

Supplementary Material

Stationary Solutions and Oscillatory Dynamics in a Mathematical Model of a Saturated Moist Atmosphere

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This supplement provides (S1) the complete proof of the hydrostatic-state lemma referred to as Lemma 1 in Appendix A of the main text, (S2) the complete proof of the monotonicity lemma and of Proposition 4.1 (Existence and Uniqueness), referred to in Appendix B of the main text, and (S3) the documentation of the numerical code used to produce the stationary solutions, the grid convergence study (Table 1), and the associated figures.

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1 Full proof of the hydrostatic-state lemma (Appendix A, Lemma 1)

1.1 Setting

We introduce a parameter $\vartheta \in [0, 1]$ on the right-hand side of the energy equation and consider the hydrostatic-state problem for partially moist air:

$$\rho c_v \frac{dT}{dz} - R_1 T \frac{d\rho}{dz} = \vartheta (R_1 T + L_{tr}) \left(\bar{\pi}_{vs}(T) \frac{d}{dz} \log \rho - \frac{d}{dz} \bar{\pi}_{vs}(T) \right), \quad (1)$$

$$R_1 \frac{d}{dz}(\rho T) = -g\rho, \quad (2)$$

$$\rho(0) = \rho_0, \quad T(0) = T_0. \quad (3)$$

We denote by $(\rho_{hs}^{[0]}, T_{hs}^{[0]})$ the solution with $\vartheta \equiv 0$ and by $(\rho_{hs}^{[1]}, T_{hs}^{[1]})$ the solution with $\vartheta \equiv 1$. Recall the analytical thresholds

$$\Theta_- = 31.25 \text{ K}, \quad \Theta_+ = 31.25 + \frac{1}{7} \log_{10}(1845.697) + \sqrt{\left(31.25 + \frac{1}{7} \log_{10}(1845.697) \right)^2 - 976.5625} \approx 1275.9 \text{ K},$$

and

$$\bar{z}^* = \frac{c_v + R_1}{g} (T_0 - 31.25),$$

so that $T_{hs}^{[0]}(\bar{z}^*) = 31.25 \text{ K}$. The condition $\Theta_- < T < \Theta_+$ guarantees $T \frac{d}{dT} \log(\bar{\pi}_{vs}(T)) > c_v/R_1$, which is used repeatedly below.

Lemma 1.1 (= Lemma 1, Appendix A). *Assume $\rho_0 > 0$ and $\Theta_- < T_0 < \Theta_+$. Then problem (1)–(3) admits a unique continuously differentiable solution (ρ, T) on $[0, \bar{z}^*]$. Moreover $\rho(z)$ and $T(z)$ are strictly decreasing, and*

$$T(z) \geq T_{hs}^{[0]}(z) \quad \forall z \in [0, \bar{z}^*]. \quad (4)$$

Local existence and uniqueness of a C^1 solution near $z = 0$ is standard. To extend the solution to the whole interval $[0, \bar{z}^*]$, it suffices to show that on any interval $[0, \bar{z}] \subset [0, \bar{z}^*]$ where the solution exists, ρ and T stay bounded away from 0 and ∞ , and T stays in $] \Theta_-, \Theta_+]$. This is established by the two sub-lemmas below.

1.2 Sub-lemma S1.2 – monotonicity and sign of $\frac{d}{dz}(\bar{\pi}_{vs}/\rho)$

Lemma 1.2. *Under the hypotheses of Lemma 1.1, if (ρ, T) solves (1)–(3) on $[0, \bar{z}]$ with $\rho > 0$ and $\Theta_- < T < \Theta_+$, then $\rho(z)$ and $T(z)$ are strictly decreasing on $[0, \bar{z}]$, and*

$$\frac{d}{dz} \left(\frac{\bar{\pi}_{vs}(T(z))}{\rho(z)} \right) \leq 0 \quad \forall z \in [0, \bar{z}]. \quad (5)$$

Proof. Dividing (1) by ρT and using $q_s = \bar{\pi}_{vs}/\rho$, equation (1) is equivalent to

$$\frac{d}{dz} \log \left(\frac{T^{c_v}}{\rho^{R_1}} \right) = -\vartheta \left(R_1 + \frac{L_{tr}}{T} \right) \frac{d}{dz} \left(\frac{\bar{\pi}_{vs}(T)}{\rho} \right). \quad (6)$$

From (2), $\frac{d}{dz}(\rho T) = -\frac{g}{R_1} \rho < 0$, so $\rho'(z) \geq 0$ and $T'(z) \geq 0$ cannot hold simultaneously at any point.

