



1 **Review article: The future of the physics-based NWP methodology: a**  
2 **critical review on applications of catastrophe theory to the**  
3 **atmospheric science**

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27 **Abstract.** *The physics-based NWP (Numerical Weather Prediction) methodology*  
28 *appears increasingly difficult to enhance the NWP model's forecasting capabilities.*  
29 *This is caused largely by catastrophe/tipping phenomena in the atmosphere as well as*  
30 *the multiplicity of NWP equation solutions. Classical physics from Newtonian*  
31 *mechanics to Einstein's general relativity is essentially the theory of various kinds of*  
32 *smooth behavior other than catastrophe phenomena (water suddenly boils, a cell*  
33 *doubles cancerously etc.) while catastrophe theory (CT) provides a methodology for*  
34 *mathematical treatment of continuous action producing a discontinuous result, and*  
35 *has broad application prospects in numerous subjects. Studies of catastrophe have*  
36 *never stopped but the closely related terminology (tipping) changed subtly, and thus*  
37 *similarities and differences between catastrophe and tipping as well as the rhetoric*  
38 *course of change and development with tipping points have been ascertained in this*  
39 *article with the result showing that CT itself never declines and it gets a new replacer*  
40 *only. Both problems of the NWP development bottleneck and the highly acclaimed*  
41 *AI-weather models enslaved to its knowledge quality could be basically attributed to*  
42 *the emergence of catastrophic/tipping points owing to the nonlinearity of model*  
43 *equations. In this Critical Review, via a comprehensive review and the relevant*  
44 *analyses, we are led to the conclusion that it would be sagacious to first determine*  
45 *which one is the "physical solution" and/or incorporate (e.g.) the second law of*  
46 *thermodynamics as an additional constraint into the NWP model so as to reduce the*  
47 *number of solutions.*

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## 50 **1. Introduction**

51 The physics-based NWP methodology or system based on the principle of fluid  
52 mechanics is still one of the most advanced tools for forecasting quantitatively  
53 weather and climate events so far, and, on the other hand, such a NWP system seems to be  
54 meeting a serious challenge in terms of further enhancing its prediction accuracy, as  
55 shown in “Fig. 1 The time series of the miscellaneous forecasting scores” in this  
56 article, which is based on the relevant datasets (see Code and data availability below  
57 for further details).

58 It is found based on Fig. 1 that, from multifarious points of view, the forecasting  
59 quality of the NWP system is very difficult to make a distinct improvement any  
60 further. The general tendency of the various forecasting scores is indeed toward  
61 enhancement, especially after 2010, but in the very recent years (since 2020 or so) it  
62 has tended gradually to an evener straight line parallel to the abscissa (time axis) with  
63 minor fluctuation, suggesting no palpable improvement in terms of forecasting scores  
64 during the recent period.

65 For example, in Fig. 1 **a** and **b** showing Anomaly Correlation coefficients (ACC)  
66 and Root Mean Square Errors (RMSEs) respectively of the 500 hPa geopotential  
67 height averaged over the northern hemisphere during the period of 2010-2025 for  
68 ECMWF and DWD (see Fig.1 captions for further details), the scores of both ACC  
69 and RMSE for the 500 hPa geopotential height tend to seemingly decline with  
70 fluctuation after 2023. In Fig. 1 **c** and **d** showing the WMO-exchanged scores in terms



71 of RMSE from global forecast centres for 500 hPa geopotential height over northern,  
72 and southern extratropics respectively, it is seen that the two-day (T+48) forecast  
73 error of model runs had kept almost unchanged from 2020 to 2024. On the other hand,  
74 Fig. 1 e and f show an even variation as well in terms of RMSE for the temperature  
75 and vector wind fields respectively at 50 hPa for one-(blue) and five-(red) day  
76 forecasts. In addition, the Continuous Ranked Probability Skill Score (CRPSS)  
77 (Hersbach, 2000) is also employed for measuring the effectiveness of the model in  
78 terms of the forecast lead-time (in days) as shown in Fig. 1 g and h, which imply  
79 simply the tendency to decline after around 2016. By the way, Fig. 1 i gives the time  
80 series for a measure of inconsistency of the 500 hPa height forecasts over northern  
81 extratropics, which show a steady variation with time as well in the recent years (after  
82 2020). And, finally, it is seen in Fig. 1 j that ACC, as a measure of the effectiveness  
83 of the model in forecasting 500 hPa heights at several forecast lead times evolved  
84 over the past decades shown, tend obviously to keep unchanged with minor  
85 fluctuation during the recent period (2020-2025). As a result, looking toward the  
86 future of NWP, it seems that significant improvement amplitude of forecast accuracy  
87 in the case of employing conventional physics-based systems is nearly approaching its  
88 limitation if any (Macnally, 2025).

89 In view of that weather forecast failure happens frequently in case  
90 catastrophe/tipping phenomena emerge in the atmosphere, it would be apparently  
91 associated with the fact that the solution of equations by which the NWP is made  
92 possesses multiplicity owing to nonlinearity of these forecasting equations.



93        Indeed, catastrophe phenomena, such as water suddenly boils, a cell doubles  
94 cancerously, stock markets collapse, and Earth and its moon quake, are very often  
95 found in the atmosphere, hydrosphere or geosphere on Earth and beyond (Thom,  
96 1975; Zeeman, 1977; Poston and Stewart, 1978) and, catastrophe would be an eternal  
97 subject owing to the reality that catastrophes on Earth involve huge loss of life and  
98 property as well as their investigation is of significant scientific importance.  
99 Catastrophe theory (CT) provides a methodology for mathematical treatment of  
100 continuous action producing a discontinuous catastrophe result, and has broad  
101 application prospects in various subjects, including the atmospheric sciences covering  
102 catastrophes like (e.g.) phase transition of moisture, the monsoon onset, sudden  
103 stratospheric warmings (SSW), the northward jumping of subtropical highs during the  
104 Boreal summer, multiple transitions in the developing convection, typhoon  
105 catastrophes in its intensity and track, abrupt changes in the climate system, and many  
106 others (Liu and Tao, 1983; Chao, 1985; Miyamoyo and Takemi, 2012; Yin and Liu,  
107 2019). The formal development of CT is credited chiefly to R. Thom<sup>3</sup> and, owing to  
108 his worldwide reputation as a Fields Prize winner the theory has drawn significant  
109 attention so as to be applied to a variety of fields for dealing with their respective  
110 tricky problems. It is noticed that for the issues which were not satisfactorily settled  
111 by calculus and/or the differential equation theory (including dynamical systems  
112 theory) for a long time in the past, the predictive results were well matched with the  
113 experimental ones via using CT. It is therefore understandable why the British  
114 Encyclopedia Yearbook 1977 reads: "*Catastrophe theory has enabled humans to have*



115 *a rare weapon to overcome benighted ignorance, and to gain a deep insight into the*  
116 *universe."*

117       In contrast with the other multitudinous subjects, the atmospheric sciences have  
118 had quite few results till now in terms of CT application, though there are various  
119 catastrophic phenomena found in the Earth's atmosphere. To our understanding, the  
120 reasons for this might be largely attributed to the recondite topology background  
121 knowledge needed for reconstructing the relevant models of catastrophe to  
122 atmospheric discontinuous evolution, due to the high complexity of the atmosphere as  
123 a many-body system. In addition, a vital factor lies in that the NWP seems to be  
124 encountering the so-called development bottleneck in terms of further improving its  
125 forecasting scores, for which no reasonable connection to catastrophic phenomena in  
126 the atmosphere is ascertained so far yet.

127       More significantly, it has been noticed that investigations into catastrophe have  
128 never stopped but the closely related terminology (tipping) changed subtly in an  
129 unusual way, and forasmuch, similarities and differences between catastrophe and  
130 tipping (including the rhetoric course of change and development with tipping points,  
131 and particularly the issue on the popular emergence of "tipping points" - although  
132 objectionable ever to some scientists - before their adoption in climate change  
133 discourse around 2000) are being in details traced to its source as well in this  
134 Perspective. This would in a great extent evoke the philosophic interest on how  
135 language controls your cognition, as is delivered in Wittgenstein's 'Tractatus  
136 Logico-Philosophicus' (Wittgenstein, 1975). In parallel, we are also motivated by the



137 realization that the advance in applications of the artificial intelligence/deep learning  
138 (AI\_DL) approaches to weather/climate forecasting is indeed in the ascendant but  
139 without an in-depth understanding that weather events are in nature non-probabilistic  
140 with no possibility of Poincare recurrence as well as the knowledge in the repository  
141 database available for machine learning during its training phase is not of quality yet  
142 at the moment. That is, the problems of the NWP development bottleneck and the  
143 AI-weather models enslaved to its knowledge quality might both be attributable to the  
144 problem of catastrophic/tipping point emergence of these nonlinear model predictive  
145 systems involved.

146 It should be stressed also here that, for catastrophic events, no matter it happened  
147 in the atmosphere, hydrosphere or geosphere on Earth, the methodology used for  
148 constructing their catastrophe models as done in CT is universal. Catastrophe would  
149 be an attractive subject just because not only natural disasters would devastate local  
150 communities, but these rare events even able to trigger new ways of thinking,  
151 providing a treasure trove of data that would certainly be used to reduce effectively  
152 vulnerability (Editorial, 2012; Tsur and Withagen, 2013; Zhang *et al.*, 2017;  
153 Farnsworth, 2023).

154 Last but not least, it is plausibly thought-provoking that CT had been questioned  
155 even at the very beginning of its emerging (Zahler and Sussmann, 1977). This is just  
156 like the case of that Aristotle's logic had impacted 2000 years while every step of  
157 progress in science, logic and philosophy during the whole recent history is actually  
158 earned against the opposition of Aristotle's disciples who seem to have stifled the



159 creative thinking of their contemporaries, as narrated by B. Russell in his book  
160 *HISTORY OF WESTERN PHILISOPHY* (Russell, 1961). Consequently, we suppose,  
161 an earlier Perspective might be helpful to expedite the progress to fire up the scientists  
162 for clarifying the basic concepts as well as applying CT to their respective interests,  
163 and especially promote the enhancement of NWP forecasting accuracy so as to break  
164 its development bottleneck.

165 **2. A profile for catastrophe theory with a survey of its applications to the**  
166 **atmospheric phenomena**

167 In mathematics, CT is a branch of bifurcation theory in investigation into  
168 dynamical systems, which studies and classifies phenomena characterized by sudden  
169 shifts in behavior arising from small changes in circumstances, analyzing how the  
170 qualitative nature of equation solutions depends on the parameters that appear in the  
171 equation and change gradually. Small changes in certain parameters of a nonlinear  
172 system can cause equilibria to appear or disappear, or to change from attracting to  
173 repelling and vice versa, leading to large and sudden changes in the behavior of the  
174 system represented by a state variable.

175 The core of CT lies in Thom's classification theorem. Employing the singularity  
176 theory, Thom showed that when the number of the controlling factors in a dynamical  
177 system is not more than four, these universal, discontinuous processes take only the  
178 seven distinct types, called as the elementary catastrophes (Thom, 1975). CT is not a  
179 single thread of ideas; it resembles more closely a web, with innumerable  
180 interconnected strands. The elementary catastrophes of Ren éThom are but one strand,



181 though an important one, and, rigorous studies of partial differential equations can  
182 often lead to the elementary catastrophe.

