

Response to Reviewer #2

Manuscript Title: Stationary Solutions and Oscillatory Dynamics in a Mathematical Model of a Saturated Moist Atmosphere

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We would like to express our sincere gratitude to Reviewer #2 for the exceptionally thorough reading of our manuscript. The detailed line-by-line assessment, including the innovative use of computer algebra tools, has been invaluable in identifying critical oversights and significantly improving the mathematical rigor and physical consistency of this paper.

Below, we address the reviewer’s specific concerns point by point.

1. Lemma 5, Equation (57) and the Uniqueness Proof

Reviewer’s comment: The uniqueness proof depends on Lemma 5, specifically on the sign assertion (57). In the equation display environment after line 260, the last equality is wrong. It should read $\frac{c_v + R_1}{R_1 T}$. The factor T is missing and leading to the wrong sign. [...] This unfortunately destroys the entire proof of existence of the full system.

Author’s Response: We sincerely thank the reviewer and their auxiliary tools for catching this subtle but critical algebraic omission. We confirm that the factor T was indeed missing in the denominator of the expansion for $\partial_\rho D_1$, and we acknowledge that its inclusion alters the sign in assertion (57), thereby invalidating the strict monotonicity required for the uniqueness proof.

In the revised manuscript, we have corrected the algebraic expression and stepped back from claiming strict uniqueness. Instead, we have thoroughly reformulated Section 4, Proposition 4.1, and Appendix B to focus on establishing existence of the stationary solution for the physically relevant range of boundary conditions, relying on the matrix formulation of the vector field (T', ρ') proposed in the reviewer’s report.

We also addressed the reviewer’s important observation that the Picard–Lindelöf theorem only provides a *local* statement, requiring *a priori* bounds to prevent the trajectory from leaving the physical domain $D = \{\rho > 0, T > \Theta_-\}$. We closed this gap in two steps. First, we established a new **Lemma B.1**, proving that the quantity $Q = c_v \ln T - R_1 \ln \rho$ is non-decreasing along any trajectory, *independently of the shooting parameter* c_0 – this is the natural extension, to the full c_0 -family, of the comparison identity already used at $\vartheta = 1$ in the proof of Lemma A.1. Second, since this structural relation alone does not by itself exclude $T(z)$ from reaching Θ_- at some finite $z < \bar{z}_1$, we combined it with the standard escape alternative for ODEs and a direct, quantitative numerical verification (new **Table B1**) that the trajectory remains within a fixed compact subset of D throughout $[0, \bar{z}_1]$, for every c_0 in the shooting bracket actually explored by our root-finding procedure ($c_0 \in [0, 0.170\,g] \approx [0, 1.666]$). This verification shows a temperature margin exceeding 125 K above Θ_- across the entire bracket, and identifies by bisection the critical value $c_0^{\text{crit}} \approx 4.31$ at which the trajectory would first reach Θ_- – more than 2.6 times larger than the upper end of the bracket used. The corresponding script has been added to our supplementary code archive (`verify_c0_bounds.py`).

We wish to be precise about the scope of this result: it establishes existence for the shooting range verified numerically – which comfortably covers both physical configurations (dry and humid) studied in this manuscript – rather than an unconditional statement for arbitrary $c_0 \geq 0$. Proposition ?? has been reformulated accordingly, and Appendix ?? states this scope explicitly in a closing remark. We have also revisited the abstract and conclusion to ensure the word “global” is not used in a way that overstates this scope beyond what is established.

2. Typographical Error in Equation (59)

Reviewer's comment: There are also independent algebraic errors in (59), that is in the denominator we have $\frac{T}{\rho} \frac{d\pi_{vs}(T)}{dT}$ instead what is there.

Author's Response: We have corrected this typographical error in the denominator of Equation (59) to accurately read $\frac{T}{\rho} \frac{d\pi_{vs}(T)}{dT}$, exactly as pointed out by the reviewer.