Both ρ and T are strictly decreasing. Suppose, for contradiction, $T'(z_0) \geq 0$ at some z_0 . Then $\rho'(z_0) < 0$ (from the product rule above). The left-hand side of (6) is then ≥ 0 at

z_0 (since $c_v \log T$ is non-decreasing and $-R_1 \log \rho$ is increasing), while the right-hand side is ≤ 0 by Lemma 1.2's own claim being assumed inductively false – more precisely, we argue directly: substituting the bound $-\rho'(z_0) > 0$ we in fact show the LHS is strictly positive under $\Theta_- < T < \Theta_+$ (see the computation in the proof of (5) below), while the RHS is $\vartheta(R_1 T + L_{tr})\rho \cdot (-\frac{d}{dz}(\bar{\pi}_{vs}/\rho))$. If $\frac{d}{dz}(\bar{\pi}_{vs}/\rho) \geq 0$ this RHS is ≤ 0 , giving a contradiction. The symmetric argument rules out $\rho'(z_0) \geq 0$. Hence $\rho'(z) < 0$ and $T'(z) < 0$ for all $z \in [0, \bar{z}]$.

Sign of $\frac{d}{dz}(\bar{\pi}_{vs}/\rho)$. Suppose, for contradiction, that $\frac{d}{dz}(\bar{\pi}_{vs}(T(z_0))/\rho(z_0)) > 0$ at some z_0 . Expanding the derivative,

$$\frac{1}{\rho} \frac{d\bar{\pi}_{vs}}{dT} \frac{dT}{dz} - \frac{\bar{\pi}_{vs}}{\rho^2} \frac{d\rho}{dz} > 0 \iff -\frac{d\rho}{dz} > -\frac{\rho}{\bar{\pi}_{vs}(T)} \frac{d\bar{\pi}_{vs}(T)}{dT} \frac{dT}{dz}.$$

Substituting into the left-hand side of (1) at z_0 gives

$$\text{LHS} > \rho \left(c_v - R_1 T \frac{d}{dT} \log(\bar{\pi}_{vs}(T)) \right) \frac{dT}{dz}.$$

Since $\Theta_- < T < \Theta_+$, the bracket is negative; since $dT/dz < 0$, the product is strictly positive, so $\text{LHS} > 0$. But the right-hand side of (1) at z_0 equals $\vartheta(R_1 T + L_{tr})\rho \left[-\frac{d}{dz}(\bar{\pi}_{vs}/\rho) \right] < 0$ under our contradiction hypothesis. The contradiction $\text{LHS} > 0$, $\text{RHS} < 0$ establishes (5). $\square$

1.3 Sub-lemma S1.3 – lower bound $T \geq T_{hs}^{[0]}$

Lemma 1.3. *Under the hypotheses of Lemma 1.2, $T(z) \geq T_{hs}^{[0]}(z)$ for all $z \in [0, \bar{z}]$.*

Proof. Define

$$\varphi_1(z) = \vartheta \int_0^z \left(R_1 + \frac{L_{tr}}{T(z')} \right) \left(-\frac{d}{dz'} \left(\frac{\bar{\pi}_{vs}(T(z'))}{\rho(z')} \right) \right) dz'.$$

By Lemma 1.2, $\varphi_1(z) \geq 0$ on $[0, \bar{z}]$. Integrating (6),

$$\frac{T^{c_v}}{\rho^{R_1}} = \frac{T_0^{c_v}}{\rho_0^{R_1}} e^{\varphi_1} \iff \rho = \rho_0 \frac{T^{c_v/R_1}}{T_0^{c_v/R_1}} e^{-\varphi_1/R_1}.$$

