Author comment: response to referee n°3

Systematic deviation in wavelet-based covariance estimation and implications for eddy-covariance flux calculations

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We thank all referees for their careful and constructive evaluation of our manuscript. Their comments have helped us improve the clarity, methodological robustness and overall presentation of the study.

This document provides a detailed, point-by-point response to all comments. For transparency, responses are organized as follows: an overall assessment, followed by general comments, specific line-by-line comments and comments on the form.

All modifications made in the manuscript are indicated at the end of each response and introduced by the symbol "⇨". References to sections, figures and tables in the revised manuscript are highlighted in blue.

Overall assessment

"The manuscript "Systematic deviation in wavelet-based covariance estimation and implications for eddy covariance flux calculations" by Jonathan Bitton et al. (egusphere-2026-2881) presents a systematic deviation in commonly used formulation for wavelet-based covariance showing implications for the eddy covariance field and proposing a better formulation.

The manuscript is very clear. I praise the authors for their successful effort in making what could be a mathematically heavy subject into a digestible, direct and clear paper.

It is in Atmospheric Measurement Techniques' scope and it brings clear contributions to the flux community interested in frontier methods."

We thank the referee for the positive assessment of our manuscript and for the detailed mathematical comments.

The comments prompted us to clarify the wavelet definitions and normalization conventions, distinguish more explicitly between the reconstruction factors C_δ and C_ψ, and revisit the mathematical origin of the Torrence and Compo formulation.

The revision also includes complementary analyses requested by the other referees. These include genuine Monte Carlo experiments, direct covariance and prescribed cross-spectral tests, sensitivity to scale resolution, analysis of the discrete filter partition, scale-resolved atmospheric diagnostics and sensitivity to the treatment of signal boundaries.

We have also reorganized the Discussion and revised the figures and captions throughout the manuscript and Supplement.

General comments

Comment n°1: ***"Should you mention the actual wavelet implementation, for example*** $\boldsymbol{\psi(t) = 2\pi^{-\frac{3}{4}} e^{-\frac{t^2}{2}} e^{i\omega_0 t}}$***?"***

We agree that the implemented wavelets and their normalization conventions should be stated explicitly. Although Morlet and DOG2 are standard wavelets, their amplitudes and associated reconstruction constants depend on the adopted energy-normalization conventions.

We used L^1 normalizations for both the Morlet and DOG2 wavelets, as this normalization leads to comparable covariance values across scales and is more straightforward for eddy covariance applications. Also, the unambiguous scale–frequency correspondence is particularly critical for these applications, as periodic information is generally sought and not scale information. It also complements the new timescale analyses implemented, which relies on cross-scalograms and power spectra. We now provide the time-domain definitions used in the calculations:

$$\psi(t) = \left(\frac{2}{\sigma^2\pi}\right)^{\frac{1}{2}} e^{-\frac{t^2}{2\sigma^2}} e^{i\omega_0 t}$$

for Morlet and

$$\psi(t) = \left(\frac{e\sigma^2}{4}\right)^{\frac{1}{2}} \left(\frac{t^2}{\sigma^2} - 1\right) \frac{e^{-\frac{t^2}{2\sigma^2}}}{\sigma^2}$$

for DOG2.

One can verify that using these expressions, the L^1 norm is preserved,

$$||\psi_{a,b}(t)||_1 = ||\psi(t)||_1.$$

⇨ Changes in Manuscript: In the revised manuscript, we have provided the time- and frequency-domain definitions of the implemented Morlet and DOG2 wavelets in Sect. 2.1.6.

Comment n°2: "***It's striking the resemblance between the classical formulation and the real-valued wavelets formulation. Could we summarise the methods around using an empirical (C_δ) or theoretical ($C_\psi ln(2)$) reconstruction factor? For example in TC's C_δ for Morlet (6) is 0.776, while C_ψ is 1.11953135173, so the ratio between TC and CWT should be***

$$\frac{\frac{1}{C_\delta}}{\frac{2ln(2)}{C_\psi}},$$

so (1.2886597938144329) / (1.2382809637065228692) = 1.040684490502956708?