183 One of the most important components of the procedure for constructing a  
184 catastrophe model to understand specific catastrophic phenomena is the determination  
185 of the state variables and relevant controlling parameters as well as the proper  
186 canonical catastrophe function family. It might be settled down through an appropriate  
187 transformation of variables to convert such a function into a canonical one. We shall go  
188 into some examples below, but for the present a good relevant picture might be that of a  
189 ball rolling around a landscape and ‘seeking’ through the agency of gravitation to settle  
190 in some position which, if not the lowest possible, is at least lower than any other  
191 nearby (cf. also Fig. 3 below).

192 From the point of view of phenomenology, all the processes characterized by  
193 sudden change or abrupt change/transition within a relatively small span of time,  
194 could be viewed as catastrophes. There are a large number of catastrophic phenomena  
195 found in the Earth’s atmosphere, as mentioned in the very beginning of this  
196 Perspective. Many of the catastrophic phenomena in the atmosphere have been  
197 investigated from a general angle of view, using the various methodologies (including  
198 CT), which would concentrate largely on the following aspects: 1) Abrupt climate  
199 change; 2) regime shift in the atmospheric circulation pattern or other regime change  
200 behavior; 3) transition between the multiple equilibria; 4) burst/transition of convection;  
201 and, 5) the miscellaneous. Here, we will not go into their details owing to limited  
202 space. Among the others, the studies with applications of CT to the atmospheric



203 science will be specifically introduced and discussed below (cf. next section), which  
204 might provide us with a general impression how CT has enabled meteorologists to  
205 have a powerful weapon to overcome some blindness, and to gain a deep insight into  
206 the atmospheric behavior.

### 207 **3. Literature Review**

#### 208 **3.1 A brief introduction to applications of CT to the atmospheric sciences**

209 Since CT was created there have been numerous relevant works to its applications to be  
210 published. Over the scope covered by this article (since 1975), the results related to  
211 applications of CT to the atmospheric sciences might be briefly summarized as  
212 follows:

##### 213 3.1.1 Fold and cusp catastrophes of a climate system of zero-dimension

214 Fraedrich (1978; 1979) developed a climate system of zero-dimension with its  
215 complexity reduced to a minimum. This system can be analyzed without explicit time  
216 integration according to CT, so that the resilience of the system, due to changes in the  
217 external parameters and due to the variation of the state variable, can be determined.  
218 External parameter changes can lead to structure instability occurring at fold lines,  
219 which meet at the cusp catastrophe. It is noteworthy that at fixed external parameters  
220 the system develops deterministically but the flow depends on the initial conditions  
221 (the branch on which the flow is located originally), which suggests a new concept of  
222 the so-called “historical dimension”, implying which final solution is reached depends  
223 on its evolutionary history.

##### 224 3.1.2 The catastrophes as seen in a low-order spectral model



225 Here two cases are given.

226 1) Vickroy and Dutton (1979) investigated the steady solutions and their stability  
227 properties for a low-order spectral model of a quasi-geostrophic flow where a zonal  
228 flow is modified by two smaller scale disturbances. Their results show that the entire  
229 solution surface can be displayed graphically by a form of the well-known cusp  
230 catastrophe surface. For forcing of all the three components, the morphogenesis set is  
231 more complex, containing regions in which there are one, three or five solutions. In fact,  
232 it had been noticed earlier in 1970s that any successful theory of atmospheric climate  
233 will have to contend with multiple solutions (e.g., Charney and De Vore (1979) posed  
234 independently the concept of “multiple equilibria” as well in 1979). Later in Liu and  
235 Tao (1983), by using a modified spectral model based on that in Vickroy and Dutton  
236 (1979), the northward jumping of the subtropical highs is analyzed and its catastrophe  
237 mechanism is interpreted in terms of CT.

238 It should be pointed out here that, from the multiple time-scale perspective, the  
239 fact that an external parameter (e.g. forcing) is actually varied at a “slower” timescale  
240 (accordingly there exists a slowly evolving state like one described in Methven and  
241 Berrisford (2015)) is potential at all and thus the multiple solutions are most likely to  
242 realistically occur in a nonlinear mathematical equation such as the Navier-Stokes  
243 equation. This is also noticed recently by Ghil (2019) which argues, in a slightly  
244 different way, that “long-term climate prediction is a bit harder than Von Neumann  
245 envisaged at the time, because the forcing changes in time, too”.

246 2) In Yost and Shirer (1982) a three-component spectral model of a



247 two-dimensional shallow convection is used for displaying the different forms and  
248 stability hierarchies of the stationary solutions. The results show that the steady  
249 behavior of the three-component model is described by three cusps. The location of  
250 these cusps and Hopf bifurcation curves determines the regions of multiple equilibria  
251 and their stability. This model also exhibits steady-state hysteresis and a mechanism  
252 whereby catastrophe in the magnitude of the stationary solution can occur.

### 253 3.1.3 Sudden stratospheric warmings (SSW) viewed as a catastrophe

254 SSW is simply viewed as a catastrophe in e.g. Chao (1985) where it is mentioned  
255 that if a model stratosphere possesses multiple equilibria, then no matter how crude the  
256 model is, it is possible to generate a sudden warming. And, many qualitative results of  
257 some previous modeling studies of SSW are interpreted by just using CT with the help  
258 of its basic concepts (such as hysteresis, cusp or triggering). In principle, SSW can be  
259 attempted to construct a catastrophe model for understanding its potential mechanism  
260 as well, and, moreover, based on such a catastrophe model, to further make an  
261 application to the prediction of the chain of events in SSW.

### 262 3.1.4 Illustrations for quantitative applications of CT (Chao, 2013; Yin and Liu, 2019)

263 Once there existed the myth that CT is purely qualitative, and as a matter of fact,  
264 the catastrophe theory methodology can be used as mathematical tools to provide  
265 quantitative information that may be tested by observations or experiments.

#### 266 1) The double criteria strategy for the precipitation forecasting.

267 Based on some basic concepts of catastrophe theory Chao (2013) proposed a  
268 novel strategy for cumulus parameterization for the correction of systematic



269 errors in the precipitation diurnal cycle (PDC) over land in a general  
270 circulation model (GCM), and thus made an improvement in the performance  
271 of the GCM. The new scheme with double criteria led into is designed based  
272 on the fact that, once begun, cumulus convection continues to exist, regardless  
273 of whether the onset criterion is still met, until the termination criterion is met,  
274 with the intensity of cumulus precipitation related to how far the state is from  
275 the termination, not the onset, criterion. The catastrophe-concept-based  
276 parameterization was tested with the results showing a dramatic improvement.

277 2) The Asian summer monsoon onset (Yin and Liu, 2019).

278 Monsoon has been puzzling that its occurrence is sudden with the rapid onset  
279 observed while the evolution of the differentiating heating, as its key controlling factor,  
280 is gradual with time of year. What is the mechanism for the monsoon onset is still under  
281 controversy till now. Yin and Liu (2019), focusing on the Asian summer monsoon with  
282 identification with the fact that the Asian summer monsoon is essentially caused by  
283 temporal changes in the differentiating heating owing to surface inhomogeneity such  
284 as land-sea contrasts, uses CT to show how gradual changes in the atmospheric heating  
285 source with time will cause an abrupt variation of the relevant atmospheric circulations  
286 and then trigger the monsoon onset (Yin and Liu, 2019). They completed their  
287 argumentation, based on the National Centers for Environmental Prediction Final  
288 analysis (NCEP-FNL 2000) data, with a conceptive model worked out using the case of  
289 2010 as an example for the monsoon onset by the cusp catastrophe surface employed.

290 3.1.5 Other catastrophe-related studies to the atmospheric sciences

291 1) For studying the response of ice-sheets to climate change, Oerlemans (2003)



292 developed a simple axi-symmetrical quasi-analytical model, and equilibrium  
293 states are analyzed and the response to periodic forcing is studied to illustrate  
294 the properties and possibilities of the model. The results show that the coupling  
295 of mass balance with the surface elevation of ice-sheet leads to nonlinear  
296 behavior and branching of the equilibrium solutions, with the qualitative  
297 behavior of the system being just that of the cusp catastrophe.

298 2) Considering the evolution of particle size in a monodisperse aerosol population  
299 during the activation and deactivation of cloud condensation nuclei (CCN),  
300 Arabas and Shima (2017) reveal that the system undergoes a saddle-node  
301 bifurcation and a cusp catastrophe. It is noticed that numerical integration of  
302 the system coupled with a simple air-parcel cloud model portrays two types of  
303 activation/deactivation hysteresis: one associated with the kinetic limitations  
304 on droplet growth when the system is far from equilibrium, and another  
305 occurring close to equilibrium with the cusp catastrophe associated.

306 3) For the bifurcation analysis of a simple energy balance model (EBM) for the  
307 future climate of the Earth, Kypke *et al.* (2020) address largely the question  
308 whether the nonlinear processes intrinsic to the atmospheric system, including  
309 natural positive feedback mechanisms, is able to cause a mathematical  
310 bifurcation of the climate state, as a consequence of continued anthropogenic  
311 forcing by rising greenhouse gas emissions. Their main results show that such a  
312 bifurcation could touch off an abrupt change to a drastically different climate  
313 state in the EBM, which is warmer and more equable than any climate existing



314 on Earth since the Pliocene epoch. In previous papers, with this EBM adapted  
315 to paleoclimate conditions, it was shown to exhibit saddle-node and cusp  
316 bifurcations, as well as hysteresis (its general schematic graph might be shown  
317 as Fig. 2(a)-(b)). The EBM was validated by the agreement of its predicted  
318 bifurcations with the abrupt climate changes that are known to have occurred in  
319 the paleoclimate record. Catastrophe happens only if the increasing ocean heat  
320 transport FO reaches a certain threshold (see e.g. Fig. 2(c)) in which the EBM  
321 is adapted to fit Anthropocene climate conditions. In addition, the EBM  
322 investigates four naturally occurring nonlinear feedback processes and predicts  
323 that a bifurcation resulting in a catastrophic climate change, to a  
324 pre-Pliocene-like climate state, will occur in coming centuries.

325 4) For paleoclimates, Jahren and Sternberg (2003) have described an Eocene  
326 Arctic rain forest with RH (relative humidity, denoted herein by  $\delta$ ) estimated  
327 to be  $\delta = 0.67$  and meanwhile modern data, from a variety of sources,  
328 suggest similar values of Arctic RH. It is assumed that the greenhouse  
329 warming effect of water vapor is determined by the Clausius - Clapeyron  
330 relation which tells us that below the freezing temperature the concentration  
331 of water vapor is very low and therefore it has very little greenhouse effect.  
332 However, above freezing, if a source of water is available (e.g. oceans), then  
333 the concentration of water vapor and its greenhouse warming effect increase  
334 rapidly. This is shown clearly in Fig. 2(d), where the three curves are  
335 representative for different levels of relative humidity and on each of the



336 curves of Fig. 2(d), the dashed portions with negative slope are unstable,  
337 while portions with positive slope are asymptotically stable (in the sense of  
338 Liapunov). Bi-stability (the coexistence of two stable solutions) occurs sooner  
339 when water vapor is present, demonstrating a typical fold catastrophe. All of  
340 these results above are very informative and suggestive from the perspective  
341 of applied catastrophe theory.