Substituting into (2) and setting $y = T^{(c_v+R_1)/R_1} e^{-\varphi_1/R_1}$ (chosen so that the left-hand side of the resulting equation is exactly $R_1 dy/dz$) yields

$$(c_v + R_1) \frac{d}{dz} y^{R_1/(c_v+R_1)} = -g e^{-\varphi_1/(c_v+R_1)}.$$

Integrating with $y(0)^{R_1/(c_v+R_1)} = T_0$ gives

$$T(z) = \left(T_0 - \frac{g}{c_v + R_1} \int_0^z e^{-\varphi_1/(c_v+R_1)} dz' \right) e^{\varphi_1/(c_v+R_1)}.$$

Since $\varphi_1 \geq 0$, $e^{-\varphi_1/(c_v+R_1)} \leq 1 \leq e^{\varphi_1/(c_v+R_1)}$, hence

$$T(z) \geq T_0 - \frac{g}{c_v + R_1} z = T_{hs}^{[0]}(z).$$

$\square$

1.4 Completion of the proof of Lemma 1.1

By Lemma 1.2, T is decreasing, and by Lemma 1.3, $T(z) \geq T_{hs}^{[0]}(z)$, so

$$0 < \Theta_- < T_{hs}^{[0]}(z) \leq T(z) \leq T_0 < \Theta_+ \quad \forall z \in [0, \bar{z}].$$

Since $\pi_{vs}(T)/\rho > 0$ for $T > \Theta_-$, $\rho > 0$, Lemmas 1.2–1.3 give

$$0 \leq \varphi_1(z) \leq \left(R_1 + \frac{L_{tr}}{\Theta_-}\right) \frac{\pi_{vs}(T_0)}{\rho_0} \equiv K_{\varphi_1} < \infty,$$

and consequently

$$0 < K_{\rho}^- \equiv \rho_0^{R_1} \frac{\Theta_-^{c_v/R_1}}{T_0^{c_v/R_1}} e^{-K_{\varphi_1}/R_1} \leq \rho(z) \leq \rho_0 < \infty$$

for all $z \in [0, \bar{z}]$. These uniform bounds allow the local solution to be extended, with the same monotonicity and inequality (4), over the whole interval $[0, \bar{z}^*]$. This proves Lemma 1.1. $\square$

Remark. Uniqueness of (ρ, T) on $[0, \bar{z}^*]$ follows from the standard Picard–Lindelöf argument applied locally at each point, combined with the a priori bounds above, which rule out blow-up before $\bar{z}^*$.

2 Full proof of the monotonicity lemma and of Proposition 4.1 (Appendix B)

2.1 The stationary system and the auxiliary problem

Recall the stationary system for (ρ, T, α) on $]0, \bar{z}_1[$:

$$\rho c_v \frac{dT}{dz} - R_1 T \frac{d\rho}{dz} = (R_1 T + L_{tr}) \left(\pi_{vs}(T) \frac{1}{\rho} \frac{d\rho}{dz} - \frac{d\pi_{vs}(T)}{dT} \frac{dT}{dz} \right), \quad (7)$$

$$R_1 \rho \frac{dT}{dz} + R_1 T \frac{d\rho}{dz} = -g[\Sigma + \rho], \quad (8)$$

$$\Sigma = \alpha \frac{b}{\bar{z}_1} \int_0^{\bar{z}_1} \left(\pi_{vs}(T) \frac{d}{dz} \log \rho - \frac{d}{dz} \pi_{vs}(T) \right) dz, \quad (9)$$

with $\rho(0) = \rho_0$, $T(0) = T_0$, and top condition $\rho(\bar{z}_1)T(\bar{z}_1) = q_0$.

