Comments for both reviewers

Please note that we have changed Eq. (4) from

$$O'(R)=\frac{\beta_{m0}(R_m)X_R(R)}{\beta_{m0}(R)X_R(R_m)}\exp\left\{2\int_R^{R_m}\left[S'_{a0}(x)-S_{m0}\right]\beta_{m0}(x)dx\right\}$$
$$\exp\left\{\int_R^{R_m}\left[S_{m0}\beta_{mo}(x)-S_{mR}\beta_{mR}(x)\right]dx\right\}\exp\left\{\int_R^{R_m}\left[\alpha_{a0}(x)-\alpha_{aR}(x)\right]dx\right\}$$
$$\exp\left\{2\frac{X_R(R_m)}{X(R_m)}\int_R^{R_m}\frac{S'_{a0}(x)\beta_{m0}X(x)}{X_R(x)}\exp\left(-\int_x^{R_m}\left[S_{m0}\beta_{m0}(x')-S_{mR}\beta_{mR}(x')\right]dx'\right)\right.$$
$$\left.\exp\left(-\int_x^{R_m}\left[\alpha_{a0}(x')-\alpha_{aR}(x')\right]dx'\right)dx\right\}^{-1}$$

to

$$O'(R)=\frac{\beta_{m0}(R_m)X_R(R)}{\beta_{m0}(R)X_R(R_m)}\exp\left\{2\int_R^{R_m}\left[S'_{a0}(x)-S_{m0}\right]\beta_{m0}(x)dx\right\}$$
$$\exp\left\{\int_R^{R_m}\left[S_{m0}\beta_{m0}(x)-S_{mR}\beta_{mR}(x)\right]dx\right\}\exp\left\{\int_R^{R_m}\left[\alpha_{a0}(x)-\alpha_{aR}(x)\right]dx\right\}$$
$$\exp\left\{-2\frac{X_R(R_m)}{X(R_m)}\int_R^{R_m}\frac{S'_{a0}(x)\beta_{m0}(x)X(x)}{X_R(x)}\exp\left(-\int_x^{R_m}\left[S_{m0}\beta_{m0}(x')-S_{mR}\beta_{mR}(x')\right]dx'\right)\right.$$
$$\left.\exp\left(-\int_x^{R_m}\left[\alpha_{a0}(x')-\alpha_{aR}(x')\right]dx'\right)dx\right\}.$$

to

a) avoid the ambiguous notation in the fourth exponential function

$$\exp\left\{2\frac{X_R(R_m)}{X(R_m)}\int_R^{R_m}\frac{S'_{a0}(x)\beta_{m0}X(x)}{X_R(x)}\exp\left(-\int_x^{R_m}\left[S_{m0}\beta_{m0}(x')-S_{mR}\beta_{mR}(x')\right]dx'\right)\right.$$
$$\left.\exp\left(-\int_x^{R_m}\left[\alpha_{a0}(x')-\alpha_{aR}(x')\right]dx'\right)dx\right\}^{-1}$$

and

b) replace the missing argument in β_{m0} in the first integral in the argument of the fourth integral, i.e., we have replaced β_{m0} (missing argument) with $\beta_{m0}(x)$.

Also, the line numbers referred to in the answers correspond to the revised manuscript with revision marks.

We thank both reviewers for their time and their insightful remarks.

Answers to reviewer 1

We thank the reviewer for their thoughtful reading of the manuscript and for their valuable suggestions. We agree that, in most of the cases in practice, the overlap functions for the elastic and the Raman channels will differ at short ranges, as already acknowledged in the previous paper by the authors. Studying that case of non-equal overlap functions would be of great value, but it is beyond the scope and the purpose of the present paper. See the added text related to this issue in lines 48-57 of the revised manuscript, in particular the

sentence “In fact, the failure of the identical overlap function assumption at short ranges is acknowledged in (Comerón et al., 2023) and made evident by the unrealistic aerosol backscatter coefficient behavior (sudden drop) at short ranges visible in fig. 2 of (Comerón et al., 2023)” (lines 55-57 of the revised manuscript) and in the conclusions (lines 123-127, final paragraph) of the revised manuscript. Anyway, we take up the challenge for a possible follow-up paper.

Specific comments:

Comment: *Line 14: Is this a rotational or vibrational channel? Single channel or 2 different channels? If single, why both gases are mentioned?*

Answer: We meant either a N_2 vibro-rotational Raman channel, a vibro-rotational O_2 Raman channel, or a purely rotational Raman (which includes N_2 and O_2 rotational Raman lines). To make the meaning clearer, we rephrase: “in an aerosol lidar equipped with a Raman channel of wavelength close to the elastic one, i.e. either a O_2 or a N_2 vibro-rotational Raman channel, or a purely rotational Raman channel collecting lines of the N_2 and O_2 rotational Raman spectra” (lines 14-16 in the revised manuscript).

Comment: *Line 54: The term term β_mR / β_m0 can be, in fact, slightly dependent on the temperature because of the temperature dependence of the cross-section of the rotational Raman (RR) lines in combination with the transmission of the interference filter. The interference filter is often not taken into account in the lidar equation. Adam et al. 2009 and 2012 showed that the temperature dependence is generally small enough to be considered negligible for elastic channels. I suggest to refer to those papers here and rephrase the text indicating that the range dependence is negligible. The temperature dependence of β_mR should be also small for vibrational Raman channels. For RR channels temperature effects can be introduced if the filter is not configured properly so that cross-section changes at the transmitted lower J lines are compensated well by the higher J lines.*

Answer: We agree with the reviewer. We have clarified these points and introduced the reference Adam 2009 and Adam 2012, as well as the references Veselovskii et al. 2015 and Zenteno-Hernández et al. 2021 regarding the temperature insensitiveness of pure rotational Raman channels with adequately chosen filters. In this respect, text in lines 58-69 of the revised manuscript has been added.