### 342 **3.2 On tipping points in the atmospheric/climatic system**

343 In addition to the sheer number of related publications, the media's exaggerated  
344 wordings like "Warming hits tipping point" and "Climate nears point of no return"  
345 mentioned as well as "we are on the precipice of climate system tipping points  
346 beyond which there is no redemption" are very suggestive in terms of "the idea is  
347 spreading like a contagion" (Gladwell, 2000; Hansen, 2005; Edenhofer, 2006).  
348 Although there's no strong evidence that the climate as a whole has a point beyond  
349 which it switches neatly into a new pattern, individual parts of the system could be in  
350 danger of changing state quickly, and perhaps irretrievably.

351 The popular emergence of "tipping point" before their adoption in climate change  
352 discourse is around 2005, as mentioned in (e.g.) "The tipping point trend in climate  
353 change communication" by Russill and Nyssa (2009). Actually, aside from the  
354 rhetorically striking character of "tipping point", tipping is the mathematically  
355 well-defined generalization of the bifurcation treated in the autonomous dynamical  
356 systems to the non-autonomous setting. As a result, in the context of dynamical  
357 systems theory, tipping should be associated with bifurcation and, both would have a



358 catastrophe connotation. For the sake of coherence in terms of style or manner of  
359 writing, here we will not go into detail, including its progressive evolution as a  
360 scientific concept as well as the debate on tipping point usage in academic literature,  
361 and instead, a further relevant discussion on it will be put in the final subsection  
362 below.

363       Next we would like to introduce several illustrations about tipping points in the  
364 atmospheric system/climate system, which would be largely brought into the  
365 framework of applied catastrophe theory, so as to get a general impression on the  
366 concept of tipping point. These illustrations include: 1) Thermodynamic analysis of  
367 snowball Earth hysteresis experiment; 2) the approach for projecting the timing of  
368 abrupt winter Arctic sea ice loss; 3) another look at climate sensitivity; 4) the role of  
369 time-varying external factors in the intensification of tropical cyclones; 5)  
370 rate-induced tipping in ecosystems and climate, and 6) the tipping point of the iceberg  
371 as well as 7) overshooting tipping point thresholds in a changing climate.

372       Some illustrations of the tipping point in the weather/climate system are as  
373 follows:

#### 374       3.2.1 Thermodynamic analysis of snowball Earth hysteresis experiment

375       Generally speaking, such behavior as external parameter changes beyond its threshold  
376 leading to catastrophe is attributable to the fact that there exist multiple equilibria in  
377 the system under study. It can be foreseen in a relevant work of e.g. Lucarini *et al.*  
378 (2010) in which it is shown that, for a large range of values of the solar constant there  
379 are two possible distinct climatic steady states detected and, accordingly, the system



380 features two distinct branches, corresponding to warm (W) vs. Snowball (SB) climate  
381 conditions. Again, we see that the existence of multiple steady states or equilibria is  
382 substantial for catastrophe (i.e. the transitions of W-SB and SB-W) occurrence. In  
383 other words, at the bifurcations (tipping points) the system makes a transition from an  
384 unstable to a stable state since macro-scale destabilizing mechanisms are more easily  
385 triggered in the case of the presence of larger fluctuations. And, the hysteresis just  
386 characterizes the delaying or buffering process before catastrophe (cf. also Fig. 4 (b)  
387 below that shows when it enters, in whatever way, into  $\Delta$ -area (“the buffering area”),  
388 catastrophe happens), suggesting that forecasting the emergence of abrupt change in  
389 the atmospheric behavior would better take account of the both thresholds (double  
390 criteria).

### 391 3.2.2 Abrupt and irreversible winter Arctic sea ice loss.

392 It is observed that it may occur under anthropogenic warming due to the  
393 disappearance of a sea ice equilibrium at a threshold value of CO<sub>2</sub>, commonly referred  
394 to as a tipping point. Previous works have been unable to conclusively identify  
395 whether a tipping point in winter Arctic sea ice exists because fully coupled climate  
396 models are too computationally expensive to run to equilibrium for many CO<sub>2</sub> values.  
397 Hankel and Tziperman (2022) highlight the inefficacy of using a single experiment  
398 with slow-changing CO<sub>2</sub> to discover changes in the sea ice steady state and provides a  
399 novel alternative method that can be developed for the identification of tipping points  
400 in realistic climate models.

### 401 3.2.3 Sensitive dependence of the Earth’s climate system



402 The Earth's climate system is characterized by sensitive dependence to the  
403 parameters, and sensitive dependence exists in either the climate system or in climate  
404 models. Zaliapin and Ghil (2010) illustrate it using a classical energy balance model  
405 (EBM) with nonlinear feedbacks. The EBM exhibits two saddle-node bifurcations,  
406 more recently called "tipping points", which give rise to three distinct steady-state  
407 climates, two of which are stable. In a truly nonlinear setting, indeterminacy in the  
408 size of the response is observed only in the vicinity of tipping points. They show that  
409 small disturbances cannot result in a large-amplitude response, unless the system is at  
410 or near such a point, suggesting a fold catastrophe.

#### 411 3.2.4 The tipping points as seen in Hurricane studies

412 The role of time-varying external parameters in tropical cyclone dynamics is  
413 explored through a low-order conceptual box model where stable-to-stable state  
414 transitions may be linked to tropical cyclone intensification, dissipation, or eyewall  
415 replacement cycles (Watson and Quinn, 2024). The findings are brought together in a  
416 case study of Hurricane Irma (2017), highlighting the role of the radial vorticity  
417 gradient in behavior such as rate-induced tipping and overshoot recovery.

#### 418 3.2.5 Rate-induced tipping in ecosystems and climate

419 The climate system as well as ecosystems might undergo relatively sudden  
420 qualitative changes in the dynamics when environmental parameters or external  
421 forcings vary due to anthropogenic influences. Feudel (2023) briefly reviewed the  
422 mechanisms of the prominent tipping events and investigated rate-dependent tipping  
423 phenomena which occur in non-autonomous systems characterized by multiple



424 timescales. His detectability study shows that depending on the rate of environmental  
425 parameter drift, the saddle-node bifurcations might be masked or hidden, and they can  
426 only be detected if a critical rate of environmental drift is crossed. This analysis  
427 reveals that unstable states of saddle-type are the organizing centers of the global  
428 dynamics in non-autonomous multistable systems

### 429 3.2.6 The tipping point of the iceberg

430 According to Walker (2006), the most striking of the vulnerable components in  
431 the tipping point discourse might be in the Arctic. As pointed out by him, if  
432 disappearing ice and dying polar bears can tip public opinion over its ‘be very worried’  
433 threshold into the realms of greater action, then further tipping points in the human  
434 world might have their own, positive, role to play. Ottmar Edenhofer, an economist at  
435 PIK, found that in some economic models of responses to climate change, increasing  
436 carbon prices encourages renewable energy. Edenhofer *et al.* (2006) stresses it by  
437 saying “Our task is to find a way to kick the economic system into a new equilibrium  
438 and we can use the tipping points of the market to achieve that” and especially  
439 “Tipping points are part of the problem, but they could also be part of the solution.”

### 440 3.2.7 Overshooting tipping point thresholds in a changing climate

441 Recent in a review, Ritchie *et al.* (2021) addressed themselves to the issue on  
442 overshooting tipping point thresholds. As atmospheric greenhouse gas concentrations  
443 continue to rise as a result of fossil fuel burning, human activity could trigger tipping,  
444 and the impacts would be difficult to adapt to. Previous studies report low global  
445 warming thresholds above pre-industrial conditions for key tipping elements such as



446 ice-sheet melt. If so, high contemporary rates of warming imply that exceeding these  
447 thresholds is almost inevitable, which are widely assumed to mean that we are now  
448 committed to suffering these tipping events. Recently developed theory indicates that  
449 a threshold may be temporarily exceeded without prompting a change of system state,  
450 if the overshoot time is short compared to the effective timescale of the tipping  
451 element. To demonstrate this, transparently simple models of tipping elements with  
452 prescribed thresholds are taken into consideration, which are driven by global  
453 warming trajectories that peak before returning to stabilize at a global warming level  
454 of 1.5 degrees Celsius above the pre-industrial level. These results highlight the  
455 importance of accounting for timescales when assessing risks associated with  
456 overshooting tipping point thresholds. In short words, ongoing global warming is  
457 likely to cause tipping point thresholds to be passed, but an abrupt system change can  
458 still be avoided if the warming is reversed quickly relative to the timescale of the  
459 tipping element.

460 Often the picture of the stability landscape is translated into a specific bifurcation  
461 diagram exhibiting hysteresis, showing the two stable states and the unstable one  
462 separating those two depending on the intrinsic parameters of the system or the  
463 external forcing (Fig. 3a). And, we might illustrate the different tipping mechanisms  
464 (see Fig. 3b) based on the classification as done in Ashwin *et al.* (2012). It is noticed  
465 that Feudel (2023) focuses actually on the mechanism of rate-induced tipping caused  
466 by basin boundary crossings. They unravel the mechanism of this transition and  
467 analyze, in particular, the role of such basin boundary crossings in non-autonomous



468 systems when a parameter drift induces a saddle-node bifurcation in which new  
469 attractors and saddle points emerge, including their basins of attraction. Furthermore,  
470 they investigate the detectability of those bifurcations by monitoring a single  
471 trajectory in state space and find that, depending on the rate of environmental  
472 parameter drift, such saddle-node bifurcations might be masked or hidden, which can  
473 only be detected if a critical rate of environmental drift is crossed. As a matter of fact,  
474 the rate-induced tipping can also occur when there is only one stable state present and  
475 the system is characterized by different timescales (slow-fast system). The dynamics  
476 of such systems can be described by so-called critical manifolds in the case of a  
477 perfect timescale separation or slow manifolds, when the timescale separation is finite.  
478 Here we will not go into its details and the readers of interest may refer directly to  
479 the original works (Ashwin *et al.*, 2012; Feudel, 2023).

480 In addition, one would like to mention the fact that the terminology of “tipping  
481 point” has been most likely used to describing the tipping itself other than discussing  
482 tipping behavior and its potential mechanisms (van der Ploeg, 2016; Verendel *et al.*,  
483 2016; Barrett and Dannenberg, 2018; Kopp *et al.*, 2025; Rietkerk *et al.*, 2025).  
484 Moreover, Hansen *et al.* (2007) elaborates his use of the tipping point concept in a  
485 manner extending the definition in Lindsay and Zhang (2005).