If this is true why the gap in Figure 1 and Figure 3 between CWT and TC seems wider? In both figures it seems to be bigger than 1.1 (and not 1.04 as the previous logic would suggest)"

We thank the referee for this constructive comment. It led us to examine more explicitly the mathematical origin of C_δ and revealed that the difference is not limited to replacing an empirical constant with a theoretical one.

The Torrence and Compo reconstruction is derived for the signal itself rather than directly for a second-order moment. It exploits the redundancy of the CWT by using a synthesis function different from the analyzing wavelet. Following the reconstruction proposed by Farge (1992), the synthesis function is a Dirac distribution. Under the L^2 CWT convention and positive scales, the corresponding continuous reconstruction can be written schematically as (Daubechies, 1992; Farge, 1992):

$$f(t) = \frac{1}{C_{\psi_1,\delta}} \int_0^{+\infty} \int_{-\infty}^{+\infty} W_f(a,b)\, \delta(a,b)\, db\, \frac{da}{a^2},$$

with the modified cross-admissibility constant

$$C_{\psi_1,\delta} = \int_0^{+\infty} \frac{\overline{\hat{\psi}(\omega)}\hat{\delta}(\omega)}{\omega} d\omega = \int_0^{+\infty} \frac{\overline{\hat{\psi}(\omega)}}{\omega} d\omega < \infty.$$

After logarithmic discretization of scale and distinction between analytic and real wavelets, this expression becomes:

$$x_n = \frac{ln(2)\, \delta j \sqrt{\delta t}}{C_{\psi_1,\delta}} \sum_{j=0}^{J} \frac{Re\left[W_x\left(\frac{a_j}{\delta t}, n\right)\right]}{\sqrt{a_j}}$$

for real wavelets and

$$x_n = \frac{2\, ln(2)\, \delta j \sqrt{\delta t}}{C_{\psi_1,\delta}} \sum_{j=0}^{J} \frac{Re\left[W_x\left(\frac{a_j}{\delta t}, n\right)\right]}{\sqrt{a_j}}$$

for analytic wavelets and real-valued functions.

Torrence and Compo instead write their signal reconstruction as (Eq. (11) in their article):

$$x_n = \frac{\delta j \sqrt{\delta t}}{C_\delta \psi_0(0)} \sum_{j=0}^{J} \frac{Re\left[W_x\left(\frac{a_j}{\delta t}, n\right)\right]}{\sqrt{a_j}}$$

Comparison of these expressions establishes that $C_{\psi_1,\delta} = 2\ln(2)\,\psi_0(0)\,C_\delta$ for Morlet and $C_{\psi_1,\delta} = \ln(2)\,\psi_0(0)\,C_\delta$ for DOG2, in our application.

Using our wavelet definitions and normalization conventions gives inferred values of approximately $C_\delta = 0.7784$ for Morlet and $C_\delta = 3.616$ for DOG2, close to the values 0.776 and 3.541 reported by Torrence and Compo. The remaining differences are consistent with their numerical calibration of C_δ, which was obtained by inverting the reconstruction of a prescribed test signal and therefore depends on the minimum scale, padding and scale resolution. Using the same approach as Torrence and Compo, we verified numerically that refining these parameters makes the inferred values approach the continuous expressions above.

This development clarifies that C_δ and C_ψ originate from different identities:

- C_δ is a constant calibrated by Torrence and Compo for signal reconstruction using a Dirac synthesis function;
- C_ψ corresponds to the admissibility constant associated with the CWT inner-product identity used to reconstruct variance and covariance.

As a result, C_δ does not incorporate the multiplicative normalization $\psi_0(0)$ of the analyzing wavelet. Consequently, extending the Torrence and Compo signal-reconstruction factor directly to second-order moments introduces two differences with our formulation: (1) the normalization dependence and (2) the numerical difference between C_δ and $\frac{C_\psi}{2\ln(2)}$.

