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Stationary Solutions and Oscillatory Dynamics in a Mathematical Model of a Saturated Moist Atmosphere

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Abstract— This paper investigates the mathematical properties of a simplified atmospheric system describing vertical air flows with water condensation. We focus on two primary aspects: the long-term equilibrium and the transient oscillatory behavior. First, we provide a rigorous proof for the existence and uniqueness of the stationary solution under specific technical conditions, establishing a solid theoretical foundation for the model's stability. Secondly, we analyze the oscillatory dynamics of the evolutionary system through numerical simulations. Our results reveal damped oscillations in the air flow intensity $\alpha(t)$ and liquid water content $\Sigma(t)$, which are interpreted as a self-regulating physical cycle driven by latent heat and droplet accumulation. By establishing a qualitative analogy with a Volterra integro-differential equation, we demonstrate that these oscillations are governed by memory effects, particularly the mean droplet residence time b. The convergence of the evolutionary variables toward the calculated stationary values provides a robust validation of the model's consistency. These findings contribute to a deeper understanding of the physical mechanisms underlying storm formation and atmospheric stability.

Keywords: Atmospheric convection, Stationary solution, Volterra equation, Damped oscillations, Existence and uniqueness.
MSC (2020): 86A10, 35Q35, 35F50

1 INTRODUCTION

Atmospheric phenomena such as cumulonimbus clouds, tropical cyclones (Montgomery and Smith, 2017), and thunderstorms (Cotton et al., 2011) are primarily caused by upward air flow and water vapor condensation. Accurately understanding and modeling these complex processes is crucial for weather forecasting and climate studies. In previous works (Aouaouda et al., 2019; Bourega et al., 2018; Ghomrani et al., 2015; Yashima, 2019), researchers have examined systems of differential equations to describe these phenomena. These systems typically involve unknown functions representing temperature (T), density (ρ), vertical velocity (v_3), and the total amount of liquid water in the air (Σ).

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A limitation observed in some older models, particularly those assuming a constant cylindrical cross-section for air flow, was an unrealistic prediction of vertical velocity increasing with altitude (Ghomrani et al., 2015). This behavior is not consistent with natural observations. To address this, an improved model was introduced in (Bourega et al., 2018), which considers air flow in a cylinder with a larger cross-section at higher altitudes, effectively being "more open upwards." In this corrected model, the vertical flow velocity varies with time but does not change significantly with altitude.

Numerical calculations of this improved evolutionary model in (Bourega et al., 2018) revealed an unexpected and significant finding: oscillations in the vertical flow velocity, accompanied by corresponding oscillations in the amount of liquid or solid water contained in the air. Although oscillations of torrential rains have been previously addressed by physicists, often in relation to gravitational waves (Plougonven and Zhang, 2014; Richiardone and Manfrin, 2003), our calculation results suggest an alternative mechanism. We propose that the oscillation of rain during a thunderstorm may be caused by the cyclic process of ascending moist air. This process leads to water vapor condensation, releasing latent heat that warms the air and accelerates its ascent. Subsequently, the accumulation of droplets produced by this process slows down the vertical air flow due to their weight. This deceleration of air ascent reduces droplet production. The gradual disappearance of droplets as they fall then allows the ascending flow of moist air to resume, creating a self-sustaining oscillatory cycle.

We believe that such a calculation result, suggesting a new mechanism for rain oscillation, warrants further investigation. This prompted us to focus on the stationary state, or more precisely, the stationary solution, of this system. Understanding the existence and uniqueness of stationary solutions is fundamental, especially when the evolutionary system exhibits complex behaviors such as damped oscillations, as observed in (Bourega et al., 2018). Recent advances in the qualitative behavior of such atmospheric models have highlighted the importance of establishing rigorous stationary states (Kouadri, 2022). The mathematical well-posedness of moisture dynamics with phase changes in atmospheric systems has also been rigorously studied (Hittmeir et al., 2017), further motivating the analysis of stationary solutions in the present work. The stationary state represents the physical equilibrium toward which the system tends to gravitate over the long term.

The main objective of this article is to prove the existence and uniqueness of the stationary solution for a simplified version of this system of equations. In this framework, the vertical velocity v_3 is characterized by a parameter α, independent of altitude z. Technically, our approach is inspired by the comparison theorem for ordinary differential equations. This type of stability analysis for non-linear dynamical systems is essential in fluid flow applications to ensure the physical consistency of the model (Mansour, 2021). However, its direct application to a system of equations is not straightforward and requires specific elaborations, which constitute the core of our mathematical contribution.

Beyond this theoretical result, we present numerical examples to illustrate the validity of our proofs. Furthermore, we provide an in-depth analysis of the oscillatory dynamics by drawing a qualitative analogy with the Volterra integro-differential equation. By introducing the concept of a memory effect– modeled by the mean droplet residence time – we

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offer a deeper mathematical understanding of the mechanisms governing these oscillations and their convergence toward the stationary state.

2 THE FULL EVOLUTIONARY MODEL

In our previous article (Bourega et al., 2018), to describe the upward flow of air accompanied by water vapor condensation in the atmosphere, we proposed a system of equations for density (ρ), temperature (T), intensity coefficient (α), and normalized velocity (w). This model was developed by adopting the approximation $v(t,z) = \alpha(t)w(t,z)$, where v denotes the vertical component of the velocity. We assume quasi-stationary conditions for the spatial structure, neglecting the time derivatives of w, ρ, and T:

$$\partial_t w(t,z) \approx 0, \qquad \partial_t \rho(t,z) \approx 0, \qquad \partial_t T(t,z) \approx 0.$$

The system, defined in a cylindrical domain $\Omega_3 = \{(x,y,z) \in \mathbb{R}^3 \,|\, 0 \le z \le \overline{z}_1,\ (x^2+y^2)\pi \le S(z)\}$, is based on the fundamental equations of fluid mechanics (Landau and Lifshitz, 1989). The main physical quantities are the vertical velocity v, temperature T, and density ρ. Pressure P is given by the ideal gas law:

$$P = R_1 \rho T, \qquad R_1 = \frac{R_0}{\mu_a}, \tag{1}$$

where R_0 is the universal gas constant and μ_a is the molar mass of air.

Before presenting the complete system, we recall the quantities involved in the process of condensation or inverse sublimation of water vapor in the atmosphere. For simplicity, we approximate the latent heat of transition from water vapor to solid state by that from vapor to liquid state. Thus, L_{tr} denotes the latent heat of phase transition of water from gaseous to liquid or solid state. The density of saturated water vapor is denoted $\overline{\pi}_{vs}(T)$. Its dependence on temperature plays a crucial role. According to (Matveev, 2000), its expression is:

$$\overline{\pi}_{vs}(T) = \frac{E_0}{R_1 T} \cdot 10^{\frac{7.63(T-273.15)}{T-31.25}}, \qquad E_0 = 6.107\,\text{mbar}. \tag{2}$$

For physical details on condensation or inverse sublimation (hereinafter simply referred to as *condensation*), see (Matveev, 2000; Prodi and Battaglia, 2004; Kikoine and Kikoine, 1979).

The amount of condensation, H_{tr}, in air containing saturated water vapor and rising with a vertical velocity $v > 0$, is given by:

$$H_{tr} = \left(\overline{\pi}_{vs}(T)\frac{d}{dz}\log\rho - \frac{d}{dz}\overline{\pi}_{vs}(T)\right) v. \tag{3}$$

Derivation of the Turbulent Condensation Rate H_{tr}

To maintain saturation, the saturated mass concentration $q_s = \overline{\pi}_{vs}(T)/\rho$ must vary along the vertical motion v. The condensation rate H_{tr} is defined by the product of the

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mass flow rate ρv and the spatial derivative of q_s: $H_{tr} = -v\rho\frac{d}{dz}\left(\frac{\overline{\pi}_{vs}}{\rho}\right)$. Applying the quotient rule for the derivative and factoring out ρ:

$$\frac{d}{dz}\left(\frac{\overline{\pi}_{vs}}{\rho}\right) = \frac{1}{\rho}\left(\frac{d\overline{\pi}_{vs}}{dz} - \overline{\pi}_{vs}\frac{1}{\rho}\frac{d\rho}{dz}\right).$$

Substituting this result into the expression for H_{tr} and using the identity $\frac{1}{\rho}\frac{d\rho}{dz} = \frac{d}{dz}\log\rho$, we obtain:

$$H_{tr} = -v\rho\cdot\frac{1}{\rho}\left(\frac{d\overline{\pi}_{vs}}{dz} - \overline{\pi}_{vs}\frac{d}{dz}\log\rho\right)$$
$$H_{tr} = v\left(\overline{\pi}_{vs}(T)\frac{d}{dz}\log\rho - \frac{d}{dz}\overline{\pi}_{vs}(T)\right).$$

By decomposing ρ into $\rho = \rho_a + \rho_h$ (dry air density ρ_a, water vapor density ρ_h), and considering that phase transition only affects water vapor, we have:

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = -H_{tr}.$$

The heat released by condensation is $L_{tr}H_{tr}$. Let Σ be the total amount of liquid or solid water in the air. Neglecting viscosity and thermal conductivity, the air movement in the "chimney" is described by:

$$S(z)\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial z}(S(z)\rho v) = -S(z)H_{tr}, \tag{4}$$
$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z}\right) = -R_1\frac{\partial}{\partial z}(\rho T) - g[\Sigma + \rho], \tag{5}$$
$$\rho c_v\left(\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial z}\right) - R_1T\left(\frac{\partial\rho}{\partial t} + v\frac{\partial\rho}{\partial z}\right) = (R_1T + L_{tr})H_{tr}, \tag{6}$$

where g is the acceleration due to gravity and c_v is the specific heat. For Σ in (5), we define:

$$\Sigma(t) = \frac{1}{\overline{z}_1}\int_0^t\varphi(t-s)\int_0^{\overline{z}_1}\left(\overline{\pi}_{vs}(T(s,z))\frac{\partial}{\partial z}\log\rho(s,z) - \frac{\partial}{\partial z}\overline{\pi}_{vs}(T(s,z))\right)v(s,z)dzds, \tag{7}$$

where $\varphi(\tau)$ represents the probability that a droplet remains in the air after time τ from its creation (Ghomrani et al., 2015).

