging bracket $\langle T \rangle$ is explicitly placed on the left-hand side of Equations (10)–(11), and the streamfunction ψ, Rossby deformation radius L_D, and Laplace tidal operator L_M are now formally introduced.
- **Changes in Manuscript:**
    - **Sections 3 & 4:** Systematically audited all variable definitions and corrected the averaging notation in Equations (10) and (11).

7. Vector Hough Harmonics vs. Scalar Eigenfunctions (Line 233)

RC1: *"In the non-rotating limit, the eigenfunctions reduce to spherical harmonics... Hough harmonics are 3-component objects (u, v, h)... L_M cannot have spherical harmonics as eigenfunctions."*

- **Author Response:** The text has been corrected to distinguish between vector Hough harmonics acting on the state vector $(u, v, \phi)^T$ and scalar Laplace-Beltrami eigenfunctions Y_l^m acting on streamfunction or geopotential scalar fields.
- **Changes in Manuscript:**
    - **Section 4:** Corrected Line 233 to explicitly clarify the distinction between 3-component vector Hough modes and scalar spherical harmonics Y_l^m.

8. Figure 2 Revision

RC1: *"Figure 2. The figure caption specifies neither which of the Rossby Hough modes is shown... eigenmodes do not need to vanish at 30N and 60N, which makes Figure 2 unnecessary."*

- **Author Response:** Figure 2 and its caption will be revised to eliminate misleading artificial boundary nodes. The figure will displays the lowest-order planetary Rossby Hough mode, matching the analytical projections presented in Section 4.
- **Changes in Manuscript:**
    - **Figure 2:** Updated plot