Using the normalized Morlet wavelet in the manuscript, the ratio between the Torrence and Compo multiplier and the proposed multiplier is:

$$\frac{\frac{1}{C_\delta}}{\frac{2\ln(2)}{C_\psi}} \approx \frac{\frac{1}{0.776}}{\frac{2\ln(2)}{1.1988}} \approx 1.114.$$

For DOG2, the ratio becomes 0.870, predicting a TC reconstruction approximately 13.0 % lower than that of the proposed formulation. The predicted multiplier ratios quantitatively match the differences observed in Sect. 3.1 for the deterministic sinusoidal signal: +11.4 % for Morlet and −13.0 % for DOG2. The approximately 1.04 ratio mentioned by the referee results from combining constants defined under different normalization conventions.

We thank the referee for drawing attention to this point. We have added the derivation to the Appendix and now explain in the Discussion that the discrepancy involves both the reconstruction identity and the normalization dependence of the wavelet coefficients.

⇨ Changes in Manuscript: In the revised manuscript, we have:

i. clarified in Sect. 2.1.4. that the Torrence and Compo factor originates from signal reconstruction using a Dirac synthesis function;
ii. added an Appendix D introducing these mathematical developments and establishing the relationship between both constants;
iii. expanded the Discussion on the difference between formulations.

Comment n°3: "*Also fed from the previous comment, the C_ψ reconstruction factor in section 2.1.2 seems to be a key piece and deserves more detail. In here or in appendix A, you should develop a bit more in detail how do you get $C_\psi \approx 1.1988$ for Morlet(6) and $C_\psi \approx 2.1349$ for DOG(2). You may want to include Morlet's mother wavelet, for example $\psi(t) = 2\pi^{-\frac{3}{4}} e^{-\frac{t^2}{2}} e^{i\omega_0 t}$*"

We agree that the calculation and interpretation of these constants require a clearer presentation.

We write the multiplicative wavelet-normalization factor as K, because its value depends on the selected normalization convention. If ψ_u denotes the unnormalized mother wavelet and $\psi = K\psi_u$ denotes the mother wavelet after normalization, the admissibility constant writes:

$$C_\psi = K^2 \int_0^{+\infty} \frac{\left|\hat{\psi}_u(\omega)\right|^2}{\omega} d\omega = K^2 C_{\psi_u}.$$

For DOG of order m, with Gaussian-width parameter σ, C_{ψ_u} can be evaluated analytically:

$$C_{\psi_u} = \frac{(m-1)!}{2\sigma^{2m}}.$$

The implemented DOG2 wavelet, with $m = 2$ and $\sigma = 1$, gives $C_{\psi_u} = 0.5$. Application of the normalization factor K adopted in our implementation (see multiplying factors of both wavelets in response to Comment n°1) gives the corresponding C_ψ value reported in the manuscript.

The calculation is more subtle for the Morlet kernel. Omitting the multiplicative normalization and for any finite central angular frequency ω_0, the integrand of C_{ψ_u} is equal to

$$\frac{e^{-\sigma^2(\omega-\omega_0)^2}}{\omega}.$$

The numerator approaches the non-zero value $e^{-\omega_0^2}$ as $\omega \to 0^+$. Since the Fourier transform does not vanish exactly at zero frequency, the continuous integral is logarithmically divergent ($1/\omega$ singularity). However, for $\omega_0 = 6$, the numerator in the integrand is $e^{-{\omega_0}^2} = e^{-36} \approx 2.3 \times 10^{-16}$. The divergent contribution therefore remains below the estimated numerical-integration error (and default absolute tolerance of standard double-precision quadrature) over all practically accessible lower bounds. The numerical integration consequently returns the stable effective value $C_{\psi_u} \approx 0.299696$, although this value should not be interpreted as a strictly finite continuous admissibility constant. Application of the adopted normalization gives the effective value used in the finite discrete reconstruction.