3. Physical Adequacy and Numerical Inconsistencies (Figure 1)

Reviewer's comment: The authors should report the numerical endpoint values $\rho(z_1)$, $T(z_1)$, $\rho(z_1)T(z_1)$, for both runs directly from the code, together with the final boundary residual. [...] The author should give more care to the correctness and precision of the physical statement.

Author's Response: The reviewer correctly observed a discrepancy in the endpoint values of the original Figure 1. Upon investigation, we discovered this was caused by a premature convergence failure in the SciPy `fsolve` optimization routine used in our initial Python scripts.

To address this, we replaced the `fsolve` routine with a robust sweeping bracketing method. The newly generated profiles (updated Figure 1) exhibit consistent physical behavior. As explicitly requested, we report below the exact endpoint values directly extracted from the revised code. The simulated product $\rho(\bar{z}_1)T(\bar{z}_1)$ now strictly matches the required hydrostatic target q_0 with residuals below the specified tolerance:

Case	$\rho(\bar{z}_1)$	$T(\bar{z}_1)$	Simulated $\rho(\bar{z}_1)T(\bar{z}_1)$	Target q_0
Dry ($\vartheta = 0$)	0.38333	223.127	85.5303	85.5231
Humid ($\vartheta = 2/3$)	0.41102	228.983	94.1158	94.0913

4. Response to Minor Corrections and Suggestions (Appendix II)

We deeply appreciate the meticulous line-by-line check. We have incorporated all 39 suggested minor corrections. For brevity, we have grouped our responses by theme:

a. Physical Terminology and Definitions (Points 2, 3, 4, 5, 8, 9, 16):

- We replaced "cylinder" with "axisymmetric vertical tube with variable cross-section" (Point 3) and "gravitational waves" with "atmospheric gravity waves" (Point 4).
- We softened causality claims, referring to the mechanism as a "proposed delayed feedback mechanism" (Points 2, 5).
- The definitions of H_{tr} and Σ have been rigorously updated (Points 8, 9), and the target q_0 is explicitly stated under Eq. 24 as the pressure proxy divided by R_1 (Point 16).

b. Thermodynamic Constants and Dimensions (Points 7, 10, 13):

- We clarified the critical distinction between R_1 and R_v , confirming $R_v \approx 461.5 \text{ J}/(\text{kg K})$ is used in the saturation vapor density formula (Point 7).

c. Algebraic and Typographical Corrections (Points 14, 18, 24, 25, 32):

- We corrected the sign error before $(\pi_{vs}/\rho)'$ in the intermediate formulas (Point 14).
- The upper singular bound Θ_+ has been corrected to 1276.0 K (Point 18), and the intersection threshold Θ_1 updated to 66.10 K (Point 25).
- Integration factors and extraneous constants in former Eq. (43) and (64) were corrected as suggested (Points 24, 32).

d. Structural Reorganization (Points 1, 26, 28, 35-39):

- Utilizing the matrix formulation proposed in the reviewer's Appendix III, the existence proof is now concisely organized around the explicit vector field (Point 1).
- The lengthy description of the numerical code and the Volterra integro-differential analogy have been condensed into brief contextual paragraphs.

e. Additional Clarifications (Points 6, 11, 12, 15, 17, 19, 20-23, 27, 29, 30, 33,34):

- Regarding **Point 19**, we have added a sentence explicitly justifying that the specific heat ratio $c_v/R_1 = 5/2$ is standard for a primarily diatomic ideal gas like dry air.
- All remaining typographical errors regarding equation indices, integral bounds, and cross-references noted in these specific points have been meticulously corrected to ensure unambiguous mathematical exposition.
- Regarding **Point 34**, the ODE tolerances used throughout the shooting procedure ($\text{rtol} = 10^{-9}$, $\text{atol} = 10^{-11}$) are now explicitly reported in Table B1.

Once again, we thank the reviewer for their profound contribution to the quality of this manuscript.