For this model to represent air movement in a thunderstorm, internal and external pressures must be equal at the boundaries. We use a hydrostatic state system as a reference for the external environment:

$$\rho c_v\frac{dT}{dz} - R_1T\frac{d\rho}{dz} = \vartheta\left(R_1T + L_{tr}\right)\left(\overline{\pi}_{vs}(T)\frac{d}{dz}\log\rho - \frac{d}{dz}\overline{\pi}_{vs}(T)\right), \tag{8}$$
$$R_1\frac{d}{dz}(\rho T) = -g\rho\,. \tag{9}$$

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Following (Ghomrani et al., 2015), we introduce the approximation $v(t,z) = \alpha(t)w(t,z)$. Substituting (3) into (4)–(6) yields:

$$S(z)\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(S(z)\rho w) = -S(z)\left(\overline{\pi}_{vs}(T)\frac{d}{dz}\log\rho - \frac{d}{dz}\overline{\pi}_{vs}(T)\right)w, \quad (10)$$

$$\rho c_v \frac{dT}{dz} - R_1 T\frac{d\rho}{dz} = (R_1 T + L_{tr})\left(\overline{\pi}_{vs}(T)\frac{1}{\rho}\frac{d\rho}{dz} - \frac{d\overline{\pi}_{vs}(T)}{dT}\frac{d}{dz}T\right), \quad (11)$$

$$\rho w D + \alpha(t)^2 \rho w \frac{dw}{dz} + R_1\rho\frac{dT}{dz} + R_1 T\frac{d\rho}{dz} = -g[\Sigma(t) + \rho], \quad (12)$$

$$\Sigma(t) = \frac{1}{\overline{z}_1}\int_0^t \varphi(t-s)\alpha(s)\int_0^{\overline{z}_1}\left(\overline{\pi}_{vs}(T)\frac{d}{dz}\log\rho - \frac{d}{dz}\overline{\pi}_{vs}(T)\right)w dz ds, \quad (13)$$

$$\frac{d}{dt}\alpha(t) = D(t). \quad (14)$$

The auxiliary unknown $D(t)$ allows the system to be treated as a control problem. The memory kernel φ is given by:

$$\varphi(\tau) = \exp\left(-\frac{\pi\tau^2}{4b^2}\right), \quad (15)$$

where the parameter b represents the mean droplet residence time.

3 THE STATIONARY SYSTEM AND ITS SIMPLIFICATION

This article focuses on the stationary solution of the system. The stationary solution must satisfy the system derived from (10)–(14) by setting $\frac{d}{dt}\alpha(t) = 0$ (hence $D(t) = 0$) and assuming $\alpha(t) = \alpha$ (a constant intensity coefficient) and $\Sigma(t) = \Sigma$ (a constant amount of total liquid water). Equations (12) and (13) are then replaced by:

$$\alpha^2\rho w\frac{dw}{dz} + R_1\rho\frac{dT}{dz} + R_1 T\frac{d\rho}{dz} = -g[\Sigma + \rho], \quad (16)$$

$$\Sigma = \alpha\frac{b}{\overline{z}_1}\int_0^{\overline{z}_1}\left(\overline{\pi}_{vs}(T)\frac{d}{dz}\log\rho - \frac{d}{dz}\overline{\pi}_{vs}(T)\right)w dz, \quad (17)$$

where $b = \int_0^\infty \varphi(\tau)d\tau$. It should be noted that Σ represents a density, so its unit is in g/m^3.

In (Ghomrani et al., 2015), the assumption of a constant cylindrical cross-section $S(z)$ led to an unrealistic increase in velocity with altitude. To correct this, (Bourega et al., 2018) proposed choosing $S(z)$ such that the normalized velocity w remains close to 1. Following this idea, if we assume that $S(z)$ is not a priori given but is determined a posteriori by the solution (ρ, T) and by setting $w \equiv 1$, we can find ρ, T, α satisfying the simplified system.

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We now derive the explicit expression for $S(z)$ by imposing the condition of a uniform normalized velocity, $w(t,z) \equiv 1$. Substituting $w = 1$ into the quasi-stationary continuity equation (10) yields:

$$S(z)\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(S(z)\rho) = -S(z)H_{tr}^{\text{cond}}, \tag{18}$$

where $H_{tr}^{\text{cond}} = \left(\overline{\pi}_{vs}(T)\frac{d}{dz}\log\rho - \frac{d}{dz}\overline{\pi}_{vs}(T)\right)$ is the term in parenthesis in Equation (3). For the fully stationary case $(\partial_t \rho = 0)$, the equation simplifies to the conservation law:

$$\frac{d}{dz}(S\rho) = -S \cdot H_{tr}^{\text{cond}}.$$

Expanding the derivative on the Left-Hand Side (LHS) and substituting the expression for H_{tr}^{cond} gives:

$$\rho\frac{dS}{dz} + S\frac{d\rho}{dz} = -S\left(\overline{\pi}_{vs}(T)\frac{d}{dz}\log\rho - \frac{d}{dz}\overline{\pi}_{vs}(T)\right).$$

Dividing by $S\rho$ and reorganizing the terms to separate the S dependence leads to:

$$\frac{1}{S}\frac{dS}{dz} = -\frac{1}{\rho}\frac{d\rho}{dz} - \frac{1}{\rho}\left(\overline{\pi}_{vs}(T)\frac{d}{dz}\log\rho - \frac{d}{dz}\overline{\pi}_{vs}(T)\right)$$
$$\frac{d}{dz}\log S = -\frac{d}{dz}\log\rho + \frac{1}{\rho}\left(\frac{d}{dz}\overline{\pi}_{vs}(T) - \overline{\pi}_{vs}(T)\frac{d}{dz}\log\rho\right)$$

Using the identity $\frac{d}{dz}\left(\frac{\overline{\pi}_{vs}}{\rho}\right) = \frac{1}{\rho}\left(\frac{d\overline{\pi}_{vs}}{dz} - \overline{\pi}_{vs}\frac{d}{dz}\log\rho\right)$ (derived in Section 2, but without the ρ factor), the expression simplifies to the logarithmic derivative of the term $\frac{\overline{\pi}_{vs}}{\rho}$:

$$\frac{d}{dz}\log S = -\frac{d}{dz}\log\rho - \frac{d}{dz}\left(\frac{\overline{\pi}_{vs}(T)}{\rho}\right).$$

Integrating this equation from $z = 0$ to z:

$$\log\left(\frac{S(z)}{S(0)}\right) = -\log\left(\frac{\rho(z)}{\rho(0)}\right) - \left[\frac{\overline{\pi}_{vs}(T(z))}{\rho(z)} - \frac{\overline{\pi}_{vs}(T_0)}{\rho_0}\right].$$

Solving for $S(z)$ finally yields:

$$S(z) = \frac{\rho_0 S(0)}{\rho(z)}\exp\left(\frac{\overline{\pi}_{vs}(T(z))}{\rho(z)} - \frac{\overline{\pi}_{vs}(T_0)}{\rho_0}\right). \tag{19}$$

Note on the Choice $w \equiv 1$: The original full evolutionary system only determines the product $v(t,z) = \alpha(t)w(t,z)$, not the spatial structure $w(t,z)$ and the temporal intensity $\alpha(t)$ separately. The choice $\mathbf{w(z)} \equiv \mathbf{1}$ in the stationary analysis is therefore a normalization made to disentangle the variables. By fixing $w(z) \equiv 1$, all variations in the flow velocity are carried by the temporal coefficient $\alpha(t)$ in the evolutionary model, and by the constant α in the stationary system. This normalization is justified because the volume $S(z)$ has been chosen precisely to make the normalized velocity w close to unity, thereby simplifying the study of the remaining equations.

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Thus, we are interested in the following system of equations for the variables (ρ, T, α) on $0 < z < \overline{z}_1$:

$$\rho c_v \frac{dT}{dz} - R_1 T \frac{d\rho}{dz} = (R_1 T + L_{tr}) \left(\overline{\pi}_{vs}(T) \frac{1}{\rho} \frac{d\rho}{dz} - \frac{d\overline{\pi}_{vs}(T)}{dT} \frac{dT}{dz} \right), \tag{20}$$

$$R_1 \rho \frac{dT}{dz} + R_1 T \frac{d\rho}{dz} = -g[\Sigma + \rho], \tag{21}$$

$$\Sigma = \alpha \frac{b}{\overline{z}_1} \int_0^{\overline{z}_1} \left(\overline{\pi}_{vs}(T) \frac{d}{dz} \log \rho - \frac{d}{dz} \overline{\pi}_{vs}(T) \right) dz, \tag{22}$$

This system is considered with the conditions:

$$\rho(0) = \rho_0, \quad T(0) = T_0, \tag{23}$$

$$\rho(\overline{z}_1) T(\overline{z}_1) = q_0. \tag{24}$$

The study of stationary solutions is a common approach in many fields of physics and engineering to analyze the long-term behavior of systems, as exemplified by works on glacier dynamics (Bogovskii et al., 2010).