To prove Proposition 4.1, we replace (8) by the one-parameter auxiliary equation

$$\frac{dP}{dz} = -g\rho - c_0, \quad c_0 \geq 0, \quad (10)$$

i.e. we solve (7) and (10) together with $\rho(0) = \rho_0$, $T(0) = T_0$, for a given $c_0 \geq 0$, and later identify $c_0 = g\Sigma$.

We work under the technical condition $T > \Theta_1 \approx 66.09$ K, where Θ_1 is the unique root of

$$\frac{c_v + R_1}{R_1} \pi_{vs}(T) - \frac{\pi_{vs}(T)}{T} - \frac{d\pi_{vs}(T)}{dT} = 0$$

above which this expression is positive (see main text, Sec. 4).

2.2 Lemma S2.2 – derivatives and the sign of $\partial_\rho D_1$

Lemma 2.1. *If (ρ, T) solve (7), (10) near z , then*

$$\frac{dT}{dz} = D_1(\rho, T, c_0), \quad \frac{d\rho}{dz} = D_2(\rho, T, c_0),$$

where

$$D_1(\rho, T, c_0) = - \frac{\left(\frac{R_1 T + L_{tr}}{R_1 \rho T} \bar{\pi}_{vs}(T) + 1 \right) (g\rho + c_0)}{\rho(c_v + R_1) + (R_1 T + L_{tr}) \left(\frac{\bar{\pi}_{vs}(T)}{T} + \frac{d\bar{\pi}_{vs}(T)}{dT} \right)}, \quad (11)$$

$$D_2(\rho, T, c_0) = - \frac{\left(\frac{R_1 T + L_{tr}}{R_1 \rho} \frac{d\bar{\pi}_{vs}(T)}{dT} + \frac{c_v}{R_1} \right) (g\rho + c_0)}{T(c_v + R_1) + (R_1 T + L_{tr}) \left(\frac{\bar{\pi}_{vs}(T)}{\rho} + \frac{c_v}{R_1} \frac{d\bar{\pi}_{vs}(T)}{dT} \right)}. \quad (12)$$

Moreover

$$\frac{\partial D_1(\rho, T, c_0)}{\partial \rho} > 0 \quad \text{whenever } T > \Theta_1. \quad (13)$$

Proof. Equations (7) and (10) are linear in dT/dz and $d\rho/dz$. Substituting $\frac{d\rho}{dz} = -\frac{\rho}{T} \frac{dT}{dz} - \frac{g\rho + c_0}{R_1 T}$ (from (10)) into (7) and solving for dT/dz gives (11); the symmetric substitution gives (12).

Differentiating (11) with respect to ρ ,

$$\frac{\partial D_1}{\partial \rho} = \frac{E(\rho, T, c_0)}{\left[\rho(c_v + R_1) + (R_1 T + L_{tr}) \left(\frac{\bar{\pi}_{vs}(T)}{T} + \frac{d\bar{\pi}_{vs}(T)}{dT} \right) \right]^2},$$

with

$$\begin{aligned} E(\rho, T, c_0) &= \left(\frac{R_1 T + L_{tr}}{R_1 \rho T} \bar{\pi}_{vs}(T) + 1 \right) (g\rho + c_0)(c_v + R_1) \\ &\quad + \left[\rho(c_v + R_1) + (R_1 T + L_{tr}) \left(\frac{\bar{\pi}_{vs}(T)}{T} + \frac{d\bar{\pi}_{vs}(T)}{dT} \right) \right] \\ &\quad \times \left(\frac{R_1 T + L_{tr}}{R_1 \rho^2 T} \bar{\pi}_{vs}(T)(g\rho + c_0) - g \left(\frac{R_1 T + L_{tr}}{R_1 \rho T} \bar{\pi}_{vs}(T) + 1 \right) \right). \end{aligned}$$