486 “Tipping point”, although objectionable to some scientists, conveys aspects of  
487 climate change that have been an impediment to public appreciation of the urgency of  
488 addressing human-caused global warming. It is a valid concept: as climate forcing and  
489 global warming increase, a point can be reached beyond which part of the climate



490 system changes substantially with only small additional forcing. Such a concept is  
491 actually another version of bifurcation set in the parameter plane for e.g. the cusp  
492 catastrophe in terms of CT in nature. Tipping points are a focus of the NASA (2007)  
493 press release for the paper. It emphasizes “critical tipping points”, and suggests that,  
494 “Tipping points can occur during climate change when the climate reaches a state  
495 such that strong amplifying feedbacks are activated by only moderate additional  
496 warming” (NASA, 2007, paragraph 3). The press release encourages the impression  
497 that the research is centrally concerned with tipping points, and at least one news  
498 story focused on this aspect primarily.

#### 499 **4. Discussion**

##### 500 **4.1 Nature and implication of the catastrophic behavior in the atmosphere**

501 Since Charney and De Vore (1979) and Vickroy and Dutton (1979) posed  
502 independently the concept of multiple equilibria<sup>19-20</sup>, many works have contributed to  
503 the investigation into the issue of multiple solutions existed in a nonlinear equation. As  
504 for the cases in nature it is very common that there exist multiple stable states. Taking  
505 bi-stability as an example, we may give a couple of simple illustrations: 1) fluorescent  
506 lamp - in the operating voltage range of 220V the lamp would be in one of the two  
507 possible states, depending on the original state (light-on state or an unlit one); and, 2) an  
508 object under pushing with the pushing force greater than kinetic frictional force but  
509 smaller than static one, there would be two possible effects to occur, depending on the  
510 original state (static or dynamic state) as well. Both of these two illustrations show that  
511 initial conditions (or alternatively, the history dimension) determine the terminal state:



512 for the case of fluorescent lamp, in the same external forcing of the operating voltage  
513 range of 220V without starter working, the lamp will be in an unlit (light-on) state if it is  
514 switched off (switched on) originally; and for the case of an object under pushing, with  
515 the same external forcing of the pushing force greater than kinetic frictional force but  
516 smaller than static one, the object will keep moving (static) if it was in motion (stillness)  
517 originally. Obviously, these two cases suggest that the multiple stable states or multiple  
518 equilibria in a dynamic system could be found at one and the same value of a certain  
519 external forcing (parameter). On the other hand, a certain parameter in an equation  
520 might be actually varied at a “slower” timescale (so “slow” that it can be regarded as  
521 almost unvaried, and thus a corresponding quasi-steady state or quasi-equilibrium as a  
522 function of time would emerge, as is implied by the terminology of “quasi-equilibrium”  
523 that might be a slowly evolving background state in a nonlinear dynamic system  
524 described by a nonlinear equation) and, the multiple solutions (as functional of time)  
525 are therefore most likely to realistically occur (cf. also Fig. 4a).

526 The factor that allows a time-asymmetric differential equation such as the  
527 Navier-Stokes equation to give birth to both order and chaos is nonlinearity. As we  
528 saw in thermodynamics, this basically means ‘getting more than you bargained for’.  
529 As a matter of fact, when we contend with a nonlinear equation, except for no analytic  
530 solution existing, one of the hardest issues is about the uniqueness of solution. As  
531 already mentioned above, the multiple steady states or multiple equilibria in a dynamic  
532 system could be found at one and the same value of a certain external parameter, which  
533 is just the intractable problem caused by nonlinearity. Chaos is caused by sensitivity to



534 initial-conditions while catastrophe is by multiple stable states coexistence though both  
535 are attributable to nonlinearity.

536 Here, it is also necessary to point out that both of these two illustrations for  
537 bi-stability above show that initial conditions determine the terminal state. In terms of  
538 the history dimension the terminal state (the solution) is determined by the branch the  
539 system is located initially at (initial conditions; see e.g. Fig. 4a). Indeed, the problems  
540 dealt with by Newtonian mechanics are initial-value ones. However, as far as NWP is  
541 concerned, the conventional initial conditions for a GCM model are not the ones in the  
542 sense of Newtonian mechanics at all. For a set of predictive equations, the initial  
543 conditions in the sense of Newtonian mechanics should in nature be the ones at the  
544 moment when the so-called big bang takes place. Apparently, such data do not exist at  
545 all. Moreover, the conventional initial conditions for a GCM model at, say, 0000 UTC  
546 1 January 2000, evolved actually from 0000 UTC 31 December 1999 or earlier, which  
547 are definitely not relevant to the initial conditions in the sense of Newtonian mechanics  
548 and thus will absolutely impact the forecasting accuracy of a NWP set of equations.

549 To put it bluntly, the occurrence of catastrophes might be attributable to multiple  
550 stable states on account of the set of controlling equations itself having non-unique  
551 solution. Such an argumentation of “quantitative might cause qualitative change”  
552 simply cannot hold water for a linear system without multiple quasi-equilibria existing,  
553 as implied by CT. The bifurcation of real-valued functions does describe the simplest  
554 kinds of essentially nonlinear behavior (see e.g. Fig. 4c).



555           What needs to be emphasized here is that the non-unique solution of a set of  
556 equations is not caused only by nonlinearity but also by insufficiency of the set of  
557 controlling equations for the atmosphere itself in terms of “controlling”. The set of  
558 atmospheric equations on which NWP is based, covers the mass conservation law (the  
559 mass continuity equation), momentum conservation law (the Newtonian second law),  
560 and energy conservation law (the first law of thermodynamics, as the special form of  
561 the total energy conservation law in thermodynamics), but without (say) the second  
562 law of thermodynamics incorporated into. As known to all, in classical  
563 thermodynamics and statistical physics, the second law of thermodynamics is a  
564 fundamental law that controls the evolutionary direction of an irreversible process in a  
565 many-body system (a system of numerous degrees of freedom) like the atmosphere.  
566 Surely, insufficiency of “controlling” would lead to insufficiency of restriction on the  
567 atmospheric motion, causing eventually multiple solutions to appear (Liu Chongjian  
568 *et al.*, 2002; Liu and Liu, 2005; Liu *et al.*, 2006; Liu *et al.*, 2007; Liu Ying and  
569 Chongjian Liu, 2007; Liu *et al.*, 2011).

570           As a matter of fact, the earth-atmospheric system, as a many-body or complex  
571 system (Altman, 2018; Lambiotte *et al.*, 2019; Battiston *et al.*, 2021), should be  
572 contended with via both deterministic and statistical methodologies. Early in the 1990s  
573 a unified vision of time had been suggested (Coveney and Highfield 1992) in which the  
574 temporal evolution of a large dissipative system is divided uniquely into two totally  
575 independent components (sub-dynamics). There is a ‘kinetic’ part which describes how  
576 the system evolves over long intervals of time, and contains the approach to



577 thermodynamic equilibrium, while the remaining ‘non-kinetic’ part describes the  
578 transient behavior starting from the initial conditions, but which vanishes during the  
579 evolution. The precise conditions that govern this division between long- and  
580 short-term behavior remain unclear, that is at what point the molecules ‘forget’ the  
581 initial conditions and head for equilibrium under the influence of the arrow of time. In  
582 other words, initial conditions are able to only exert a limited influence on solutions of  
583 an (e.g. differential) equation within a certain span of time, especially in the case of  
584 multiple equilibria emergence leading possibly to catastrophe happening.

585

586 **4.2 Regarding the possibility of CT for promoting progress on key atmospheric**  
587 **science problems.**

588 Firstly, it is noticed that, as implied in some illustrations above, in some  
589 circumstances only employing the basic concepts of CT such as (for e.g. cusp  
590 catastrophe) bi-stability, hysteresis and catastrophe can suggest new lines for the  
591 weather/climate prediction and research strategy. For example, in the case of multiple  
592 equilibria existing, temporal changes in a meteorological variable might be sometimes  
593 dramatic since an unstable equilibrium state is unattainable or unobservable, as shown  
594 by the thick-dashed line in Fig. 4b, and then once the parameter (as function of time)  
595 exceeds a threshold, catastrophe from a stable state to another one will emerge, thus  
596 causing remarkable changes in the meteorological variable. That is to say, the  
597 weather/climate prediction might output more than a single solution at a certain  
598 forecasting moment, suggesting the possibility of multiple predictive outcomes. On



599 the other hand, double thresholds are associated with hysteresis convention in CT  
600 language, which will lead to twin criteria employed for, say, precipitation  
601 parameterization (Chao, 2013).

602 As a preliminary attempt at applying CT to understanding abrupt changes in  
603 natural phenomena, the frequently-used way is to construct a catastrophe model for  
604 investigating specific catastrophic processes. Constructing a catastrophe model is  
605 comprised of two basic elements: to determine the state variables and relevant  
606 controlling parameters; and, to select the proper canonical catastrophe function family  
607 to match the characteristics of phenomena under study.

608 A deeper relevant analysis reveals that there are some underlying regularities in  
609 the mathematical structure which permit routine calculations of how such systems  
610 behave, based on the traditional applied mathematician's use of Taylor series  
611 approximations (by the way, the Einstein's mass-energy relation  $E = c^2M$  where  $E$  and  
612  $M$  are energy and mass, respectively, with  $c$  being the light velocity is just a unique  
613 exemplification in terms of Taylor series approximations). It might be settled down  
614 through an appropriate transformation of variables to convert a function into a  
615 canonical one (that is, Thom's standard polynomials). It is seen from the illustration of  
616 applications in the monographs (Thom, 1975; Zeeman, 1977; Poston and Stewart,  
617 1978) on CT that the canonical catastrophe formula or its standard unfolding is often  
618 linked to Taylor expanding. In this context, the main mathematical trust is to develop  
619 a proper understanding of the geometric and algebraic methods used to handle Taylor  
620 series felicitously. Once developed, the methods provide powerful tools for tackling a



621 wide range of problems, opening up perspectives to which the traditional use of  
622 Taylor expansions as a source of approximations, to be justified *post hoc* by  
623 experiment, is blind.

624 As for a novel CT-based forecasting methodology suggested under way, we will  
625 organize a separate paper to specifically address it shortly.

626

#### 627 **4.3 Some enlightenment inspired by applying CT**

628 We would like to suppose the present moment as an appropriate occasion to make a  
629 Perspective about the NWP's development bottleneck, by dint of a review of  
630 applications of CT to the atmospheric sciences, since we are facing the following  
631 realism. On one hand, the NWP methodology is not so satisfactory with even  
632 nowcasts often going very wrong in terms of forecast score though the improving  
633 strategy such as ensemble forecasts is employed, suggesting that NWP seems to  
634 encounter a serious challenge owing to the problem of multiple solutions existence of  
635 a nonlinear model equation with catastrophe happening at a threshold. On the other  
636 hand, the advance in applications of the AI\_ML methodology to weather forecasting  
637 viewed as a potential paradigm shift of weather prediction to a certain extent is in the  
638 ascendant yet without a dawning realization that the knowledge in the repository  
639 database available for machine learning (ML) during its training phase is not of  
640 quality as is caused by the problem of multiple solutions emergence as well.