4 PRELIMINARIES FOR THE STATIONARY SOLUTION

In the phenomenon of water vapor condensation caused by upward air flow, the saturated vapor density function $\overline{\pi}_{vs}(T)$ plays a crucial role. Its properties influence the solution of the system of equations (20)–(22). We define the expression for $\overline{\pi}_{vs}(T)$ by:

$$\overline{\pi}_{vs}(T) = \frac{E_0}{R_1 T} \cdot 10^{\frac{7.63(T-273.15)}{T-31.25}}, \qquad E_0 = 6.107 \text{ (mbar)}. \tag{25}$$

which is given by physics (see for example (Matveev, 2000)). We consider this expression only for $T > 31.25$ (K). If, in the problem, we neglect condition (24) and set $\alpha = 0$ (and thus $\Sigma = 0$), then ρ and T can be determined by equations (20), (21) and condition (23).

Consider a parameter $\vartheta \in [0, 1]$ introduced on the right-hand side of equation (20). We define the problem of the hydrostatic state of partially moist air as follows:

$$\rho c_v \frac{dT}{dz} - R_1 T \frac{d\rho}{dz} = \vartheta \, (R_1 T + L_{tr}) \left(\overline{\pi}_{vs}(T) \frac{d}{dz} \log \rho - \frac{d}{dz} \overline{\pi}_{vs}(T) \right), \tag{26}$$

$$R_1 \frac{d}{dz}(\rho T) = -g\rho \,, \tag{27}$$

$$\rho(0) = \rho_0, \quad T(0) = T_0. \tag{28}$$

We denote by $(\rho_{hs}^{[0]}, T_{hs}^{[0]})$ the solution of problem (26)–(28) with $\vartheta \equiv 0$, and by $(\rho_{hs}^{[1]}, T_{hs}^{[1]})$ that of the case $\vartheta \equiv 1$. Before stating Lemma 1, we define $\Theta_- = 31.25$ (K),

$$\Theta_+ = 31.25 + \frac{1}{7} \log_{10}(1845.697) + \sqrt{\left(31.25 + \frac{1}{7} \log_{10}(1845.697) \right)^2 - 976.5625} \approx 1275.9 \text{ (K)}, \tag{29}$$

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and

$$\overline{z}^* = \frac{c_v + R_1}{g}(T_0 - 31.25). \tag{30}$$

The condition $\Theta_- < T < \Theta_+$ guarantees the inequality $T\frac{d}{dT}\log(\overline{\pi}_{vs}(T)) > \frac{c_v}{R_1} = \frac{5}{2}$, which is necessary in the proof. Note that $T_{hs}^{[0]}(\overline{z}^*) = 31.25$.

Physical vs. Analytical Significance of the Thresholds The defined thresholds $\Theta_- = 31.25$ K, $\Theta_1 \approx 66$ K, and $\Theta_+ \approx 1275.9$ K are purely analytical bounds derived from the structure of $\overline{\pi}_{vs}(T)$ ((25)). The lower bound Θ_- corresponds to the temperature where the denominator of the vapor density expression becomes zero. The bounds Θ_1 and Θ_+ are the analytical limits introduced to guarantee the crucial inequality $T\frac{d}{dT}\log(\overline{\pi}_{vs}(T)) > c_v/R_1$ used in the monotonicity lemmata. Since the temperatures of the physical phenomena of interest satisfy 100 K $< T <$ 373.15 K, all analytical conditions ($\Theta_- < T < \Theta_+$) are automatically satisfied. Therefore, these bounds impose no practical restriction on the physical domain of the solution, serving only as a technical requirement for the mathematical proof structure. We now state the following lemma:

Lemma 1. *We assume that $\rho_0 > 0$ and $\Theta_- < T_0 < \Theta_+$. Then the system of equations (26)–(27) with condition (28) admits a unique continuously differentiable solution (ρ, T) on the interval $[0, \overline{z}^*[$. Furthermore, the functions $\rho(z)$ and $T(z)$ are strictly decreasing, and we have*

$$T(z) \geq T_{hs}^{[0]}(z) \qquad \forall z \in [0, \overline{z}^*[\,. \tag{31}$$

Proof . It is clear that the system of equations (26)–(27) with initial data $\rho_0 > 0$ and $\Theta_- < T_0 < \Theta_+$ admits a unique C^1 solution in a sufficiently small interval. To prove the existence and uniqueness of the solution in $[0, \overline{z}^*[$, it suffices to demonstrate that if the solution exists in an interval $[0, \overline{z}[$ with $0 < \overline{z} < \overline{z}^*$, then $0 < \inf_{0\leq z<\overline{z}} \rho(z) \leq \sup_{0\leq z<\overline{z}} \rho(z) < \infty$ and $\Theta_- < \inf_{0\leq z<\overline{z}} T(z) \leq \sup_{0\leq z<\overline{z}} T(z) < \Theta_+$. For this, we use the properties illustrated in the following lemmas.

Lemma 2. *Assume that $\rho_0 > 0$ and $\Theta_- < T_0 < \Theta_+$ and that (ρ, T), with $\rho > 0$ and $\Theta_- < T < \Theta_+$, is the solution of the system of equations (26)–(27) with condition (28) in an interval $[0, \overline{z}[$, $0 < \overline{z} \leq \overline{z}^*$. Then the functions $\rho(z)$ and $T(z)$ are strictly decreasing in the interval $[0, \overline{z}[$ and we have*

$$\frac{d}{dz}\left(\frac{\overline{\pi}_{vs}(T(z))}{\rho(z)}\right) \leq 0 \quad \forall z \in [0, \overline{z}[. \tag{32}$$

Proof. It is convenient to transform equation (26) into the form:

$$\frac{d}{dz}\log\left(\frac{T^{c_v}}{\rho^{R_1}}\right) = -\vartheta\left(R_1 + \frac{L_{tr}}{T}\right)\frac{d}{dz}\left(\frac{\overline{\pi}_{vs}(T)}{\rho}\right). \tag{33}$$

Equation (33) is obtained by dividing Equation (26) by ρT and utilizing two key identities:

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1. The logarithmic derivative of the left-hand side (LHS):

$$\frac{d}{dz}(c_v \log T - R_1 \log \rho) = \frac{d}{dz} \log \left(\frac{T^{c_v}}{\rho^{R_1}} \right)$$

2. The compact form of the right-hand side (RHS) using $q_s = \overline{\pi}_{vs}/\rho$:

$$\frac{1}{\rho} \left(\overline{\pi}_{vs} \frac{d}{dz} \log \rho - \frac{d}{dz} \overline{\pi}_{vs} \right) = -\frac{d}{dz} \left(\frac{\overline{\pi}_{vs}}{\rho} \right)$$

Combining these steps yields Equation (33).
Furthermore, from equation (27), it immediately follows that:

$$\frac{d}{dz}(\rho(z)T(z)) = -\frac{g}{R_1}\rho(z) < 0 \quad \forall z \in [0, \overline{z}[\,.$$

Therefore, it is impossible that $\frac{d}{dz}\rho(z) \geq 0$ and $\frac{d}{dz}T(z) \geq 0$ simultaneously.

Now we show that ρ and T are both strictly decreasing. Suppose by contradiction that $\frac{d}{dz}T(z_0) \geq 0$ at some z_0. Then from $\frac{d}{dz}(\rho T) < 0$, we must have $\frac{d}{dz}\rho(z_0) < 0$. But then the LHS of (33) would be strictly positive, while the RHS would be non-positive, which is impossible. Similarly, if $\frac{d}{dz}\rho(z_0) \geq 0$, then $\frac{d}{dz}T(z_0) < 0$, making the LHS negative while the RHS non-negative. Thus, we must have $\frac{d}{dz}\rho(z) < 0$ and $\frac{d}{dz}T(z) < 0$ for all $z \in [0, \overline{z}[$.

To prove (32), we assume by contradiction that there exists a $z_0 \in [0, \overline{z}[$ such that:

$$\frac{d}{dz} \left(\frac{\overline{\pi}_{vs}(T(z_0))}{\rho(z_0)} \right) > 0.$$

This implies that at z_0:

$$\frac{1}{\rho} \frac{d\overline{\pi}_{vs}}{dT} \frac{dT}{dz} - \frac{\overline{\pi}_{vs}}{\rho^2} \frac{d\rho}{dz} > 0 \quad \Longleftrightarrow \quad -\frac{d\rho}{dz} > -\frac{\rho}{\overline{\pi}_{vs}(T)} \frac{d\overline{\pi}_{vs}(T)}{dT} \frac{dT}{dz}.$$

Substituting this into the LHS of (26) at z_0 gives:

$$\text{LHS} > \rho \left(c_v - R_1 T \frac{d}{dT} \log(\overline{\pi}_{vs}(T)) \right) \frac{dT}{dz}.$$

Under the condition $\Theta_- < T < \Theta_+$, the term in parenthesis is negative. Since $\frac{dT}{dz} < 0$, the product is strictly positive (LHS > 0).