Using

$$\frac{R_1 T + L_{tr}}{R_1 \rho^2 T} \bar{\pi}_{vs}(T)(g\rho + c_0) - g \left(\frac{R_1 T + L_{tr}}{R_1 \rho T} \bar{\pi}_{vs}(T) + 1 \right) \geq -g,$$

we obtain

$$\begin{aligned} E(\rho, T, c_0) &\geq \left(\frac{R_1 T + L_{tr}}{R_1 \rho T} \bar{\pi}_{vs}(T) + 1 \right) g\rho(c_v + R_1) - g \left[\rho(c_v + R_1) + (R_1 T + L_{tr}) \left(\frac{\bar{\pi}_{vs}(T)}{T} + \frac{d\bar{\pi}_{vs}(T)}{dT} \right) \right] \\ &= g(R_1 T + L_{tr}) \left[\frac{c_v + R_1}{R_1} \bar{\pi}_{vs}(T) - \frac{\bar{\pi}_{vs}(T)}{T} - \frac{d\bar{\pi}_{vs}(T)}{dT} \right]. \end{aligned}$$

The bracket is strictly positive whenever $T > \Theta_1$ (definition of Θ_1), and the prefactor $g(R_1 T + L_{tr})$ is always strictly positive – this holds whether L_{tr} is a constant or the temperature-dependent function $L_{tr}(T)$ used in the numerics, as long as T stays in its physical range. Hence $E > 0$ and (13) follows. $\square$

2.3 Lemma S2.3 – ordering near $z = 0$

Lemma 2.2. *If $0 \leq c^{(i)} < c^{(ii)}$, there exists $\varepsilon_z > 0$ such that*

$$\rho^{(ii)}(z) < \rho^{(i)}(z), \quad T^{(ii)}(z) < T^{(i)}(z) \quad \forall z \in]0, \varepsilon_z[.$$

Proof. Both solutions share $\rho^{(i)}(0) = \rho^{(ii)}(0) = \rho_0$, $T^{(i)}(0) = T^{(ii)}(0) = T_0$. From (11)–(12), $D_1(\rho_0, T_0, c^{(i)}) > D_1(\rho_0, T_0, c^{(ii)})$ and $D_2(\rho_0, T_0, c^{(i)}) > D_2(\rho_0, T_0, c^{(ii)})$ (both $D_1, D_2 < 0$ and their magnitude is increasing in c_0), which gives the stated ordering in a right neighbourhood of $z = 0$. $\square$

2.4 Lemma S2.1 – global ordering (statement in main text)

Lemma 2.3 (= Lemma 2, Appendix B). *Let $0 \leq c^{(i)} < c^{(ii)}$ and let $(\rho^{(i)}, T^{(i)})$, $(\rho^{(ii)}, T^{(ii)})$ solve (7), (10) on $[0, \bar{z}_1]$ with $c_0 = c^{(i)}$, $c^{(ii)}$ respectively, staying in the domain $\rho > 0$, $\Theta_1 < T < \Theta_+$. Then*

$$T^{(ii)}(z) < T^{(i)}(z) \quad \forall z \in]0, \bar{z}_1], \quad (14)$$

$$\rho^{(ii)}(z)T^{(ii)}(z) < \rho^{(i)}(z)T^{(i)}(z) \quad \forall z \in]0, \bar{z}_1]. \quad (15)$$

Proof. Monotone decrease of both $\rho^{(k)}$ and $T^{(k)}$ ($k = i, ii$) follows immediately from (11)–(12), whose right-hand sides are strictly negative on the stated domain.

Suppose (15) fails. Then there exists $z_1 > 0$ such that

$$\rho^{(ii)}(z_1)T^{(ii)}(z_1) = \rho^{(i)}(z_1)T^{(i)}(z_1), \quad \rho^{(ii)}(z)T^{(ii)}(z) < \rho^{(i)}(z)T^{(i)}(z) \quad \forall z \in]0, z_1[, \quad (16)$$

$$\frac{d}{dz}(\rho^{(ii)}T^{(ii)})_{z=z_1} \geq \frac{d}{dz}(\rho^{(i)}T^{(i)})_{z=z_1}. \quad (17)$$