Comment: *Eq 5: I have repeated the calculations mentioned by the authors (substituting Eq 3a and 3b in Eq 4) and I can confirm that they result to Eq 5. It might be worth showing how the last exponential term of Eq 4 becomes much simpler after substituting X and X_R (showing that the integrals from 0 to x and from x to R_m add up to an integral from 0 to R_m which is constant and can get out of the double integral). This part requires some “imagination” and might not be straightforward to all readers. The rest of algebraic calculations are straightforward.*

Answer: we thank especially for this comment, as, indeed, the original manuscript left too much to the reader’s imagination in contradiction with the promise in the abstract. To tackle this, we have added text and equations in lines 79-91 of the revised manuscript.

Comment: *Eq. 7: Please mention the equivalent equation from Comeron et al. 2023.*

Answer: We think the reviewer refers rather to Eq. (1), as Eq. (7) (Eq. (12) in the revised manuscript) is "almost" Eq. (1), but in fact is the last part of the explanation to arrive to it, and did not appear in Comerón et al. 2023. Therefore, we have modified line 29 to read "relates to the "true" overlap function, $O(R)$, through (Eq. (18) of Comerón et al., 2023)"

Technical comments:

Eq 3a: Please change β_ao to β_a0. → Done.

Eq 4: Please change β_mo to β_m0 in the second exponential term. → Done

We thank the reviewer for their careful reading and noticing these typos.

Answers to reviewer 2

We thank the reviewer for their comments, suggestions and spotted typos. In the following we respond to their comments.

Comment: *Although the present improvement of the overlap retrieval is described as "streamlined" in the abstract, chapter 3 mentions an "alternative retrieval" and states: "... we have found a way ... other than the daunting Eq. (4) ...". This gave me the impression that the two papers provide different solutions, which is not true. In fact, to my understanding, the two retrievals are the same, but the calculation of beta_a0, that seems to be additional in Eq. (5), is embedded in Eq. (4) (Eq. (15) in the previous version), while it is done externally with Eq. (8) in the present version.*

This equivalence should be made more clear.

Answer: the reviewer is right in their interpretation. Actually, the two-step method is fully equivalent to the one presented in Comerón et al. 2023. To stress this and avoid confusion, we have added a clarification in line 21 (within the abstract) of the revised manuscript ("... we propose a streamlined, yet equivalent to the one already published, two-step explicit formulation...") and in line 116-117 within the conclusions ("...we propose a two-step explicit formulation for the retrieval of the overlap function, easier to apply than that proposed in (Comerón et al., 2023), but otherwise equivalent")

Comment: *As already mentioned by reviewer 1, the assumption that the overlap function in the elastic channel is the same as in the Raman channel is critical. It should be mentioned that not only interference filters can have a considerable impact on the overlap function, but also the mechanical misalignments of the detectors and optical elements like lenses and apertures with respect to each other, and that these misalignments, as well as the transmission characteristic of the interference filters, are possibly subject to environmental conditions like temperature changes.*

Hence the equality of the two overlap functions should not be blindly assumed, but should be verified with appropriate test setups.

Answer: we agree with the reviewer. See in this respect the text added in lines 48-57 of the revised manuscript, in particular the sentence "In fact, the failure of the identical overlap function assumption at short ranges is acknowledged in (Comerón et al., 2023) and made evident by the unrealistic aerosol backscatter coefficient behavior (sudden drop) at short

ranges visible in fig. 2 of (Comerón et al., 2023)" (lines 55-57 of the revised manuscript) and in the conclusions (lines 123-127, final paragraph) of the revised manuscript.

Comment; *It would be furthermore very helpful if the present error analysis could be extended in a follow-up of these two papers to include the impact of different overlap functions in the elastic and Raman channels*.

Answer: we fully agree. We thank the reviewer for the suggestion and take up the challenge of a follow-up paper dealing with this… expecting we can find useful results.

Comment: *The figures 3 and 4 of the previous paper give the impression that the differences between the overlap functions retrieved with differently assumed aerosol lidar ratios below about 300 m become small and negligible, but this seems only because the slope of the overlap functions is so steep towards the near range. This wrong impression could be corrected in the present paper*.

Answer: We agree with the reviewer that this apparent equality of the overlap functions retrieved for different lidar ratios in Comerón et al., 2023 is an impression induced by the steep slope. However, we don't see a way of pointing this out in the present paper without introducing more confusion. Moreover, although not directly related to the reviewer's comment, note that in Comerón et al. 2023 we already warned that the results below 400 m must be considered "suspicious" anyway, because of the anomalous behavior of the aerosol backscatter coefficient retrieved through Eq. (9) of that paper (Eq. (13) of the present one's revised version), which assumes equality of overlap functions in the elastic and Raman channels. See in this respect the last paragraph (lines 123-127) in the conclusions of the revised paper.

Typos

In Eq. 4 in the fourth exponential term it should be "beta_m0(x)". → Corrected

Line 53: probably O'(R) instead of O(R). → Corrected. Note as well that we have added "resulting" in the previous phrase ("and the resulting overlap function as $O'(R)$", line 77 of the revised manuscript). We thank the reviewer for spotting this typo, which might have originated to a lot of confusion