Meanwhile, the right-hand side of (26) at z_0 is:

$$\text{RHS} = \vartheta \left(R_1 T + L_{tr} \right) \rho \left[-\frac{d}{dz} \left(\frac{\overline{\pi}_{vs}}{\rho} \right) \right] < 0.$$

The contradiction (LHS > 0 and RHS < 0) proves that $\frac{d}{dz}(\overline{\pi}_{vs}/\rho) \leq 0$. The lemma is proven. □

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Lemma 3. *Assume that* $\rho_0 > 0$ *and* $\Theta_- < T_0 < \Theta_+$ *and that* (ρ, T)*, with* $\rho > 0$ *and* $\Theta_- < T < \Theta_+$*, is the solution of the system of equations* (26)–(27) *with condition* (28) *in an interval* $[0, \overline{z}[$*,* $0 < \overline{z} \leq \overline{z}^*$*. Then we have*

$$T(z) \geq T_{hs}^{[0]}(z) \quad \forall z \in [0, \overline{z}[. \tag{34}$$

Proof . Define

$$\varphi_1(z) = \vartheta \int_0^z \left(R_1 + \frac{L_{tr}}{T(z')}\right)\left(-\frac{d}{dz'}\left(\frac{\overline{\pi}_{vs}(T(z'))}{\rho(z')}\right)\right) dz'. \tag{35}$$

By virtue of Lemma 2, we have

$$\varphi_1(z) \geq 0 \qquad \forall z \in [0, \overline{z}[\,. \tag{36}$$

Furthermore, from equation (26) (or (33)), we deduce

$$\frac{T^{c_v}}{\rho^{R_1}} = \frac{T_0^{c_v}}{\rho_0^{R_1}} e^{\varphi_1},$$

or

$$\rho = \rho_0 \frac{T^{c_v/R_1}}{T_0^{c_v/R_1}} e^{-\varphi_1/R_1}. \tag{37}$$

Substituting (37) into (27), we obtain

$$R_1 \frac{d}{dz}\left(T^{(c_v+R_1)/R_1} e^{-\varphi_1/R_1}\right) = -g T^{c_v/R_1} e^{-\varphi_1/R_1}. \tag{38}$$

Justification for the substitution $y(z)$*:* The variable $y(z)$ is chosen precisely because it is the exact expression inside the derivative on the Left-Hand Side (LHS) of (38). This enables the immediate transformation of the LHS into $R_1 dy/dz$. Simultaneously, the RHS simplifies because $T^{c_v/R_1} e^{-\varphi_1/R_1}$ can be fully expressed as a power of $y(z)$, leading to the compact form (39) in the next step. Indeed, if we set $y = T^{(c_v+R_1)/R_1} e^{-\varphi_1/R_1}$, equation (38) transforms into

$$R_1 \frac{d}{dz} y = -g y^{c_v/(c_v+R_1)} e^{-\varphi_1/(c_v+R_1)},$$

or

$$(c_v + R_1) \frac{d}{dz} y^{R_1/(c_v+R_1)} = -g e^{-\varphi_1/(c_v+R_1)}. \tag{39}$$

The solution of equation (39) with initial condition $(y(0))^{R_1/(c_v+R_1)} = T_0$ is

$$y^{R_1/(c_v+R_1)} = T_0 - \frac{g}{(c_v + R_1)} \int_0^z e^{-\varphi_1/(c_v+R_1)} dz'.$$

Therefore, $T(z)$ has the form

$$T(z) = \left(T_0 - \frac{g}{(c_v + R_1)} \int_0^z e^{-\varphi_1/(c_v+R_1)} dz'\right) e^{\varphi_1/(c_v+R_1)}. \tag{40}$$

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If $\varphi_1 \geq 0$, then we have $e^{-\varphi_1/(c_v+R_1)} \leq 1 \leq e^{\varphi_1/(c_v+R_1)}$. Therefore, by virtue of (36), we have

$$T(z) \geq T_0 - \frac{g}{(c_v+R_1)} z = T_{hs}^{[0]}(z).$$

The lemma is proven. □

Continuation of the proof of Lemma 1 Since $T(z)$ is decreasing according to Lemma 2, and considering (34), we have

$$0 < \Theta_- < T_{hs}^{[0]}(z) \leq T(z) \leq T_0 < \Theta_+ \qquad \forall z \in [0, \overline{z}]. \tag{41}$$

Furthermore, since for all $T > \Theta_-$ and for all $\rho > 0$, we have $\frac{\overline{\pi}_{vs}(T)}{\rho} > 0$, from Lemmas 2 and 3, we deduce that

$$0 \leq \varphi_1(z) \leq \left(R_1 + \frac{L_{tr}}{T_{hs}^{[0]}(z)} \right) \frac{\overline{\pi}_{vs}(T_0)}{\rho_0} \leq \left(R_1 + \frac{L_{tr}}{\Theta_-} \right) \frac{\overline{\pi}_{vs}(T_0)}{\rho_0} \equiv K_{\varphi_1} < \infty. \tag{42}$$

Consequently, from (37), (42), and Lemma 2, we obtain

$$0 < K_\rho^- \equiv \rho_0^{R_1} \frac{{\Theta_-}^{c_v/R_1}}{T_0^{c_v/R_1}} e^{-K_{\varphi_1}/R_1} \leq \rho_0^{R_1} \frac{(T_{hs}^{[0]}(z))^{c_v/R_1}}{T_0^{c_v/R_1}} e^{-K_{\varphi_1}/R_1} \leq \rho(z) \leq \rho_0 < \infty \tag{43}$$

for all $z \in [0, \overline{z}]$. Inequalities (41) and (43) allow us to extend the solution (ρ, T) with its properties of strict decrease and inequality (34) over the entire interval $[0, \overline{z}^*[$. The proposition is proven. □

5 EXISTENCE AND UNIQUENESS OF THE STATIONARY SOLUTION

We will prove the existence and uniqueness of the solution to problem (20)–(24), under a technical condition:

$$T > \Theta_1, \tag{44}$$

where Θ_1 is defined by the following relations:

$$\frac{c_v + R_1}{R_1} \overline{\pi}_{vs}(T) - \frac{\overline{\pi}_{vs}(T)}{T} - \frac{d\overline{\pi}_{vs}(T)}{dT} > 0 \quad \text{if } T > \Theta_1, \tag{45}$$

$$\frac{c_v + R_1}{R_1} \overline{\pi}_{vs}(T) - \frac{\overline{\pi}_{vs}(T)}{T} - \frac{d\overline{\pi}_{vs}(T)}{dT} < 0 \quad \text{if } \Theta_- < T < \Theta_1; \tag{46}$$

we estimate the value of Θ_1 to be 66.09 (K). Condition (44) simplifies the proof of existence and uniqueness of the solution. It is possible to study the existence and uniqueness of the solution in a more general case, but this would require much more complicated arguments. Furthermore, the case $T \leq \Theta_1$ ($\Theta_1 \approx 66.09$ (K)) is infrequent (at least concerning Earth's atmosphere). For this reason, in the present article, we will show the existence and uniqueness of the solution under condition (44).

Then for problem (20)–(24) we have the following result (in the statement, $\rho_{hs}^{[1]}$ and $T_{hs}^{[1]}$ are the functions defined by problem (26)–(28) and condition (44)).

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Proposition 5.1. *Consider two real numbers $\rho_0 > 0$ and $T_0 \in]\Theta_1, \Theta_+[$. Let $\overline{z}_1 \in]0, \overline{z}^*[$ (where $\overline{z}^*$ is the number defined in (30)) under the condition $T_{hs}^{[1]}(\overline{z}_1) > \Theta_1$. Then there exists a number $\overline{Q}_0$ with $0 < \overline{Q}_0 < \rho_{hs}^{[1]}(\overline{z}_1) T_{hs}^{[1]}(\overline{z}_1)$, and for each $q_0 \in]\overline{Q}_0, \rho_{hs}^{[1]}(\overline{z}_1) T_{hs}^{[1]}(\overline{z}_1)]$ there exists a unique solution (ρ, T, α) to problem (20)–(24) such that*

$$\inf\{T(\overline{z}_1) \quad | \quad (\rho, T) = Sol(q_0)\} = \Theta_1, \tag{47}$$

where $Sol(q_0)$ is the solution of problem (20)–(24) with $q_0 \in]\overline{Q}_0, \rho_{hs}^{[1]}(\overline{z}_1) T_{hs}^{[1]}(\overline{z}_1)]$.

Proof . To prove the proposition, consider the auxiliary equation:

$$\frac{dP}{dz} = -g\rho - c_0 \tag{48}$$

with a constant $c_0 \geq 0$. We therefore have the following lemma:

Lemma 4. *Consider the numbers ρ_0, T_0, $\overline{z}_1$ as in Proposition 5.1. If (ρ, T) is a solution in the interval $[0, \overline{z}_1]$ of problem (20), (48), (23) with some $c_0 \geq 0$, then $\rho(z)$ and $T(z)$ are decreasing functions.*

On the other hand, let $c^{(i)}$ and $c^{(ii)}$ be two numbers such that $0 \leq c^{(i)} < c^{(ii)}$. If $(\rho^{(i)}, T^{(i)})$ and $(\rho^{(ii)}, T^{(ii)})$ are the solutions in the interval $[0, \overline{z}_1]$ of problem (20), (48), (23) with $c_0 = c^{(i)}$ and $c_0 = c^{(ii)}$ respectively, solutions that satisfy the relations:

$$\rho^{(i)}(z) > 0, \quad \rho^{(ii)}(z) > 0, \quad \Theta_1 < T^{(i)}(z) < \Theta_+, \quad \Theta_1 < T^{(ii)}(z) < \Theta_+.$$

Then we have

$$T^{(ii)}(z) < T^{(i)}(z) \qquad \forall z \in]0, \overline{z}_1], \tag{49}$$

$$\rho^{(ii)}(z) T^{(ii)}(z) < \rho^{(i)}(z) T^{(i)}(z) \qquad \forall z \in]0, \overline{z}_1]. \tag{50}$$

The proof of Lemma 4 will be given in Section 6. In this section, we assume it to be true and continue the proof of Proposition 5.1.