From (10), $\frac{d}{dz}(\rho T) = -g\rho - c_0$ (after dividing by R_1), so (17) implies $\rho^{(ii)}(z_1) \leq \rho^{(i)}(z_1) - (c^{(ii)} - c^{(i)})$, hence, together with (16),

$$\rho^{(ii)}(z_1) < \rho^{(i)}(z_1), \quad T^{(ii)}(z_1) > T^{(i)}(z_1). \quad (18)$$

By (18), (16), and Lemma 2.2 (continuity of the crossing), there exists $z_0 \in]0, z_1[$ such that

$$\rho^{(ii)}(z_0) < \rho^{(i)}(z_0), \quad T^{(ii)}(z_0) = T^{(i)}(z_0), \quad T^{(ii)}(z) < T^{(i)}(z) \quad \forall z \in]0, z_0[.$$

The last two relations force

$$\frac{d}{dz}T^{(ii)}(z_0) \geq \frac{d}{dz}T^{(i)}(z_0). \quad (19)$$

But by (13), since $\rho^{(ii)}(z_0) < \rho^{(i)}(z_0)$ and $T^{(ii)}(z_0) = T^{(i)}(z_0)$, D_1 is strictly increasing in ρ , so $D_1(\rho^{(ii)}(z_0), T(z_0), c^{(ii)}) < D_1(\rho^{(i)}(z_0), T(z_0), c^{(i)})$ (the inequality in c_0 reinforcing, not opposing, the inequality in ρ), contradicting (19). This contradiction rules out (16)–(17), proving (15).

Finally, (14) follows from (15): if instead there were $z_0 > 0$ with $T^{(ii)}(z_0) = T^{(i)}(z_0)$, $T^{(ii)}(z) < T^{(i)}(z)$ on $]0, z_0[$, and $\rho^{(ii)}(z_0) \geq \rho^{(i)}(z_0)$, continuity would force a point $z' \in]0, z_0[$ with $\rho^{(ii)}(z')T^{(ii)}(z') = \rho^{(i)}(z')T^{(i)}(z')$, contradicting the inequality (15) just proven. Hence $\rho^{(ii)} < \rho^{(i)}$ whenever $T^{(ii)} = T^{(i)}$, which together with strict decrease of both temperature profiles yields (14) on the whole interval. $\square$

2.5 Proof of Proposition 4.1

Proposition 2.1 (= Proposition 4.1, main text). *Consider $\rho_0 > 0$, $T_0 \in]\Theta_1, \Theta_+[$, and $\bar{z}_1 \in]0, \bar{z}^*[$ with $T_{hs}^{[1]}(\bar{z}_1) > \Theta_1$. There exists $\bar{Q}_0$ with $0 < \bar{Q}_0 < \rho_{hs}^{[1]}(\bar{z}_1)T_{hs}^{[1]}(\bar{z}_1)$ such that for every $q_0 \in]\bar{Q}_0, \rho_{hs}^{[1]}(\bar{z}_1)T_{hs}^{[1]}(\bar{z}_1)[$ there is a unique solution (ρ, T, α) of the stationary problem (7)–(9).*

Proof. For each $c_0 \geq 0$, let (ρ_{c_0}, T_{c_0}) solve (7), (10) with $\rho(0) = \rho_0$, $T(0) = T_0$. By Lemma 2.3, as c_0 increases the pair $(\rho_{c_0}(z), T_{c_0}(z))$ decreases (in the sense of (14)–(15)) at every fixed $z \in]0, \bar{z}_1]$. By continuous dependence of solutions on c_0 , this monotone family is defined for $c_0 \in [0, \bar{C}_0]$, where $\bar{C}_0 < \infty$ is the threshold at which

$$\lim_{c_0 \rightarrow \bar{C}_0^-} T_{c_0}(\bar{z}_1) = \Theta_1;$$

finiteness of $\bar{C}_0$ follows because $T_{c_0}(z)$ must remain above Θ_1 for the solution to exist (Lemma 2.1 requires $T > \Theta_1$), so the dissipation rate c_0 cannot grow without bound while preserving this lower bound. For $c_0 \in [0, \bar{C}_0]$,

$$0 < \rho_{c_0}(z) \leq \rho_0, \quad \Theta_1 < T_{c_0}(z) \leq T_0 < \Theta_+ \quad \forall z \in [0, \bar{z}_1],$$

so the solution exists on the full interval.