641 Numerous catastrophe phenomena have been observed in the Earth atmosphere,  
642 but many of these discontinuous phenomena still defy analysis. From the angle of CT,



643 the existence of multiple solutions of a model equation might become, in the present  
644 stage, a bottleneck for further improving the prediction since the conventional  
645 numerical weather prediction can only output a single forecasting outcome and, the  
646 set of equations that controls the atmospheric motion is nonlinear, which would  
647 encounter the possibility to have more than one solution.

648 As a result, it is implied that a significant issue concerning improving the  
649 forecasting accuracy of a NWP model should first solve the problem of multiple  
650 mathematical solutions with the potential physical solution that most likely matches  
651 the realistic observation to be fixed or determined reasonably, and then embark on the  
652 issues concerning the improvement of prediction via, say, ensemble forecasting,  
653 AI\_DL methodologies, or the other significant ones (Luo and Liu, 2006; Liu *et al.*,  
654 2007; Liu and Liu, 2007; Xu and Liu, 2008; Liu *et al.*, 2009), including those arisen  
655 recently in Alley *et al.* (2019).

656 As for the problem how to find the potential physical solution from the possible  
657 multiple mathematical solutions, it could be solved with the help of, say, the  
658 qualitative theory of ordinary differential equations and the bifurcation theory. Surely,  
659 the intermediate procedure therein involves how to determine, among these multiple  
660 solutions, which solution(s) is stable and which else is unstable based on the theory of  
661 motion stability (Liapunoff, 1949; Thompson and Michael, 1982). In addition, using  
662 the numerical techniques for bifurcation solutions selecting, it might be expected to  
663 further solve the issue how the bifurcation solutions switch from one branch to  
664 another. All of these potential strategies above address themselves to contribute to



665 reduce the multiplicity of solution and thus to break through the NWP development  
666 bottleneck.

667

## 668 **5. Summary and perspectives**

669 The factors impacting the NWP's development bottleneck are omnifarious and the  
670 multiplicity of nonlinear equation solution is only one of them though significant one.  
671 As mentioned in *ecmwf-strategy-2025-2034* (ECMWF, 2024), we need robust  
672 numerical weather prediction science (the best science) to exploit data-driven methods  
673 anchored on physics-based modelling, which involve achievements in comprehensive  
674 math-physics and interdisciplinary subjects, especially the novel results from  
675 thermo-dynamics (Liu and Liu *et al.*, 2009; Liu and Luo *et al.*, 2010; Liu and Ma *et*  
676 *al.*, 2010; Liu *et al.*, 2011).

677       Though CT has enabled humans to have a rare weapon to overcome benighted  
678 ignorance and studies on catastrophe phenomena have never stopped (only the  
679 relevant terminology changed to tipping subtly in an unusual way), a heated  
680 controversy over CT started even at the very beginning of its emerging (Zahler and  
681 Sussmann, 1977). This can't help remind one of Einstein's famous quote "Since  
682 mathematicians have invaded the theory of relativity, I don't understand it myself  
683 anymore". The point might lie in that although CT seems to be, in some extent, not so  
684 rigorous mathematically, it does not however affect its practical application. As a  
685 matter of fact, the techniques of CT become more firmly embedded in the  
686 consciousness of the scientific community (Liu *et al.*, 2009; Liu and Luo *et al.*, 2010;



687 Liu and Ma et al., 2010; Liu et al., 2011; Marín-Aguilar *et al.*, 2023; Zhou *et al.*,  
688 2023). This theory is viewed as a powerful new weapon but not a replacement for  
689 existing methods with an addition to them only. Nowadays the point is that,  
690 increasingly in the hard-sciences CT will be assimilated into the generally known  
691 calculus and/or mathematical physics, including e.g. the atmospheric dynamics and  
692 the physics-based NWP methodology, and the prediction of weather/climate events is  
693 however subject usually to about fail in the case of catastrophe or tipping's coming  
694 but in the present NWP models the solution to handle multiple solutions seems to  
695 have not been brought up on the agenda yet.

696 More significantly, it would be beneficial, at the moment, to recall similarities  
697 and differences between the terms of catastrophe and tipping as well as the rhetoric  
698 course of change and development with tipping points. We would like to first pay  
699 attention to the temporal points with regard to emergence of terminology of “tipping”.  
700 Although the abrupt climate change occurrence is found in even paleoclimatic  
701 recordings, no terms such as tipping points were used till 2000 or so (Russill and  
702 Nyssa, 2009) while catastrophe was only met occasionally after 2000 (say, Chao, 2013;  
703 Yin and Liu, 2019) despite the fact that the terminology “catastrophe” in CT has been  
704 used since 1975. What is particularly striking, it is found in Russill and Nyssa (2009)  
705 that “commentators observe that the idea behind tipping points is not new; the editors  
706 of Nature, for example, suggest it is old wine in new bottles”. Or alternatively, it can  
707 be assumed that catastrophe and tipping are synonyms to a great extent.

708 As a matter of fact, Ashwin *et al.* (2012) have ever called this transition tipping



709 and, it is pointed out that, aside from the rhetorically striking character of tipping  
710 points, tipping is the mathematically well-defined generalization of the bifurcations  
711 treated in the autonomous dynamical systems to the non-autonomous with random  
712 setting addressed herein. It seems actually to not be the critical point and, here we will  
713 prefer to emphatically point the fact that the time cell between the CT emergence  
714 (1975) and popular adoption of the term “tipping points” (around 2005) is about 30  
715 years, a time interval of exactly one generation of human being! This is easy to call  
716 the fluctuation history of “three ups and two declines” experienced in AI-field since  
717 its advent around the 1950s to mind, and to naturally associate it with a specious CT  
718 decline. On one hand, the idea is spreading like a contagion, and on the other hand,  
719 “this is old wine in new bottles” (both views were published in Nature and in 2006  
720 (Nature, 2006; Walker, 2006). Since that’s the case, CT, as a rare weapon to overcome  
721 benighted ignorance, can then be employed as an existing methodology to handle the  
722 issues related to tipping points (Nature, 2006; Lucarini and Chekroun, 2023)with  
723 innumerable great achievements expected.

724 In view of that the existence of catastrophe/tipping phenomena in the atmosphere  
725 is a serious challenge to its prediction and/or predictability (as shown in Fig. 1,  
726 temporal variation of the forecast scores is being to tend almost evenly to a  
727 constant-line suggesting a certain limitation’s approaching), and that the prediction of  
728 weather/climate events is usually subject to about fail in the case of catastrophe or  
729 tipping’s coming, much effort has been contributed to improving the NWP accuracy,  
730 including the novel methodologies such as the ensemble forecasts (EF) approach



731 (Palmer, 2006; Palmer, 2019)and/or AI-ML methodologies (AI\_M) (Gilpin, 2024;  
732 Bracco et al., 2025). Here it is necessary to point that, for the former (EF), its  
733 improvement potentiality is largely dependent on the quality of ensemble members  
734 themselves in terms of their respective original performance. The earth-atmospheric  
735 system is a many-body system of numerous degrees of freedom which should be  
736 contended with via both deterministic and statistical methods, and the EF approach  
737 does be an attempt at reconciling the deterministic and stochastic points of view (as  
738 seen from the fact that it arose just at the historic moment soon after chaos dynamics  
739 and the theory of dissipative structures came into being). As a matter of fact, the EF  
740 approach seems to be “to treat the symptoms, not to cure the disease” since it  
741 combines simply the deterministic and statistical methodologies that are in nature  
742 opposed to each other in terms of methodology. As for the latter (AI\_M) its  
743 contribution to enhancing prediction accuracy depends substantially on whether the  
744 knowledge in the repository database available during its training phase is of quality.  
745 In fact, AI\_M is in the ascendant as a whole, but it is limited only to its own fields of  
746 advantage in which (e.g.) playing chess and/or AI\_M are able to contribute to  
747 enhancing the opportunity to win just because, to a great extent, the knowledge in the  
748 repository database available during its training phase for deep learning is of quality.  
749 Unfortunately, NWP does not belong to the so-called fields of advantage owing to the  
750 relatively poorer knowledge provided for learning, which in nature like the case of “if  
751 you have no hand you can’t make a fist” (the issue on the limitation of AI-application  
752 to weather forecast, and particularly on AI-application’s claim about the support of



753 physical principles, will be discussed in another article but would not intend to go  
754 further for it here). So strictly speaking, the underlying logic behind it is that we'd  
755 first improve the NWP forecasting accuracy through, say, determining which one is  
756 the most realistic “physical solution” among the multiple mathematical solutions from  
757 a set of nonlinear model equations, and/or incorporating (e.g.) the second law of  
758 thermodynamics as a constraint into the NWP model for refraining from the  
759 multiplicity of solution. The current evidence (shown e.g. Fig. 1 above) suggests that  
760 we are reaching a point to save NWP as a whole (according to the present evolutionary  
761 tendency toward a straight line parallel generally to the abscissa - time axis (Haiden *et*  
762 *al.*, 2024), it is implied that there is no possibility to dramatically enhance the NWP  
763 forecasting accuracy hereafter). In other words, the physics-based NWP methodology  
764 would be about to approach its potential end in the case of no creating innovative  
765 technologies (say, incorporating e.g. the second law of thermodynamics as an  
766 additional constraint into the NWP model so as to reduce the number of solutions  
767 (Horowitz and Gingrich, 2020; Du *et al.*, 2024; Liu *et al.* 2024; Mawalagedara *et al.*,  
768 2025), or no triggering a paradigm shift.

769

770

#### 771 **Code and data availability**

772 No code was developed in the current study. And, this work is aimed at  
773 reviewing/providing perspectives on the relevant topic with no new datasets generated  
774 during its creation.

775



776 **Author contributions**

777 C.L. and J.Y. defined the problem and conceptualized the paper with contributions  
778 from other authors. C.L. wrote the paper. C.L. and X.W. designed and created Figure  
779 1. C.L. and J.Y. designed and created Figure 4. All authors prepared the manuscript  
780 together and contributed to revision.