Continuation of the proof of Proposition 5.1 Let (ρ_{c_0}, T_{c_0}) be the solution (ρ, T) of problem (20), (48), (23) with c_0. The strict monotonicity of the solution $\nu(c_0)$ is guaranteed by Lemma 4. Specifically, the ordering of the solutions ($T^{(ii)} < T^{(i)}$ for $c^{(ii)} > c^{(i)}$) is established by the sign of the derivative $\partial_\rho \mathbf{D_1} > \mathbf{0}$ (proven in Section 6 under the condition $T > \Theta_1$), which prevents the crossing of the temperature profiles $T^{(i)}$ and $T^{(ii)}$.

By virtue of the strict inequalities (49), (50), and the continuous dependence of the solution on c_0, we define the domain of c_0 as $[\mathbf{0}, \overline{\mathbf{C}}_\mathbf{0}]$, where $\overline{C}_0$ is a finite value such that:

$$\lim_{c_0 \to \overline{C}_0^-} T_{c_0}(\overline{z}_1) = \Theta_1 \tag{51}$$

The finiteness of $\overline{C}_0$ is guaranteed by the lower bound of the temperature domain, $T > \Theta_1$. Since $T(z)$ must remain above Θ_1 for the solution to exist (otherwise the rate of decrease

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would become singular), the rate of dissipation (proportional to c_0) cannot increase indefinitely.

The relation (51), the decrease of functions $\rho(z)$ and $T(z)$, and (49), guarantee that, for any $c_0 \in [0, \overline{C}_0[$, we have

$$0 < \rho_{c_0}(z) \leq \rho_0, \qquad \Theta_1 < T_{c_0}(z) \leq T_0 < \Theta_+$$

for $z \leq \overline{z}_1$. This implies that for any $c_0 \in [0, \overline{C}_0[$ the solution (ρ_{c_0}, T_{c_0}) exists over the entire interval $[0, \overline{z}_1]$.

Now, we define the function $\nu(c_0)$ representing the value of the product ρT at the top boundary $z = \overline{z}_1$. This quantity is directly proportional to the pressure $P(\overline{z}_1) = R_1 \rho(\overline{z}_1) T(\overline{z}_1)$ at that height:

$$\nu(c_0) = \rho_{c_0}(\overline{z}_1) T_{c_0}(\overline{z}_1). \tag{52}$$

By virtue of the continuous dependence of the solution on the data, the function $\nu(c_0)$ is continuous in c_0 and, according to Lemma 4, it is strictly decreasing. Therefore, if q_0 is given in the admissible interval, then there exists a unique c_0 given by:

$$c_0 = \nu^{-1}(q_0). \tag{53}$$

With the determined value of c_0, we can solve problem (20), (48), (23) and obtain the solution $(\rho, T) = (\rho_{c_0}, T_{c_0})$. We can then determine the value of the integral:

$$\int_0^{\overline{z}_1} \left(\overline{\pi}_{vs}(T) \frac{d}{dz} \log \rho - \frac{d}{dz} \overline{\pi}_{vs}(T) \right) dz. \tag{54}$$

By setting:

$$\alpha = \frac{c_0 \overline{z}_1}{gb} \left[\int_0^{\overline{z}_1} \left(\overline{\pi}_{vs}(T) \frac{d}{dz} \log \rho - \frac{d}{dz} \overline{\pi}_{vs}(T) \right) dz \right]^{-1}, \tag{55}$$

we will have constructed the solution (ρ, T, α) of problem (20)–(24). The uniqueness of the solution follows from the strict monotonicity of the function $\nu(c_0)$ and the uniqueness of the local solution of the system of equations (20), (48). Characterization (47) is also justified by Lemma 4. The Proposition 5.1 is proven. □

6 PROOF OF LEMMA 4

Lemma 5. *If the functions $\rho(z)$ and $T(z)$ satisfy equations (20) and (48) in the neighborhood of z, then*

$$\frac{dT}{dz} = D_1(\rho, T, c_0), \qquad \frac{d\rho}{dz} = D_2(\rho, T, c_0), \tag{56}$$

$$\frac{\partial D_1(\rho, T, c_0)}{\partial \rho} > 0 \qquad \textit{with} \quad T > \Theta_1, \tag{57}$$

where

$$D_1(\rho, T, c_0) = -\frac{\left(\frac{R_1 T + L_{tr}}{R_1 \rho T} \overline{\pi}_{vs}(T) + 1 \right) (g\rho + c_0)}{\left[\rho(c_v + R_1) + (R_1 T + L_{tr}) \left(\frac{\overline{\pi}_{vs}(T)}{T} + \frac{d\overline{\pi}_{vs}(T)}{dT} \right) \right]}, \tag{58}$$

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$$D_2(\rho,T,c_0) = -\frac{\left(\dfrac{R_1T+L_{tr}}{R_1\rho}\dfrac{d\overline{\pi}_{vs}(T)}{dT}+\dfrac{c_v}{R_1}\right)(g\rho+c_0)}{\left[T(c_v+R_1)+(R_1T+L_{tr})\left(\dfrac{\overline{\pi}_{vs}(T)}{\rho}+\dfrac{c_v}{R_1}\dfrac{d\overline{\pi}_{vs}(T)}{dT}\right)\right]}. \tag{59}$$

Proof. Since equations (20) and (48) are linear with respect to $\frac{dT}{dz}$ and $\frac{d\rho}{dz}$, by substituting

$$\frac{d\rho}{dz} = -\frac{\rho}{T}\frac{dT}{dz} - \frac{g\rho+c_0}{R_1T}$$

and

$$\frac{dT}{dz} = -\frac{T}{\rho}\frac{d\rho}{dz} - \frac{g\rho+c_0}{R_1\rho}$$

which are obtained from (48) into (20), we obtain (56) with (58) and (59).

We also have

$$\frac{\partial D_1(\rho,T,c_0)}{\partial\rho} = \frac{E(\rho,T,c_0)}{\left[\rho(c_v+R_1)+(R_1T+L_{tr})\left(\dfrac{\overline{\pi}_{vs}(T)}{T}+\dfrac{d\overline{\pi}_{vs}(T)}{dT}\right)\right]^2}, \tag{60}$$

where

$$\begin{aligned}E(\rho,T,c_0) = &\left(\frac{R_1T+L_{tr}}{R_1\rho T}\overline{\pi}_{vs}(T)+1\right)(g\rho+c_0)(c_v+R_1)\\ &+\left[\rho(c_v+R_1)+(R_1T+L_{tr})\left(\frac{\overline{\pi}_{vs}(T)}{T}+\frac{d\overline{\pi}_{vs}(T)}{dT}\right)\right]\\ &\times\left(\frac{R_1T+L_{tr}}{R_1\rho^2T}\overline{\pi}_{vs}(T)(g\rho+c_0)-g\left(\frac{R_1T+L_{tr}}{R_1\rho T}\overline{\pi}_{vs}(T)+1\right)\right).\end{aligned}$$

Now, since

$$\frac{R_1T+L_{tr}}{R_1\rho^2T}\overline{\pi}_{vs}(T)(g\rho+c_0)-g\left(\frac{R_1T+L_{tr}}{R_1\rho T}\overline{\pi}_{vs}(T)+1\right)\geq -g,$$

we have

$$\begin{aligned}E(\rho,T,c_0) &\geq \left(\frac{R_1T+L_{tr}}{R_1\rho T}\overline{\pi}_{vs}(T)+1\right)g\rho(c_v+R_1)-g\left[\rho(c_v+R_1)+(R_1T+L_{tr})\left(\frac{\overline{\pi}_{vs}(T)}{T}+\frac{d\overline{\pi}_{vs}(T)}{dT}\right)\right]\\ &= g(R_1T+L_{tr})\left[\frac{c_v+R_1}{R_1}\overline{\pi}_{vs}(T)-\frac{\overline{\pi}_{vs}(T)}{T}-\frac{d\overline{\pi}_{vs}(T)}{dT}\right],\end{aligned}$$

Therefore, considering condition (45), we obtain (57). □

Note on the Latent Heat L_{tr}: It is important to highlight that the factor $g(R_1T+L_{tr})$ remains strictly positive. This holds true regardless of whether L_{tr} is treated as a constant (as simplified in some analytic proofs) or as a temperature-dependent function $L_{tr}(T)$ (as used in the numerical scheme), provided that T remains within its physical domain. Crucially, the dependence of L_{tr} on T does not alter the sign of the derivative $\partial_\rho D_1$ which is essential for establishing the monotonicity of the solution and thus the uniqueness (Lemma 4).

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We recall the behavior of the solution in the neighborhood of $z = 0$.

Lemma 6. *If $0 \leq c^{(i)} < c^{(ii)}$, then there exists a number $\varepsilon_z > 0$ such that*

$$\rho^{(ii)}(z) < \rho^{(i)}(z), \quad T^{(ii)}(z) < T^{(i)}(z) \quad \forall z \in]0, \varepsilon_z[\,. \tag{61}$$

Proof . According to condition (23), we have

$$\rho^{(ii)}(0) = \rho^{(i)}(0) = \rho_0, \qquad T^{(ii)}(0) = T^{(i)}(0) = T_0.$$

On the other hand, from the expressions for $D_1(\rho, T, c_0)$ and $D_2(\rho, T, c_0)$ given by (58) and (59), we can see that

$$D_1(\rho_0, T_0, c^{(i)}) > D_1(\rho_0, T_0, c^{(ii)}), \qquad D_2(\rho_0, T_0, c^{(i)}) > D_2(\rho_0, T_0, c^{(ii)}),$$

which implies that (61) is satisfied in the neighborhood of $z = 0$. The lemma is proven.□

Now, we will prove Lemma 4.