This distinction does not affect the empirical comparison among reconstruction formulas on the same finite CWT grid, but it is important for the mathematical interpretation and portability of the constant. A strictly admissible zero-mean Morlet definition would remove the formal divergence and yield a genuinely finite continuous C_ψ.

We have revised the manuscript to avoid presenting this Morlet value as an exact finite continuous admissibility constant. It is now identified as the effective constant corresponding to the finite discrete implementation. We also document the negligible negative-frequency leakage of the implemented Morlet filter.

⇨ Changes in Manuscript: In the revised manuscript, we have:

i. added the integral defining C_ψ and its dependence on K^2 in Sect. 2.1.4;
ii. added the analytical DOG result and explained that the conventional Morlet kernel is only approximately admissible for $\omega_0 = 6$ and that its formal continuous integral diverges logarithmically in Sect. 2.1.6;
iii. identified the reported Morlet value as an effective finite-grid constant rather than an exact continuous admissibility constant in Sect. 2.1.6;
iv. added a table reporting K, C_δ, C_ψ and C_{ψ_u} in Sect. 2.1.6.

Line-by-line comments

Figure 3: "*In Figure 3 (now **Fig. 4**), it's unclear the non-stationarity (b, d) role here. The interesting comparison is within EC plausible range, one would not use EC under non-stationarity. All is explained by the authors in the text (l. 240-247), but this paragraph adds little to the paper's message.*"

We agree that the original presentation did not sufficiently explain the purpose of these panels. The non-stationary cases were not intended to validate the CWT covariance against a true EC flux. They were included to examine how the agreement between block covariance and wavelet covariance changes when the assumptions underlying a single block-averaged Reynolds decomposition are not satisfied.

We have retained this comparison because the new scale-resolved analysis shows that the divergence is associated with stronger and more variable contributions near the 30 min upper-period limit. However, we have shortened the corresponding text and reframed the panels as diagnostics of covariance composition rather than validation against an operational reference.

⇨ Changes in Manuscript: In the revised manuscript, we have:

i. revised the Fig. 4 caption to clarify the diagnostic role of the non-stationary panels;
ii. removed language suggesting that conventional EC covariance is a valid quantity under non-stationary conditions;
iii. referred readers to the new scale-resolved analysis in Sect. 3.4.2.

L265: "*Can you be more concrete "long-range dependent geophysical processes"? Possibly add (eg. ...)*"

Long-range dependence may arise in atmospheric and land-surface records through persistent boundary-layer organization, slowly evolving ecosystem exchange, hydrological storage and other geophysical processes whose autocorrelation extends across multiple timescales.

⇨ Changes in Manuscript: We removed this sentence in the revised Discussion.

**L272: "W*here does this come from: "Under non-stationary conditions, covariance contributions become intermittent and strongly scale-dependent.""*

We agree that this statement required direct support. We have now added a scale- and time-resolved analysis of the atmospheric records.

Compared with stationary records, the selected non-stationary records showed stronger and more variable contributions at intermediate and long periods. Representative cross-scalograms showed persistent large-scale structures, localized events and alternating positive and negative contributions. The same pattern remained after excluding records close to day/night transitions and after replacing observed neighboring data with reflected extension.

The statement is now presented as an empirical result from the revised analysis rather than as a general premise.

⇨ Changes in Manuscript: In the revised manuscript, we have:

i. added scale-resolved and time-scale covariance results in Sect. 3.4.2;
ii. revised related statements in the Discussion (Sect. 4.4) to refer explicitly to these results.

L272–276: "*Do you mean non-stationary conditions and pink-noise are somewhat similar?*"

No. We thank the referee for identifying this ambiguity.