781

782 **Competing Interests**

783 The authors have no relevant financial or non-financial interests to disclose.

784

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## References

- 798  
799 Alley, R. B., Emanuel, K. A. and Zhang, F.: Advances in weather prediction, *Science*, 363, 342,  
800 doi: 10.1126/science.aav7274, 2019.
- 801 Altman, E.: Many-body localization and quantum thermalization, *Nature Phys* 14, 979–983.,  
802 <https://doi.org/10.1038/s41567-018-0305-7>, 2018.
- 803 Arabas, S. and Shima, S.: On the CCN (de)activation nonlinearities, *Nonlin. Processes Geophys.*,  
804 24, 535–542, <https://doi.org/10.5194/npg-24-535-2017>, 2017.
- 805 Ashwin, P., S. Wicczorex, R. Vitolo and P. Cox: Tipping points in open systems: bifurcation,  
806 noise-induced and rate-dependent examples in the climate system, *Phil. Trans. R. Soc. A*, 370,  
807 1166–1184, doi:10.1098/rsta.2011.0306, 2012.
- 808 Barrett, S. and Dannenberg, A.: Sensitivity of collective action to uncertainty about climate  
809 tipping points, *Nature Climate Change*, 8, 2–12, 2018.
- 810 Battiston, F., Amico, E., Barrat, A. *et al.*: The physics of higher-order interactions in complex  
811 systems, *Nat. Phys.*, 17, 1093–1098, <https://doi.org/10.1038/s41567-021-01371-4>, 2021.
- 812 Bracco, A., Brajard, J., Dijkstra, H.A. *et al.*: Machine learning for the physics of climate, *Nat Rev*  
813 *Phys*, 7, 6–20. <https://doi.org/10.1038/s42254-024-00776-3>, 2025.
- 814 Chao, W. C.: Sudden Stratospheric Warmings as Catastrophes, *J. Atmos. Sci.*, 42, 1631-1646,  
815 doi:10.1175/1520-0469(1985)042<1631:SSWAC>2.0.CO;2, 1985.
- 816 Chao, W. C.: Catastrophe-concept-based cumulus parameterization: Correction of systematic  
817 errors in the precipitation diurnal cycle over land in a GCM, *J. Atmos. Sci.*, 70, 3599-3614,  
818 doi:10.1175/JAS-D-13-022.1, 2013.
- 819 Charney, J. G., and J. G. De Vore: Multiple flow equilibria in the atmosphere and blocking, *J.*  
820 *Atmos. Sci.*, 36, 1205-1216, doi:10.1175/1520-0469(1979)036<1205:MFEITA>2.0.CO;2,  
821 1979.
- 822 Editorial: Climate change dialogues: *Nature Geosci.*, 5, 301, <https://doi.org/10.1038/ngeo1474>,  
823 2012.
- 824 Coveney, P., and Highfield, R.: The arrow of time: A voyage through science to solve time's  
825 greatest mystery, *Science and Society*, 56, 501-504, 1992.
- 826 Du, Y., Calzavarini, E. and Sun, C.: The physics of freezing and melting in the presence of flows.  
827 *Nat. Rev. Phys.*, 6, 676–690, <https://doi.org/10.1038/s42254-024-00766-5>, 2024.
- 828 ECMWF: ecmwf-strategy-2025-2034-summary-en: *a summary of the full ECMWF Strategy*,  
829 <https://doi.org/10.21957/05dd9b657e>, from  
830 <https://www.ecmwf.int/en/about/what-we-do/strategy.2024>.
- 831 Edenhofer, O., Lessmann, K., Kemfert, C., Grubb, M., and Köhler, J.: Induced Technological  
832 Change: Exploring its Implications for the Economics of Atmospheric Stabilization:  
833 Synthesis Report from the Innovation Modeling Comparison Project, *The Energy Journal*, 27,  
834 57-107, 2006.
- 835 Editorial: Climate change dialogues, *Nature Geosci.*, 5, 301, <https://doi.org/10.1038/ngeo1474>,  
836 2012.
- 837 Farnsworth, A., Lo, Y.T.E., Valdes, P.J. *et al.*: Climate extremes likely to drive land mammal  
838 extinction during next supercontinent assembly, *Nat. Geosci.*, 16, 901–908,  
839 <https://doi.org/10.1038/s41561-023-01259-3>, 2023.
- 840 Feudel, U.: Rate-induced tipping in ecosystems and climate: the role of unstable states, basin  
841 boundaries and transient dynamics, *Nonlin. Processes Geophys.*, 30, 481–502,



- 842 <https://doi.org/10.5194/npg-30-481-2023>, 2023.
- 843 Feudel, U.: Rate-induced tipping in ecosystems and climate: the role of unstable states, basin  
844 boundaries and transient dynamics, *Nonlin. Processes Geophys.*, 30, 481–502,  
845 <https://doi.org/10.5194/npg-30-481-2023>, 2023..
- 846 Fraedrich, K.: Structural and stochastic analysis of a zero-dimensional climate system, *Quart. J.*  
847 *Roy. Meteor. Soc.*, 104, 461–474, 1978.
- 848 Fraedrich, K.: Catastrophes and resilience of a zero-dimensional climate system with ice-albedo  
849 and greenhouse feedback, *Quart. J. Roy. Meteor. Soc.*, 105, 147–167, 1979.
- 850 Ghil, M.: A Century of Nonlinearity in the Geosciences, *Earth and Space Science*, 6, 1007–1042,  
851 doi:10.1029/2019EA000599, 2019.
- 852 Gilpin, W.: Generative learning for nonlinear dynamics, *Nat. Rev. Phys.* 6, 194–206,  
853 <https://doi.org/10.1038/s42254-024-00688-2>, 2024.
- 854 Gladwell, M.: *The Tipping Point: How Little Things Can Make a Big Difference*, 1st ed., Boston,  
855 MA: Little, Brown and Company, viii + 279 pp., ISBN: ISBN 0 - 316 - 31696 - 2, 2000.
- 856 Jahren, A. H. and Sternberg, L. S. L.: Humidity estimate for the middle Eocene Arctic rain forest,  
857 *Geology*, 31, 463–466, 2003.
- 858 Thomas Haiden, Martin Janousek, Frederic Vitart, Maliko Tanguy, Fernando Prates, Matthieu  
859 Chevallier: Evaluation of WCMWF forecasts, *Technical Memo* 918, September 2024.
- 860 Thomas Haiden and Matthieu Chevallier: Forecast performance 2024, ECMWF Newsletter,  
861 Number 182 - Winter 2025,  
862 [https://www.ecmwf.int/en/newsletter/182/news/forecast-performance-2024\\_2015](https://www.ecmwf.int/en/newsletter/182/news/forecast-performance-2024_2015).
- 863 Hankel, C., Tziperman, E.: An approach for projecting the timing of abrupt winter arctic sea ice  
864 loss, *Nonlinear Processes in Geophysics Discussions*, 2022.
- 865 Hansen, J. Is there still time to avoid “dangerous anthropogenic interference” with global climate?  
866 *A tribute to Charles David Keeling*, American Geophysical Union, San Francisco, December  
867 6, <http://www.columbia.edu/jeh1/>, 2005.
- 868 Hansen, J. et al.: Dangerous human-made interference with climate: a GISS model E study,  
869 *Atmospheric Chemistry & Physics*, 7, 2287–2312, 2007.
- 870 Hersbach, H.: Decomposition of the continuous ranked probability score for ensemble prediction  
871 systems, *Wea. Forecasting*, 15, 559–570, 2000.
- 872 Horowitz, J.M. and Gingrich, T.R.: Thermodynamic uncertainty relations constrain  
873 non-equilibrium fluctuations, *Nat. Phys.*, 16, 15–20, <https://doi.org/10.1038>, 2020.
- 874 Kopp, R. E., E. Gilmore, R. Shwom, et al.: ‘Tipping points’ confuse and can distract from urgent  
875 climate action, *Nature Climate Change*, 15, 29–36, 2025.
- 876 Kypke K. L. and Langford W. F., Willms A R.: Anthropocene climate bifurcation, *Nonlin.*  
877 *Processes Geophys.*, 27, 391–409, <https://doi.org/10.5194/npg-27-391-2020>, 2020.
- 878 Lambiotte, R., Rosvall, M., and Scholtes, I.: From networks to optimal higher-order models of  
879 complex systems, *Nat. Phys.*, 15, 313–320, <https://doi.org/10.1038>, 2019.
- 880 Liapunoff, M. A.: Probleme general de la stabilite du mouvement, volume 17 of *Ann. Math.*  
881 *Studies*, Princeton Univ. Press, 1949.
- 882 Lindsay, R.W. and Zhang, J.: The thinning of Arctic sea ice, 1988–2003: have we passed a tipping  
883 point? *Journal of Climate*, 18, 4879–4894, 2005.
- 884 Liu, C., and S. Tao: Northward jumping of subtropical highs and cusp catastrophe, *Scientia Sinica*,  
885 *Series B*, 26, 1065–1074, 1983.



- 886 Liu Chongjian, Liu Ying and Kang Hongwen: A new technique of physical dissipation and its  
887 application to a mesoscale numerical weather prediction model, *Science in China (Series D)*,  
888 Vol.45, No.9, 769-780, 2002.
- 889 Liu, C. and Liu, Y.: An attempt at improving a global spectral model by incorporating the second  
890 law of thermodynamics, *Geophys. Res. Lett.*, 32, L03806, doi:10.1029/2004GL021602, 2005.
- 891 Liu, C., Liu, Y. and Xu H.: A physics-based diffusion scheme for numerical models, *Geophys. Res.*  
892 *Lett.*, 33, L12805, doi:10.1029/2006GL025781, 2006..
- 893 Liu, C., Xu, H., and Liu, Y.: Linking entropy flow with typhoon evolution: a case-study, *Journal*  
894 *of Physics D: Applied Physics*, 40, 6694-6704, 2007.
- 895 Chongjian Liu, Ying Liu, and Zhexian Luo: Improving numerical simulation of a hurricane by  
896 irreversible thermodynamic operators, *Atmos. Res.*, 93, 44-49,  
897 doi:10.1016/j.atmosres.2008.10.012, 2009.
- 898 Liu, C., Liu, Y. and Xu H.: A New Diffusion Scheme for Numerical Models Based on Full  
899 Irreversibility, *Weather and Forecasting*, 24, 595-600, DOI: 10.1175/2008WAF2222145.1,  
900 2009.
- 901 Liu, C.; Luo, Z.; Liu, Y.; Yu, H.; Zhou, X.; Wang, D.; Ma, L.; Xu, H.: Implication of entropy flow  
902 for the development of a system as suggested by the life cycle of a hurricane, *Modern*  
903 *Physics Letters B (MPLB)*, 24(16), 1747-1757, DOI: 10.1142/S0217984910024122, 2010.
- 904 Liu, C; Ma, L.; Liu, Y.; Luo, Z.; Lei, X.; Zhou, X.; Wang, D.; Xu, H.: Linking tropical cyclone  
905 number over the western north pacific with sea surface temperatures, *Hurricanes and*  
906 *Climate Change* (eds: Elsner, J.B., Hodges, R.E., Malmstadt, J.C., Scheitlin, K.N.), Springer:  
907 New York, NY, USA, Volume 2, doi: 10.1007/978-90-481-9510-712, 2010.
- 908 Chongjian Liu, Ying Liu, Zhexian Luo, Xiaotu Lei, Donghai Wang and Xiaogang Zhou: Studies of  
909 the Hurricane Evolution Based on Modern Thermodynamics, *Recent Hurricane Research:*  
910 *Climate, Dynamics, and Societal Impacts* by A. Lupo (ed.), InTech, ISBN  
911 978-953-307-238-8, 2011.
- 912 Liu Ying and Chongjian Liu: On the entropy flow properties of a severe tropical storm, *Applied*  
913 *Physics Letters*, 91(1), DOI: 10.1063/1.2753538, 2007.
- 914 Liu, Ying; Liu, Chongjian; Wang, Donghai: Understanding Atmospheric Behaviour in Terms of  
915 Entropy: A Review of Applications of the Second Law of Thermodynamics to Meteorology,  
916 *Entropy*, 13(1), 211-240, 2011.
- 917 Liu, Z., Jin, P., Lei, M. *et al.* Topological thermal transport. *Nat. Rev. Phys.*, 6, 554-565,  
918 <https://doi.org/10.1038/s42254-024-00745-w>, 2024.
- 919 Lucarini V., Fraedrich, K., and Lunkeit, F.: Thermodynamic analysis of snowball Earth hysteresis  
920 experiment: Efficiency, entropy production and irreversibility, *Quart. J. Roy. Meteor. Soc.*,  
921 136, 2-11, doi:10.1002/qj.543, 2010.
- 922 Lucarini, V. and Chekroun, M.D.: Theoretical tools for understanding the climate crisis from  
923 Hasselmann's programme and beyond, *Nat. Rev. Phys.*, 5, 744-765.,  
924 <https://doi.org/10.1038/s42254-023-00650-8>, 2023.
- 925 Luo, Z., and Liu, C.: An investigation into the sensitivity of idealised vortex interactions to initial  
926 conditions and island topography, *Geophys. Res. Lett.*, 33, L01809,  
927 doi:10.1029/2005GL024543, 2006.
- 928 Macnally, A.: ECMWF in its 50th year, what is the future of Numerical Weather Prediction?  
929 *EUMETSAT Meteorological Satellite Conference 2025*, Lyon, France, 15-19 September