Proof of Lemma 4 The functions $\rho(z)$ and $T(z)$ that satisfy equations (20) and (48) are decreasing; this follows immediately from (56), (58) and (59). Suppose that relation (50) is not satisfied. This is equivalent to assuming that there exists a $z > 0$ such that

$$\rho^{(ii)}(z)T^{(ii)}(z) \geq \rho^{(i)}(z)T^{(i)}(z).$$

This hypothesis implies that there exists a $z_1 > 0$ such that

$$\rho^{(ii)}(z_1)T^{(ii)}(z_1) = \rho^{(i)}(z_1)T^{(i)}(z_1), \qquad \rho^{(ii)}(z)T^{(ii)}(z) < \rho^{(i)}(z)T^{(i)}(z) \quad \forall z \in]0, z_1[\,, \tag{62}$$

$$\frac{d}{dz}(\rho^{(ii)}T^{(ii)})_{z=z_1} \geq \frac{d}{dz}(\rho^{(i)}T^{(i)})_{z=z_1}. \tag{63}$$

This implies that

$$\rho^{(ii)}(z_1) \leq \rho^{(i)}(z_1) - (c^{(ii)} - c^{(i)})$$

and according to (62) we conclude that

$$\rho^{(ii)}(z_1) < \rho^{(i)}(z_1), \qquad T^{(ii)}(z_1) > T^{(i)}(z_1). \tag{64}$$

Relations (64) and (62), in addition to Lemma 6, imply that there exists a $z_0 \in]0, z_1[$ such that

$$\rho^{(ii)}(z_0) < \rho^{(i)}(z_0), \qquad T^{(ii)}(z_0) = T^{(i)}(z_0), \qquad T^{(ii)}(z) < T^{(i)}(z) \quad \forall z \in]0, z_0[\,. \tag{65}$$

The last two relations give

$$\frac{d}{dz}T^{(ii)}(z) \geq \frac{d}{dz}T^{(i)}(z) \quad \text{at } z = z_0. \tag{66}$$

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By virtue of (57), the inequality $\rho^{(ii)}(z_0) < \rho^{(i)}(z_0)$ implies that inequality (66) cannot be satisfied. Therefore, our hypothesis of the existence of a $z > 0$ satisfying $\rho^{(ii)}(z)T^{(ii)}(z) \geq \rho^{(i)}(z)T^{(i)}(z)$ cannot be satisfied. Thus, inequality (50) is proven.

We have already proven that there is no $z_0 > 0$ for which

$$T^{(ii)}(z_0) = T^{(i)}(z_0), \qquad T^{(ii)}(z) < T^{(i)}(z) \quad \forall z \in]0, z_0[\,, \qquad \rho^{(ii)}(z_0) < \rho^{(i)}(z_0).$$

If

$$T^{(ii)}(z_0) = T^{(i)}(z_0), \qquad T^{(ii)}(z) < T^{(i)}(z) \quad \forall z \in]0, z_0[\,, \qquad \rho^{(ii)}(z_0) \geq \rho^{(i)}(z_0),$$

there would exist a $z' \in]0, z_0]$ such that

$$\rho^{(ii)}(z')T^{(ii)}(z') = \rho^{(i)}(z')T^{(i)}(z'),$$

which contradicts (50), thus establishing (49). Lemma 4 is proven. □

7 NUMERICAL METHOD FOR THE STATIONARY SOLUTION

This section details the numerical methodology employed to determine the stationary solution of the model, building upon the theoretical framework established in the preceding sections.

Robustness and Efficiency The numerical resolution of the parameter c_0 is performed using a numerical root-finding function, such as `fsolve` from Python's `scipy.optimize` library. As defined by the model's physics, $\mathbf{c_0}$ is directly related to the total stationary quantity of liquid/solid water $\mathbf{\Sigma}$ by the relation $\mathbf{c_0} = \mathbf{g\Sigma}$. This function seeks the root of an objective function, i.e., the value of c_0 for which the pressure condition at the top $(\rho(\overline{z}_1)T(\overline{z}_1) - q_0 = 0)$ is satisfied.

The root-finding algorithm efficiently converges to the required value of c_0. For example, for the partially humid state, the algorithm identifies a value of c_0 that makes the objective function very close to zero, ensuring precise satisfaction of the physical condition at the top boundary.

Interpretation of the Process: The convergence process is robust and rapid. The algorithm progressively adjusts the value of c_0 until the objective function is minimized to a value very close to zero. This ensures that the final calculated profiles of density and temperature satisfy the top boundary pressure condition with high precision.

This process illustrates the robustness of the numerical method used to determine the parameters of the stationary solution, ensuring a precise convergence to the imposed physical conditions.

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Description of the solveStationary Implementation The `solveStationary.py` file implements the methodology for finding the stationary solution of the model, as described in Proposition 5.1 of the article. The objective is to determine a set of stationary profiles for density $\rho(z)$, temperature $T(z)$, as well as constants α (intensity coefficient) and Σ (total quantity of liquid/solid water), satisfying the stationary equations and boundary conditions.

The key steps of the process, as coded, are as follows:

Initialization of constants and boundary conditions at $z = 0$ The code begins by defining the physical constants (`g`, `R1`, `cv`, `z_bar_1`, `b_phi`) as well as the initial conditions at the base of the "chimney" ($z = 0$): the initial density ρ_0 and the initial temperature T_0. These values serve as a starting point for the spatial integration towards $z = z_1$.

Automated calculation of the top boundary condition (q_0) A crucial aspect is the automation of the q_0 calculation, defined as $q_0 = \rho(z_1)T(z_1)$ for the external hydrostatic state. The code uses the `getHydrostaticProfiles.py` function, which models this state with $\vartheta = 2/3$ (partially humid case), by receiving the same initial values (ρ_0, T_0) as input.

This automatically provides the target value q_0^{target} (external top pressure at $z_1 = 10\,000\,\text{m}$), to which the internal solution must be adjusted.

Search for c_0 via a numerical root-finding function The resolution relies on finding the parameter c_0 (such that $c_0 = g\Sigma$) that satisfies the pressure condition at the top boundary. This condition ensures that the calculated product $\rho(\overline{z}_1)T(\overline{z}_1)$ matches the target value q_0, where $R_1\rho(\overline{z}_1)T(\overline{z}_1)$ represents the pressure $P(\overline{z}_1)$ of the internal solution. The stationary system (equations (20) and (48)) leads to the following formulation:

$$f(c_0) = \rho(\overline{z}_1)T(\overline{z}_1) - q_0 = 0.$$

A numerical root-finding function is used to solve this equation. It relies on an objective function, `get_rhoT_at_z1.py`, which calculates $\rho(\overline{z}_1)T(\overline{z}_1)$ for each proposed c_0 value. This process numerically implements the theoretical relation $c_0 = \nu^{-1}(q_0)$ by identifying the c_0 value that makes $\nu(c_0)$ equal to the target q_0.

Resolution of solveSpatialProfileshelper The `solveSpatialProfiles_helper.py` function performs an explicit numerical integration (Euler type) of the system equations from $z = 0$ to $z = \overline{z}_1$, for a given c_0 value. At each spatial step, a 2×2 linear system is solved (matrix M and vector V derived from equations (20) and (48)) to obtain $\frac{dT}{dz}$ and $\frac{d\rho}{dz}$, and update the profiles.

This function also calculates the integral necessary for the evaluation of α, namely:

$$\int_0^{\overline{z}_1} \left(\overline{\pi}_{\text{vs}}(T) \frac{d}{dz} \log \rho - \frac{d}{dz} \overline{\pi}_{\text{vs}}(T) \right) dz.$$

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Robustness tests are integrated to detect singular matrices, negative densities, or non-physical values (`NaN`, `Inf`), and to interrupt the procedure with a failure signal for the root-finding function.

Calculation of stationary Σ and α Once c_0 is determined by the root-finding function, the stationary value of Σ is obtained by the defining relation $\mathbf{c_0 = g\Sigma}$:

$$\Sigma^{\mathrm{stat}} = \frac{c_0}{g}.$$

The value of α is then calculated from equation (55):

$$\alpha^{\mathrm{stat}} = \frac{\Sigma^{\mathrm{stat}}}{\frac{b}{\bar{z}_1}\int_0^{\bar{z}_1}\left(\overline{\pi}_{\mathrm{vs}}(T)\dfrac{d}{dz}\log\rho - \dfrac{d}{dz}\overline{\pi}_{\mathrm{vs}}(T)\right)dz}.$$

This final step uses the Σ^{stat} value derived from c_0 and the integral evaluated from the (ρ, T) profiles obtained with that c_0.

Advantages of the Approach This method provides a stationary solution in a robust and automated manner. The use of a numerical root-finding function ensures precise satisfaction of the pressure condition at the top. The automation of q_0 calculation, as well as the modularity of the functions (`getHydrostaticProfiles`, `solveSpatialProfiles_helper`), allows flexible exploration of different initial conditions, facilitating model validation.

8 NUMERICAL EXAMPLES

This section presents examples of numerical solutions obtained for the systems of equations studied in this article. We first illustrate the results for the stationary solution, whose existence and uniqueness have been established in the preceding sections. The numerical implementation of these models can benefit from specialized scientific computing software packages (Petrov et al., 2018). Then, we revisit the full evolutionary model to highlight the observed oscillatory behaviors, providing insight into the temporal dynamics of the system.