Pink noise in the synthetic experiment is a stationary colored process used to test sensitivity to a spectrum dominated by lower frequencies and to finite-record sampling of long timescales. The atmospheric non-stationary records instead exhibit evolving covariance structure within the analyzed interval. The two cases may both reduce the effective number of independent large-scale contributions, but they should not be regarded as physically or statistically equivalent.

⇨ Changes in Manuscript: We have separated the discussion of stationary colored-noise sampling from the discussion of atmospheric non-stationarity and removed wording suggesting equivalence between them.

L288–289: "*Replace "with the one-to-one relationship." By "(Fig. 3) (now **Fig. 4)""***

⇨ Changes in Manuscript: The sentence now refers directly to Fig. 4.

L338–340: **“*A too long concluding remark. Suggestion to split it "These results underline the importance of theoretically consistent formulations in multi-scale analysis. They indicate that wavelet-based estimates of geophysical variability and turbulent fluxes may benefit from a careful reassessment within this framework."”***

⇨ Changes in Manuscript: We agree and have adopted the suggested clearer wording.

L329–331: **“*The phrasing can be improved using something more down to earth like "help identify the temporal scales driving observed surface–atmosphere exchanges." or by given a real-world example before the final sentence.”***

We agree that a more direct formulation is preferable for the target community.

⇨ Changes in Manuscript: We adopted the referee’s proposed structure.

Suggestions on the form

Section 4.3: "***Some changes in the discussion section may help highlight the paper's ideas. Section 4.3 is too long. The most direct discussion to the flux community you are targeting is the last but 4.3's length gets in the way."***

Detailed suggestions on L219–296; L280–286 and L297–306.

We thank the referee for the numerous propositions regarding Sect. 4.3. We have incorporated them into a reorganized discussion of wavelet choice.

The revised Sect. 4.3 distinguishes three separate properties:

1. frequency localization and discrete filter partition;
2. temporal localization and wavelet support;
3. the two-degree-of-freedom interpretation for complex coefficients under appropriate Gaussian assumptions;

The revised Monte Carlo experiments showed lower reconstruction-error dispersion for Morlet across the Gaussian stochastic experiments. The discrete filter-partition analysis also showed a more uniform Morlet partition over the resolved frequency range under the selected scale parametrization. For DOG2, the residual difference observed for white noise was quantitatively predicted by the full-band partition for a spectrally flat input.

In the atmospheric application, DOG2 showed lower sensitivity to discontinuous or absent signal extension because of its more compact temporal support, whereas Morlet showed larger differences near the upper-period limit. Reflected extension nevertheless remained close to the adjacent-data calculation for both wavelets.

Wavelet choice is discussed as a trade-off among frequency localization, temporal localization, discrete spectral coverage, boundary sensitivity and the objective of the analysis.

⇨ Changes in Manuscript: In the revised manuscript, we have shortened and reorganized Sect. 4.3 to:

i. qualify the two-degree-of-freedom argument so that it applies only under the appropriate Gaussian assumptions;
ii. replace the speculative discussion of inertial-range compensation with the quantitative discrete filter-partition analysis;
iii. distinguish the white-noise partition result from the behavior expected for spectrally colored signals;
iv. conclude that wavelet choice should reflect the analytical objective rather than a universal ranking.

Sect. 4.4: "*Section 4.4 could also be reworded for readability.*"

Detailed suggestion on L308–311;

We agree and have rephrased Sect. 4.4. The revised text emphasizes the principal methodological implication: the proposed formulation provides a normalization-consistent framework for integrating variance and covariance across wavelet scales, while the interpretation of the retained scales remains a separate physical question.

⇨ Changes in Manuscript: In the revised manuscript, we reorganized Sect. 4.4 to distinguish the reconstruction of scale-resolved covariance from the physical interpretation of the retained contributions as turbulent transport. We also revisited the statement citing Aubinet et al. (2012), Seo et al. (2023) and Stull (1988) following the referee's suggestion, to improve readability and provide a more concrete description of the relevance to surface–atmosphere exchanges.