- 930 2025.
- 931 Marín-Aguilar, S., Camerin, F., and van der Ham, S. et al.: A colloidal viewpoint on the sausage  
932 catastrophe and the finite sphere packing problem, *Nat Commun*, 14, 7896, 2023.
- 933 Mawalagedara, R., Ray, A., and Das, P. et al.: Non-linear dynamical approaches for characterizing  
934 multi-sector climate impacts under irreducible uncertainty, *npj Climate and Atmospheric  
935 Science*, 8, 329, <https://doi.org/10.1038/s41612-025-01208-4>, 2025.
- 936 Methven, J., and Berrisford, P.: The slowly evolving background state of the atmosphere, *Quar. J.  
937 Roy. Meteor. Soc.*, 141, 2237-2258, doi:10.1002/qj.2518, 2015.
- 938 Miyamoto, Y., and Takemi, T.: A transition mechanism for the spontaneous axisymmetric  
939 intensification of tropical cyclones, *J. Atmos. Sci.*, 70, 112-129,  
940 doi:10.1175/JAS-D-11-0285.1, 2012.
- 941 Nature: Reaching a tipping point, *Nature*, 441, 785, <https://doi.org/10.1038/441785a>, 2006.
- 942 Oerlemans J.: A quasi-analytical ice-sheet model for climate studies, *Nonlinear Processes in  
943 Geophysics*, 10, 441-452, 2003.
- 944 Palmer, T.: Predictability of weather and climate: From theory to practice, *Predictability of  
945 Weather and Climate* (Eds.: T. Palmer & R. Hagedorn), pp. 1-29, Cambridge University Press,  
946 <https://doi.org/10.1017/CBO9780511617652.002>, 2006.
- 947 Palmer, T.N.: Stochastic weather and climate models, *Nat. Rev. Phys.*, 1, 463-471,  
948 <https://doi.org/10.1038/s42254-019-0062-2>, 2019.
- 949 van der Ploeg, F.: Reacting to multiple tipping points, *Nature Climate Change*, 6, 442-443, 2016.
- 950 Poston, T. and Stewart, I.: *Catastrophe Theory and Its Applications*, xviii+491 pp, Pitman  
951 Publishing Limited, 1978.
- 952 Rietkerk, M., Skiba, V. and Weinans, E. et al.: Ambiguity of early warning signals for climate  
953 tipping points, *Nature Climate Change*, 15, 479-488, 2025.
- 954 Ritchie, P.D.L., Clarke, J.J., Cox, P.M. et al.: Overshooting tipping point thresholds in a changing  
955 climate. *Nature*, 592, 517-523. <https://doi.org/10.1038/s41586-021-03263-2>, 2021.
- 956 Russell, B.: *HISTORY OF WESTERN PHILOSOPHY*, George Allen & Unwin Ltd, 1961.
- 957 Russill, C., and Nyssa, Z.: The tipping point trend in climate change communication, *Global  
958 Environmental Change*, 19, 336-344, doi:10.1016/j.gloenvcha.2009.04.001, 2009.
- 959 Thom, R.: *Structural Stability and Morphogenesis*, W. A. Benjamin, 1975.
- 960 Thompson, J., and Michael, T.: *Instabilities and Catastrophes in Science and Engineering*, 226 pp,  
961 New York: Wiley, 1982.
- 962 Tsur, Y., and Withagen, C.: Preparing for catastrophic climate change, *Journal of Economics*, 110,  
963 225-239, doi:10.1007/s00712-012-0331-3, 2013.
- 964 Verendel, V. Johansson, D. J. A. and Lindgren, K.: Strategic reasoning and bargaining in  
965 catastrophic climate change games, *Nature Climate Change*, 6, 265-269, 2016.
- 966 Vickroy, J. G., and Dutton, J. A.: Bifurcation and catastrophe in a simple, forced, dissipative  
967 quasi-geostrophic flow, *J. Atmos. Sci.*, 36, 42-52,  
968 doi:10.1175/1520-0469(1979)036<0042:BACIAS>2.0.CO;2, 1979.
- 969 Walker, G.: The tipping point of the iceberg, *Nature*, 441, 802-805,  
970 <https://doi.org/10.1038/441802a>, 2006.
- 971 Watson, S., Quinn, C.: The role of time-varying external factors in the intensification of tropical  
972 cyclones, *Nonlinear Processes in Geophysics*, 31(3), 2024.
- 973 Wittgenstein, L.: *Tractatus Logico-Philosophicus* (English translation of Logisch-Philosophische



974 Abhandlung, by David Pears, Brian McGuinness), Routledge, 112pp, ISBN: 0415028256,  
975 1975.

976 WMO-LCDNV: WMO Lead Centre for Deterministic Forecast Verification,  
977 [https://wmo/cdnv.ecmwf.int/scores/time\\_series](https://wmo/cdnv.ecmwf.int/scores/time_series), 2011.

978 Xu, H. and Liu, C.: Entropy properties of a typhoon as simulated by a meso-scale model,  
979 *Europhysics Letters*, 83, 18001, DOI: 10.1029/0295-5075/83/18001, 2008.

980 Yin, J., and Liu, C.: Understanding the Asian summer monsoon onset in terms of catastrophe  
981 theory, *Theor. Appl. Climatology*, doi: 10.1007/s00704-019-02880-2, 2019.

982 Yost, D. A., and Shirer, H. N.: Bifurcation and stability of low-order steady flows in horizontally  
983 and vertically forced convection, *J. Atmos. Sci.*, 39, 114-125,  
984 doi:10.1175/1520-0469(1982)039<0114:BASOLO>2.0.CO;2, 1982.

985 Zahler, R. and Sussmann, H.: Claims and accomplishments of applied catastrophe theory, *Nature*,  
986 269, 759–763, <https://doi.org/10.1038/269759a0>, 1977.

987 Zaliapin, I. and Ghil, M.: Another look at climate sensitivity, *Nonlinear Processes in Geophysics*,  
988 17(2), 2010.

989 Zeeman, E. C.: Catastrophe theory. Selected papers 1972-1977, Addison-Wesley Educational  
990 Publishers Inc., 1977.

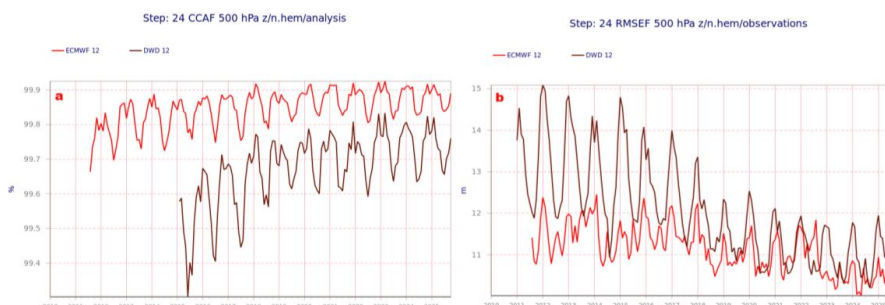
991 Zhang, X., Zwiers, F., Li, G. et al.: Complexity in estimating past and future extreme  
992 short-duration rainfall, *Nature Geosci*, 10, 255–259, <https://doi.org/10.1038/ngeo2911>, 2017.

993 Zhou, X., Ren, X. and Xiao D., et al.: Higher-order singularities in phase-tracked  
994 electromechanical oscillators, *Nat Commun*, 14, 7944, 2023.

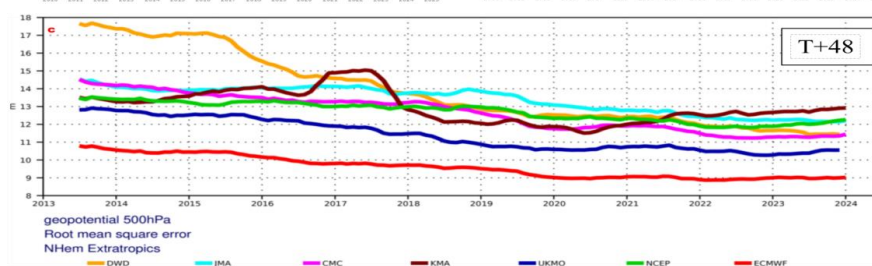
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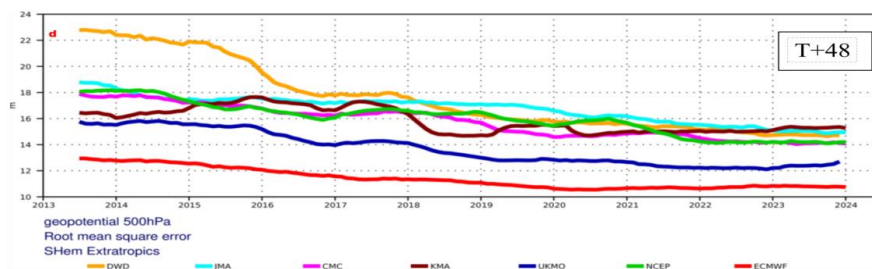


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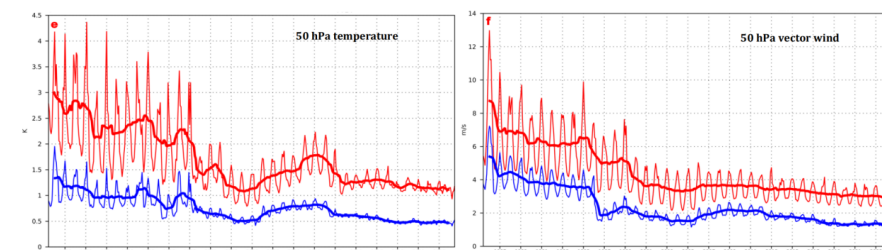