Numerical Results for the Stationary Solution We present examples of numerical solutions for the system (20)–(22), whose existence and uniqueness have been established. For the physical coefficients, we use: $g = 9.8\,\mathrm{m/s^2}$, $R_0 = 8.314\,\mathrm{J/(mol\,K)}$, $\mu_a = 28.96\,\mathrm{g/mol}$, and the temperature-dependent latent heat $L_{tr}(T) = (3244 - 2.72\,T) \times 10^3$ J/kg. We recall $R_1 = R_0/\mu_a$, $c_v = \frac{5}{2}R_1$, and the expression for $\overline{\pi}_{vs}(T)$ specified in (25). The value $\mathbf{c_v = \frac{5}{2}R_1}$ is consistent with the standard value for a diatomic gas.

Note on the Latent Heat L_{tr}: The analytic proofs (notably Lemma 4.3 and the existence proof) treat the latent heat L_{tr} as a constant parameter for simplification,

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which is sufficient to establish the necessary monotonicity and uniqueness. However, in the numerical implementation, we utilize the temperature-dependent expression $L_{tr}(T)$, as defined above, for enhanced physical realism and precision in the computed profiles. This approach ensures the robustness of the theoretical proof while maintaining the accuracy of the numerical results.

We set $\overline{z}_1 = 10\,000\,\mathrm{m}$, $\rho_0 = 1.2055\,\mathrm{kg/m^3}$ and $T_0 = 293\,\mathrm{K}$. Two cases are analyzed: a dry state ($\vartheta = 0$) with $q_0 = 85.51\,\mathrm{kg\,K/m^3}$ and a moist state ($\vartheta = 2/3$) with $q_0 = 94.08\,\mathrm{kg\,K/m^3}$. For these cases, the values of Σ obtained are respectively $\Sigma = 0.0562\,\mathrm{kg/m^3}$ and $\Sigma = 0.0153\,\mathrm{kg/m^3}$. Each q_0 value corresponds to the product $\rho(\overline{z}_1) \cdot T(\overline{z}_1)$ derived from the solution of the hydrostatic state problem (26)–(28). The distribution of $\rho(z)$ and $T(z)$ is consistent with natural observations. The temperature profile for the humid case deviates significantly from the dry case due to latent heat release, while density profiles remain nearly identical as the atmosphere stays in hydrostatic equilibrium. Figure 2 illustrates the continuity and strict decrease of the function $\nu(\Sigma)$ defined in (52), which is crucial for the existence proof.

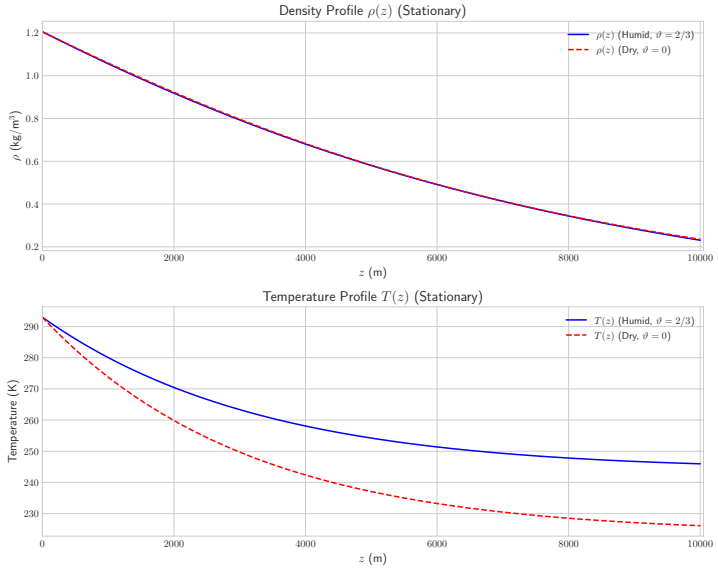

Figure 1: Distribution of density and temperature for the stationary solution (dry case, $\vartheta = 0$ and humid case, $\vartheta = 2/3$).

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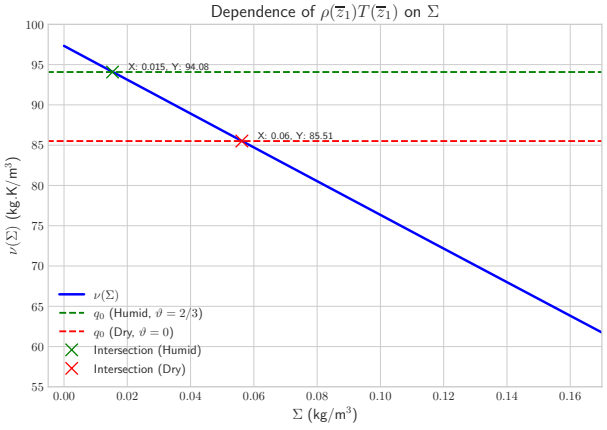

Figure 2: Dependence of $\rho(\overline{z}_1)T(\overline{z}_1)$ as a function of Σ.

Numerical Results of the Evolutionary Model: Oscillations As discussed in the motivation, the stationary solution's existence is vital when the evolutionary system (10)–(13) exhibits oscillations. Following the finite difference method detailed in (Bourega et al., 2018), we illustrate a damped oscillatory behavior using $b = 450$, $\vartheta = 2/3$, and discretization steps $h_t = 10\,\mathrm{s}$ and $h_z = 10\,\mathrm{m}$. This value of b, representing the mean droplet residence time, ensures the numerical stability and the damping of the oscillations toward the stationary state.

Figure 3 presents the results for $\alpha(t)$ and $\Sigma(t)$ with $\rho_0 = 1.2055\,\mathrm{kg/m^3}$, $T_0 = 293\,\mathrm{K}$, and $\overline{z}_1 = 10\,000\,\mathrm{m}$, showing damped oscillations over 7400 s. The strong correlation between $\alpha(t)$ and $\Sigma(t)$ suggests a cyclical mechanism: ascending humid air produces water, which accumulates and slows the ascent; the resulting decrease in water production then allows the air to resume a faster ascent.

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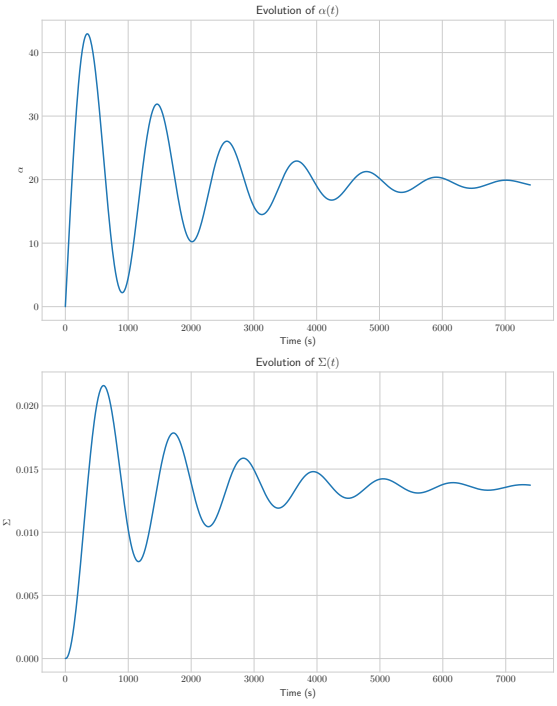

Figure 3: Damped oscillations of $\alpha(t)$ and $\Sigma(t)$.

A significant finding is that $\Sigma(t)$ and $\alpha(t)$ stabilize at constant values remarkably consistent with the humid stationary solution ($\vartheta = 2/3$). This agreement provides a robust validation: the variables converge precisely toward $\Sigma_{\text{stat}} \approx 0.0153$ and $\alpha_{\text{stat}} \approx 32.87$ (Table 1). Furthermore, the comparison highlights the role of latent heat: in the dry case, the coupling parameter α_{stat} must be significantly higher (≈ 120.03) to sustain the convective cell, whereas moist processes facilitate upward motion with a lower forcing threshold.

Table 1: Summary of Stationary Solution Parameters.

Case	c_0	Σ_{stat} **(kg/m^3)**	α_{stat}
Humid ($\vartheta = 2/3$)	0.150241	0.015331	32.872555
Dry ($\vartheta = 0$)	0.550882	0.056212	120.034341

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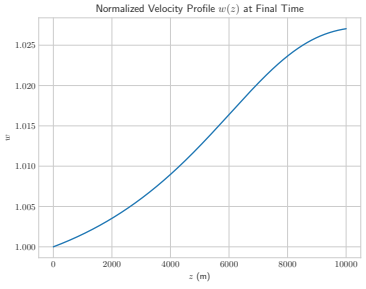
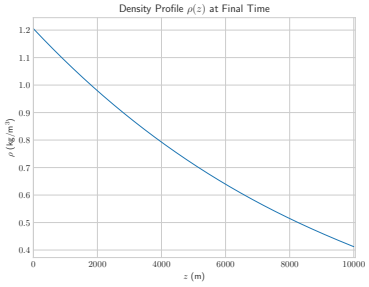
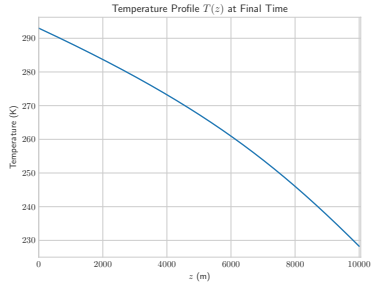


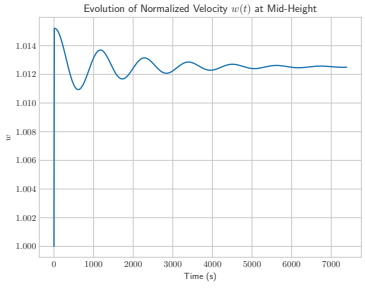
(a) Normalized velocity profile $w(z)$ at final time.



