Review of *"Implementation and evaluation of the lognormal prior probability distribution in a variational atmospheric inversion framework"* by Vojta et al.

Vojta et al. present an implementation of a lognormal prior probability distribution in an atmospheric inversion framework. They use European SF_6 emissions as a case study. They go through a case with pseudo-data and then an example with real data and they look at the impact of the choice of using the mean, median, or mode for a lognormal distribution. The paper is clearly written and the implementation seems correct, however the paper does not seem particularly novel. As far as I can tell, most of this is textbook material from a statistics course. This does not seem sufficient for a full paper. I suggest the paper be revised into a short technical note that emphasizes new derivations. The math seems more relevant than the case study.

Comments

Key results follow trivially from known statistical work

The text is well written and all seems fine. However, the work does not seem novel. Pretty much the entirety of this is material that would be covered in introductory courses on topic and, in some cases, textbooks on the topic. As an example, the Fletcher and Zupanski (2006) reference described how to construct the cost function, Jacobian, and Hessian for data assimilation problems using log-normal distributions 20 years ago. I don't see how this work goes much beyond Fletcher and Zupanski or textbooks/standard course material on the topic.

The case study itself is not presented as a new or novel result, it is purely presented as a prototypical inverse problem. Therefore, in my opinion, the novelty needs to be in the mathematical analysis. I do not find new or novel information there. In my opinion, this should be framed as a short technical note that gives the brief extension to show that the mean and mode solutions are unstable due to the additional log-term. That said, it is already common practice to use the median when working with log-normal $P(\mathbf{x})$ or $P(\mathbf{y}|\mathbf{x})$.

The conclusions of the current manuscript are presented in a bullet point form. These are the key points from this manuscript:

- **Votja et al.:** Gridded emission inventories, which are often used as prior information in inversions, frequently exhibit lognormal distributions, as demonstrated for the example of SF6. In such cases, it is likely that the associated uncertainties are also better represented by a lognormal rather than a normal distribution.

  Comment: Emissions being lognormally distributed has been known for decades. This is unsurprising. The assumption that the associated uncertainties (or errors) are therefore lognormally distributed is often assumed. This has been done by lots of previous work. I do not find anything compelling one way or the other in Votja et al to change what people are already doing regarding the assumption of lognormality of errors.

- **<u>Votja et al.</u>:** For a lognormal distribution mean, median, and mode do not fall together making it necessary to choose explicitly which of these parameters to optimize for in the inversion.

  <u>Comment</u>: The fact that the mean, median, and mode are distinct for a lognormal distribution is textbook material that is covered in basic statistics.

- **<u>Votja et al.</u>:** Optimizing for the mode and the mean can yield improved emission estimates when the observational constraints are strong. However, under weak observational constraints, these choices can lead to unstable and strongly biased inversion results caused by a decrease in the cost in the state space due to the extra logarithm term in the cost functions.
  …
  Optimizing for the median consistently improved prior emissions across all tested cases, and thus provides the most robust estimator to optimize for.

  <u>Comment</u>: This material is covered in textbooks on inverse modeling. The median of the posterior distribution is typically used when lognormal distributions are used for either the $P(\mathbf{x})$ or $P(\mathbf{y}|\mathbf{x})$.

- **<u>Votja et al.</u>:** The lognormal based inversion produced a posterior emission pattern that is overall very similar to the one obtained using a normal distribution. However, it prevented non-physical negative emission values and may have also avoided the overestimation of nearby emissions, which might occur in normal-based inversions as a compensatory effect for negative values. Further, the lognormal prior distribution occasionally allowed for stronger positive emission adjustments compared to the Gaussian, likely due to the long tail of the distribution.

  <u>Comment</u>: This seems entirely unsurprising. A lognormal is by definition going to yield positive constraints and it has a long tail. Again, not a new finding.

- **<u>Votja et al.</u>:** The posterior uncertainties can be estimated using a Monte Carlo approach by creating an ensemble of inversions, each with prior emissions perturbed by lognormally distributed errors. However, we find that the resulting distributions are strongly influenced by the sign of the inversion increment. Negative increments typically lead to narrow, nearly Gaussian posterior distributions, while positive increments are associated with broader and more skewed, lognormal-shaped distributions. To account for this asymmetry, we recommend characterizing posterior uncertainty using interpercentile ranges derived from the ensemble, rather than the standard deviation.

  <u>Comment</u>: This seems unsurprising. Doing perturbations in log-space is appropriate if one has log-normal distributions.

- **<u>Votja et al.</u>:** The error reduction in linear space is highly asymmetric with respect to the sign of the inversion increments, which complicates its interpretation as a measure of the information content provided by the observations. We therefore recommend assessing error reduction in log space, which exhibits much weaker asymmetries, and is therefore better suited to reflect the observational constraints.

Comment: Fairly textbook. The information content is going to be skewed if you are working in log-space. The information content the authors are considering is formulated from Gaussians, so things are skewed in log-space.

In my opinion, there is not enough new or novel information to justify publication as a full paper. I recommend the manuscript be rejected and a more focused analysis be resubmitted as a technical note. I do not think the manuscript warrants publication as a full paper in its current form.