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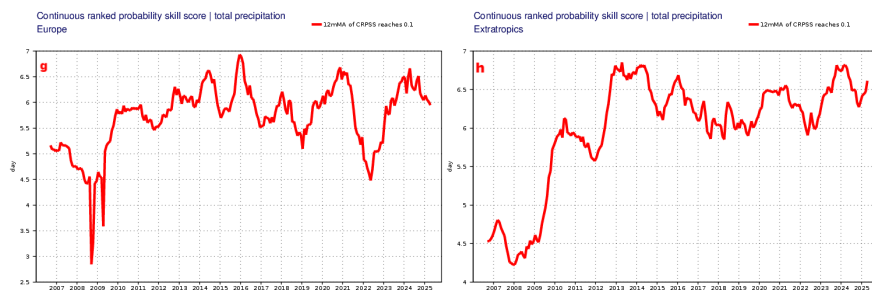
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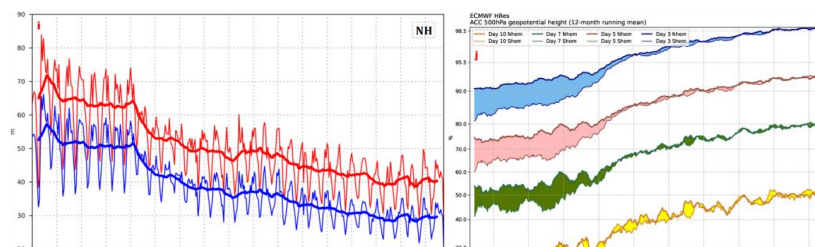
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1026 **Fig. 1 | The time series of the miscellaneous forecasting scores.** In this figure the global forecast  
 1027 centres mentioned include: ECMWF = European Center for Medium-range Weather Forecasts,  
 1028 JMA = Japan Meteorological Agency, CMC = Canadian Meteorological Centre, UKMO = the UK  
 1029 Met Office, KMA = Korea Meteorological Administration, NCEP = U.S. National Centers for  
 1030 Environmental Prediction, DWD = Deutscher Wetterdienst. **a** Anomaly Correlation coefficients  
 1031 (ACC) and **b** Root Mean Square (RMS) errors for the 500 hPa geopotential height averaged over  
 1032 the northern hemisphere during the period of 2010–2025 for ECMWF and DWD. **c**  
 1033 WMO-exchanged scores from global forecast centres: RMS error of 500 hPa geopotential height  
 1034 over northern, and **d** southern extratropics. The plot shows the two-day (T+48) forecast error of  
 1035 model runs initiated at 12UTC. Each model is verified against its own analysis. **e** Model scores for  
 1036 temperature and **f** for wind in the northern extratropical stratosphere. Curves show the monthly  
 1037 average RMS temperature and vector wind error at 50 hPa for one-day (blue) and five-day (red)  
 1038 forecasts, verified against analysis. 12-month moving average scores are also shown (in bold). **g**  
 1039 a measure of the effectiveness of the model in terms of the forecast lead-time (in days) at which the  
 1040 Continuous Ranked Probability Skill Score (CRPSS) of 24 hour precipitation forecasts falls below  
 1041 10 % for twelve-month mean centred on each month (red line) over Europe and **h** over  
 1042 extratropics. **i** a measure of inconsistency of the 500 hPa height forecasts over northern  
 1043 extratropics. Curves show the monthly average RMS difference between forecasts for the same  
 1044 verification time but initialised 24 h apart, for 96–120 h (blue) and 120–144 h (red). 12-month  
 1045 moving average scores are also shown (in bold). And **j** a measure of the effectiveness of the model  
 1046 in forecasting 500 hPa heights at several forecast lead times and with this effectiveness evolved  
 1047 over the past decades shown. The ACC (in %, note non-linear scale) are shown at: day 3 - blue,  
 1048 day 5 - red, day 7 - green, day 10 – yellow for Northern Hemisphere (thick lines) and Southern  
 1049 Hemisphere (thin lines).

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1051 **a-b** from [https://wmo/cdnv.ecmwf.int/scores/time\\_series](https://wmo/cdnv.ecmwf.int/scores/time_series) (ECMWF, 2011) and

1052 **c-f, i** from <http://www.ecmwf.int/publications/> (Haiden et al., 2024)

1053 **g-h, j** from

1054 <https://charts.ecmwf.int/?facets=%7B%22Type%22%3A%5B%22Verification%22%5D%2C%22Parameters%22%3A%5B%5D%7D> (Haiden and Chevallie, 2025)

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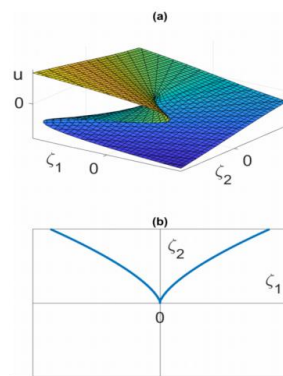
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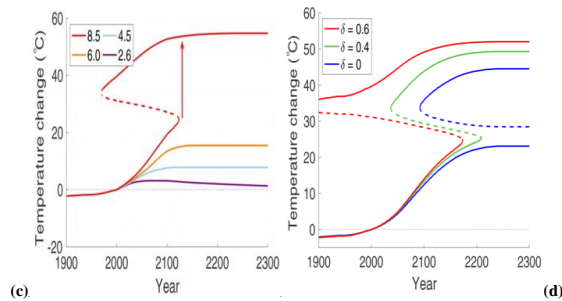
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**Fig. 2 | Multifarious catastrophe patterns.** (a) Cusp bifurcation diagram for the EBM in normal form coordinates. Graph of the equilibrium surface with normal form unfolding parameters ( $\zeta_1$ ,  $\zeta_2$ ). (b) Projection of this surface in 3-D onto the ( $\zeta_1$ ,  $\zeta_2$ ) plane. The blue semi-cubic parabola represents fold bifurcations, and it separates the ( $\zeta_1$ ,  $\zeta_2$ ) plane into two open 2-D regions. Inside the cusp region, that is between the two branches of the semi-cubic parabola, there exist three equilibrium solutions. This figure is shown in terms of eigenbasis coordinates ( $u$ ,  $v$ ); and for  $u$ , it is seen that two stable and the middle one unstable. Outside of the semi-cubic parabola there exists only one unique equilibrium solution  $u$  and it is stable. (c) Fold catastrophe. Arctic surface temperature change projected to year 2300 (relative to year 2000 temperature of  $-28.90$  °C), with linearly increasing ocean heat transport FO. With the constant atmospheric heat transport FA = 104, the jump in temperature for RCP 8.5 now occurs nearly 40 years earlier than for the case of constant FO. (d) fold catastrophe for Arctic surface temperature change with increasing relative humidity of water vapor,  $\delta = 0.0, 0.4, 0.6$ , and with fixed FO and FA, projected to year 2300 (temperature change is relative to the year 2000 temperature). On all three curves, CO2 is increasing in time according to RCP 8.5. The red curve is for  $\delta = 0.6$ . The blue and green curves have water vapor fixed at  $\delta = 0.0$  and  $\delta = 0.4$ , respectively. For temperatures significantly below freezing, water vapor makes little contribution to temperature change. However, above freezing, the greenhouse warming effect of water vapor is dramatic. See the text for further details. Adapted from Kypke et al. (2020)<sup>27</sup>.

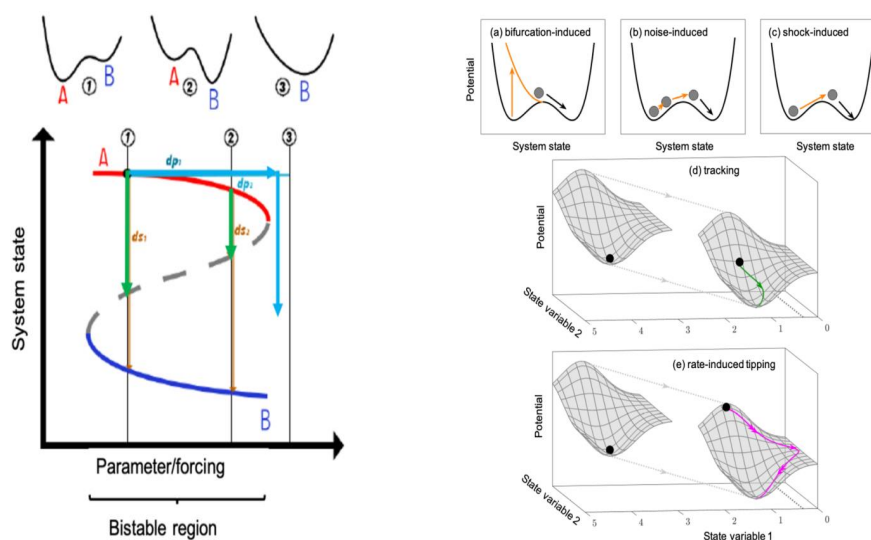
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1089 **Fig. 3 | Rate-induced tipping in ecosystems and climate.** [Left] Relationship between the  
 1090 stability landscape and the control parameters/the forcing of the system. Cyan arrows indicate the  
 1091 changes in control parameters or forcings; the light-green and lightbrown arrows refer to changes  
 1092 in the state variables. [Right] Illustration of the four different tipping mechanisms: (a)  
 1093 bifurcation-induced tipping; (b) noise-induced tipping; (c) shock-induced tipping; and (d, e)  
 1094 rate-induced tipping in the case where no alternative states are present, but the rate-induced  
 1095 tipping refers to a very large excursion in state space with different properties. (d) Rate-induced  
 1096 tipping with a tracking trajectory and (e) rate-induced tipping with a tipping trajectory. See the text  
 1097 for further details. Adapted from Feudel (2023)<sup>37</sup>.

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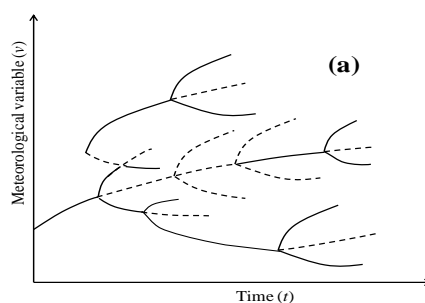
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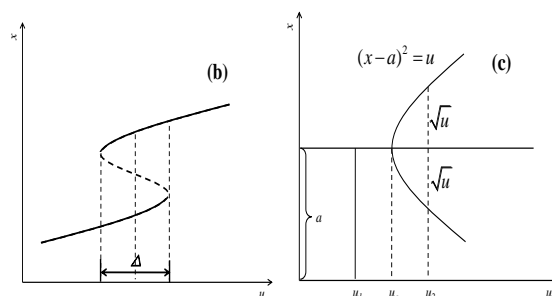
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1106 **Fig. 4 | Illustration of multiple equilibria/solutions and catastrophes.** (a) Schematic  
 1107 bifurcation diagram for temporal changes of a meteorological variable  $v$  with multiple  
 1108 equilibria/solutions emerging when the parameter as function/functional of time reaches a  
 1109 certain threshold. (b) Hysteresis: when it enters, in whatever way, into  $\Delta$ -area, catastrophe  
 1110 happens. Solid lines stand for the stable equilibrium/solution, and thick-dashed lines for the  
 1111 unstable one. And, (c) an illustration of an example of quadratic polynomial equation whose  
 1112 analytic solution number depends on the value of  $u$ : there is only a single solution and at  $u =$   
 1113  $u_1$  two analytic solutions at  $u = u_2$ , with  $u_c$  as its bifurcation point.