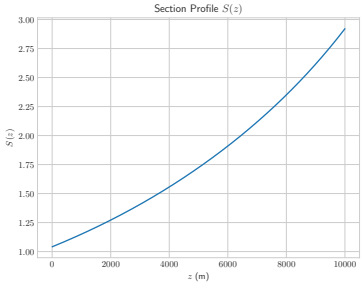
(b) Density profile $\rho(z)$ at final time of the evolutionary simulation. This shows the convergence of the evolutionary model to a stable density distribution, consistent with the humid stationary state.



(c) Temperature profile $T(z)$ at final time of the evolutionary simulation. This profile demonstrates the convergence of the evolutionary model towards a stable state.



(d) Evolution of normalized velocity $w(t)$ at mid-height.



(e) Section profile $S(z)$.

Figure 4: Results of the evolutionary simulation of the model. These figures complement the oscillation analysis presented in Figure 3.

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9 OSCILLATORY DYNAMICS: THE VOLTERRA MODEL

As indicated in the introduction, the oscillatory behavior observed in our mathematical thunderstorm model, particularly for the variables $\alpha(t)$ and $\Sigma(t)$, exhibits striking analogies with the solutions of Volterra integro-differential equations. These equations are fundamental for describing dynamic systems that incorporate "memory" effects, where the future state depends not only on the current state but also on the entire past history of the system. The phenomenon of oscillation in these equations is a well-established area of mathematical literature, as noted by Volterra himself in his pioneering work on population dynamics (Volterra, 1931), and further developed by more recent studies such as those by Gopalsamy (Gopalsamy, 1987).

To explore this analogy, we consider the following Volterra integro-differential equation, which is a specific form relevant to our comparison:

$$y'(t) = -a \int_0^t K(t-s)(y(s) - c)ds + d, \tag{67}$$

where $y(t)$ is the state variable, and $K(\tau) = \exp\left(-\frac{\pi\tau^2}{4b^2}\right)$ is the convolution kernel (a positive Gaussian function that decays rapidly, modeling the influence of past events). The parameters a, b, c, and d are positive constants. The equation is complemented by an initial condition $y(0) = \eta$.

To facilitate a qualitative comparison with the $\Sigma(t)$ variable of our thunderstorm model, which represents a physical quantity that must always be positive, we define a function $\Psi(t)$ as the integral of the product of the kernel $K(t-s)$ by the solution $y(s)$, modulated by the coefficient a:

$$\Psi(t) = a \int_0^t K(t-s)y(s)ds. \tag{68}$$

Since $a > 0$, $K(\tau) > 0$, and we expect $y(t)$ to remain positive (like $\alpha(t)$ in our model), this function $\Psi(t)$ is designed to always be positive, thus reflecting the behavior of $\Sigma(t)$.

Equation (67) can then be rewritten to highlight the relationship between $y'(t)$, $\Psi(t)$, and the other terms:

$$y'(t) = -a \int_0^t K(t-s)y(s)ds + ac \int_0^t K(t-s)ds + d$$
$$y'(t) = -\Psi(t) + \left(ac \int_0^t K(t-s)ds + d\right).$$

The term in parentheses, $f(t) = ac \int_0^t K(t-s)ds + d$, acts as a forcing function.

Challenges of Numerical Simulation The numerical solution of integro-differential equations like (67) is not trivial. Unlike classical ordinary differential equations where the derivative depends only on the current state, here, the derivative $y'(t)$ depends on an integral that accumulates the influence of all past values of $y(s)$ up to time t. This

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requires an iterative approach where, at each time step, the integral must be recalculated by summing over all previous time points. We used a numerical method based on temporal discretization (Euler's method for the ODE) and a Riemann sum approximation for the integral term, which demands careful management of temporal dependencies.

Simulation Results and Theoretical Justification For our simulation, we used parameters chosen to reinforce the analogy with our evolutionary storm model, specifically setting the memory parameter to $b = 450$. The values used are:

$$a = 3 \times 10^{-5}, \qquad b = 450, \qquad c = 1.2, \qquad d = 1 \times 10^{-5}, \qquad \eta = 0.01.$$

The simulation was conducted over a duration of 7200 s to match the timescale of the physical model. This choice of parameters allows for a precise physical analogy:

- Coupling (a): Represents the weight or negative buoyancy effect, dictating how strongly the accumulated water Σ slows down the air intensity α.
- Memory (b): Represents the mean droplet residence time, governing the time delay between condensation and the mechanical braking effect.
- Target (c): Corresponds to the stationary state α_{stat} toward which the system naturally gravitates.
- Forcing (d): Represents the thermal instability (the energy engine) that drives the convective motion.
- Initial Pulse (η): Reflects the initial updraft intensity $\alpha(0)$ required to trigger the cycle.

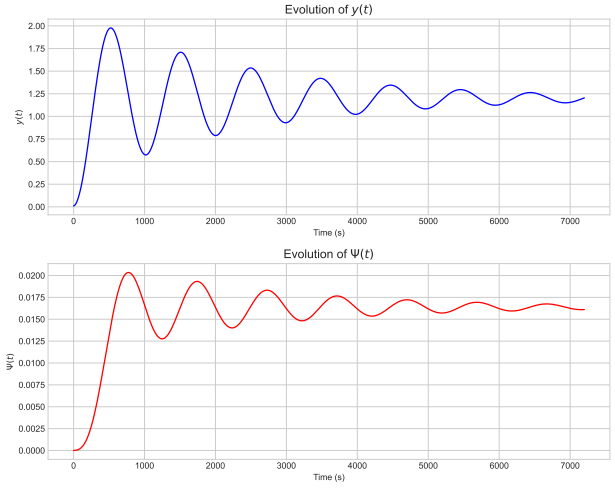

Figure 5: Damped oscillations of the Volterra equation solution. Upper panel: Evolution of $y(t)$. Lower panel: Evolution of $\Psi(t)$.

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We clearly observe that $y(t)$ exhibits damped oscillations that converge towards an equilibrium value, mirroring the behavior of $\alpha(t)$. The function $\Psi(t)$ remains always positive and also displays damped oscillations, consistent with the physical nature of $\Sigma(t)$. This dynamic correlation–where high $y(t)$ leads to an increase in $\Psi(t)$ –illustrates the feedback mechanism where past states influence future dynamics via the Gaussian kernel $K(\tau)$.

The works of Gopalsamy (Gopalsamy, 1987) and others provide conditions for the existence of oscillations in integro-differential equations. While our specific formulation (67) with positive parameters may not satisfy every strict constraint of existing theorems (such as integrability over an infinite interval), the qualitative behavior observed is fully consistent with the expected dynamics for systems with memory effects and restoring forces.

In conclusion, this simulation of the Volterra equation serves as a powerful mathematical analogy for understanding the damped oscillations of our thunderstorm model. It highlights that systems with memory and feedback mechanisms naturally generate such dynamics. The synchronized qualitative analogy between (y, Ψ) and (α, Σ), both governed by a similar residence time parameter $b = 450$, strengthens the understanding of the complex physical phenomena at play.

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10 CONCLUSION

The complete evolutionary system (10)–(14), previously studied in (Bourega et al., 2018), describes complex atmospheric phenomena driven by ascending air flow and condensation. This article has significantly advanced our understanding by focusing on the stationary solution of a simplified version of this system.

The main contributions of this work are as follows:

1. We have successfully demonstrated the existence and uniqueness of the stationary solution for the system (20)–(24) under the technical condition $T > \Theta_1$. This theoretical result provides a rigorous mathematical basis for the long-term behavior of such atmospheric models.

2. Numerical examples illustrate the validity of these results, showing that the moist case ($\vartheta = 2/3$) yields higher temperatures than the dry case due to latent heat release, while maintaining consistent density distributions in hydrostatic equilibrium.

3. The analysis of the full evolutionary model revealed damped oscillatory behaviors of the intensity coefficient $\alpha(t)$ and liquid water content $\Sigma(t)$. The fact that these variables stabilize at the exact values predicted by the stationary solution provides a robust validation of our analytical framework.

4. The qualitative analogy with the Volterra integro-differential equation, using the consistent residence time parameter $b = 450$, highlighted striking mathematical similarities. This suggests that the oscillations are a self-regulating cycle driven by memory effects and restoring forces.

In conclusion, this study contributes to the mathematical modeling of atmospheric processes by providing both a rigorous analytical framework for stationary states and deeper insights into oscillatory dynamics. Future work could involve extending the existence and uniqueness proofs to more general conditions and exploring the implications of these oscillatory behaviors for real-world weather events.

DATA AVAILABILITY

No external or experimental data sets were used in this theoretical study. The numerical simulations presented in Sections 7, 8, and 9 were generated using custom Python scripts. The code is available from the corresponding author upon reasonable request.

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COMPLIANCE WITH ETHICAL STANDARDS

This article does not contain any studies with human participants or animals performed by the author.

AUTHOR CONTRIBUTIONS

D. R. Bourega designed the study, developed the mathematical proofs, performed the numerical simulations, and wrote the manuscript.

COMPETING INTERESTS

The author declares that there are no competing interests.

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