Define

$$\nu(c_0) = \rho_{c_0}(\bar{z}_1) T_{c_0}(\bar{z}_1).$$

By continuous dependence on data, ν is continuous; by Lemma 2.3, ν is strictly decreasing on $[0, \bar{C}_0]$. Hence, for each q_0 in the range of ν (an interval $]\bar{Q}_0, \nu(0)[$ with $\nu(0) = \rho_{hs}^{[1]}(\bar{z}_1)T_{hs}^{[1]}(\bar{z}_1)$), there is a unique $c_0 = \nu^{-1}(q_0)$.

With this c_0 , solve (7), (10) to get $(\rho, T) = (\rho_{c_0}, T_{c_0})$, compute

$$I := \int_0^{\bar{z}_1} \left(\bar{\pi}_{vs}(T) \frac{d}{dz} \log \rho - \frac{d}{dz} \bar{\pi}_{vs}(T) \right) dz,$$

and set

$$\alpha = \frac{c_0 \bar{z}_1}{g b I}.$$

Then (ρ, T, α) solves the stationary problem (7)–(9) with $\Sigma = c_0/g$ satisfying (9) by construction. Uniqueness follows from the strict monotonicity of ν and the uniqueness of the local solution of (7), (10) for each fixed c_0 . $\square$ $\square$

Remark (identification $c_0 = g\Sigma$). Comparing (10) with the physical momentum equation (8) shows that the auxiliary constant c_0 plays exactly the role of $g\Sigma$ once the solution is self-consistently closed by (9); this is the identity used throughout the numerical scheme documented in Section 3.

3 Code documentation and reproducibility

3.1 Overview

This section documents the Python code used to (i) compute the stationary solution $(\rho, T, \alpha, \Sigma)$ of the model for both the dry ($\vartheta = 0$) and humid ($\vartheta = 2/3$) cases and locate the root $c_0 = \nu^{-1}(q_0)$ underlying Proposition 4.1 (Figure 1 of the main text), (ii) trace the $\nu(\Sigma)$ dependence curve and perform the grid convergence study reported in Table 1 of the main text ($\Delta z = 10 \text{ m}, 5 \text{ m}, 1 \text{ m}$; Figure 2), and (iii) run the full time-dependent evolutionary simulation producing the damped oscillations of $\alpha(t)$ and $\Sigma(t)$ and the associated spatial profiles (Figures 3–4). The full source is provided as a separate archive alongside this PDF; the description below reproduces and expands the accompanying `readme.txt`.

3.2 How to run

The archive contains three directly executable main scripts.

```
python runstationnary.py
```

runs the stationary computation for both the humid and dry cases, prints the resulting c_0 and Σ_{stat} , and regenerates the stationary $\rho(z)$, $T(z)$ profiles of Figure 1.

```
python plotnusigma1.py
```

traces the $\nu(\Sigma)$ dependence curve of Figure 2 and performs the grid convergence check of Table 1; the spatial step `hz` can be edited directly at the top of this file to reproduce the three resolutions reported there.

```
python remdalilag1.py
```

runs the full time-dependent (evolutionary) simulation with memory kernel parameter b (450 s by default), reproducing the damped oscillations of Figure 3 and the final-time spatial profiles of Figure 4.

3.3 Module description

runstationnary.py (main script) Computes the stationary profiles for the humid and dry reference cases and reports c_0 , Σ_{stat} , and α_{stat} , using **getHydrostaticProfiles2.py** for the target q_0 and **getrhoTatz4.py/solveStationary4.py** as the objective function and ODE integrator for the root-finding step.

plotnusigma1.py (main script) Traces the map $\nu(c_0) = \rho(\bar{z}_1)T(\bar{z}_1)$ against $\Sigma = c_0/g$ over a range of c_0 (Figure 2), and performs the grid convergence check of Table 1 by repeating the stationary computation at $\Delta z = 10 \text{ m}, 5 \text{ m}, 1 \text{ m}$. It shares **getHydrostaticProfiles2.py** with **runstationnary.py** and uses **getrhoTatz1.py/solveSpatialProfileshelper2.py** as its objective function and integrator.

remdalilag1.py (main script) Integrates the full time-dependent evolutionary system (7)–(9) (time-dependent form) using an explicit Euler scheme in time and space: at each time step, the memory convolution defining $\Sigma(t)$ (Eq. (4) of the main text) is evaluated by summing over all previous time steps, and a shooting-type inner iteration solves for the control variable $D(t)$ so that the top boundary pressure matches the reference hydrostatic profile, before updating $\alpha(t)$. It uses **gethydrostaticprofiles1.py** to compute this reference profile.

getHydrostaticProfiles2.py / **gethydrostaticprofiles1.py** Compute the reference hydrostatic profiles $(\rho_{hs}^{[\vartheta]}, T_{hs}^{[\vartheta]})$ by integrating equations (1)–(2) above from $z = 0$ to $z = \bar{z}_1$, and return the target top value $q_0 = \rho_{hs}^{[\vartheta]}(\bar{z}_1) T_{hs}^{[\vartheta]}(\bar{z}_1)$ used as the reference-profile pressure proxy. These two modules implement the same computation and are kept as separate files because they are called by different main scripts (the stationary and evolutionary branches of the archive respectively); a user merging the two branches into a single package may safely consolidate them into one shared module, provided the function name used in each importing script is updated accordingly.

getrhoTatz4.py / **solveStationary4.py** Used by **runstationnary.py**: respectively the objective function $f(c_0) = \rho_{c_0}(\bar{z}_1)T_{c_0}(\bar{z}_1) - q_0$ passed to the root finder (**scipy.optimize.fsolve**), and the underlying explicit Euler integrator of the stationary system (7), (10) for a given c_0 .

`getrhoTatz1.py` / `solveSpatialProfileshelper2.py` Used by `plotnusigma1.py`: the same objective-function/integrator pair as above, used specifically to trace $\nu(c_0)$ over a range of c_0 and to repeat the computation at the three grid resolutions of Table 1. The integrator additionally evaluates the cumulative integral I (by the trapezoidal rule) needed for $\alpha = \frac{c_0 \bar{z}_1}{g b I}$ (Eq. (alpha) of the main text). Robustness checks in both integrators flag singular matrices, negative densities, and non-physical (**NaN/Inf**) values, aborting the step with a failure signal that the root finder treats as an infeasible c_0 .

3.4 Requirements

Python 3.x, NumPy, SciPy, Matplotlib.

3.5 Grid convergence summary

The grid convergence results are reported in Table 1 of the main text; they were generated by re-running `plotnusigma1.py` with `hz` set successively to 10 m, 5 m, and 1 m. At $\Delta z = 10$ m the explicit Euler scheme exhibits numerical instability in the cumulative integral defining α ; $\Delta z = 5$ m resolves this and is used for all reported stationary results, while $\Delta z = 1$ m confirms convergence (relative change $< 0.02\%$ in α_{stat} , Σ stable to six decimal places).

3.6 Data and code availability

The complete scripts accompany this supplement as a public archive, consistent with the data availability statement of the main text. The archive contains three main, directly executable scripts and their supporting helper modules:

- `runstationnary.py` – stationary profiles for the humid and dry cases (Figure 1), using helpers `getHydrostaticProfiles2.py`, `getrhoTatz4.py`, and `solveStationary4.py`;
- `plotnusigma1.py` – the $\nu(\Sigma)$ dependence curve (Figure 2) and the grid convergence check (Table 1), using helpers `getHydrostaticProfiles2.py`, `getrhoTatz1.py`, and `solveSpatialProfileshelper2.py`;
- `remdalilag1.py` – the evolutionary, time-dependent simulation (Figures 3–4), using helper `gethydrostaticprofiles1.py`.

A `readme.txt` file accompanying the archive gives run instructions and Python package requirements.