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Revealing horizontal gravity force in geopotential coordinates via metric tensors

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Abstract. Earth gravity force $\mathbf{g}$ is represented by geopotential Φ, $\mathbf{g} = \nabla\Phi$, with ∇ the three-dimensional gradient operator. True gravity $\mathbf{g}$ has horizontal component. Oceanographic and meteorological communities use two approaches to eliminate horizontal gravity force. The first one is to replace Φ by geopotential for uniform Earth mass density ($\Phi_{uniform}$) representing no horizontal gravity force, such as spherical, spheroidal, or recently developed Realistic, Ellipsoidal, Analytically Tractable (GREAT) Φ_{GREAT} on the base of $|\Phi - \Phi_{uniform}| \ll |\Phi_{uniform}|$, however, it is not sufficient to justify because we still need to compare $\nabla(\Phi - \Phi_{uniform})$ to other forces such as Coriolis force and pressure gradient force. The second approach is to use Φ to establish geopotential coordinates. Consider local Cartesian coordinates (ξ, η, ζ) with unit vectors $(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}})$. The geopotential coordinates (x, y, Z) and corresponding unit vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{Z}})$ are defined by $x = \xi, y = \eta$, $Z = -\Phi/g_0$, $g_0 = 9.81$ m/s^2. From such a relationship, metric tensors between (ξ, η, ζ) and (x, y, Z), and in turn the horizontal gradient operator are obtained. The pressure gradient is $\nabla_C p = \partial_\xi p\hat{\boldsymbol{\xi}} + \partial_\eta p\hat{\boldsymbol{\eta}} + \partial_\zeta p\hat{\boldsymbol{\zeta}}$ in the local Cartesian coordinates, and $\nabla_G p = (\partial_x p + \partial_x N \partial_Z p)\hat{\mathbf{x}} + (\partial_y p + \partial_y N \partial_Z p)\hat{\mathbf{y}} + (\partial_Z p)\hat{\mathbf{Z}}$ in the geopotential coordinates with N the geoidal undulation. The horizontal gravity force exists in horizontal momentum equation with the geopotential coordinates. Importance of the horizontal gravity force is verified using two publicly available datasets. Five concerns with PNAS-e2416636121 are presented in Appendix. The major equation in PNAS-e2416636121, $\nabla p = \partial_\xi p\hat{\boldsymbol{\xi}} + \partial_\eta p\hat{\boldsymbol{\eta}} + \partial_\zeta p\hat{\boldsymbol{\zeta}} = \partial_x p\hat{\mathbf{x}} + \partial_y p\hat{\mathbf{y}} + |\nabla Z|\partial_Z p\hat{\mathbf{Z}}$, is valid only for the gravity with no horizontal component, i.e., for the geopotential coordinates coincident with the local Cartesian coordinates.

Short Summary

Standard approach with covariant and contravariant metric tensors between Cartesian and geopotential coordinates is used to obtain the gradient operator in the geopotential coordinates and to confirm the existence of horizontal gravity force. Importance of the horizontal gravity force is verified using two publicly available datasets, i.e., the global static gravity field model EIGEN-6C4 for the geoid undulation N and the climatological annual mean temperature and salinity from the NCEI WOA23.

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1. Introduction

The gravity force **g** in oceanography and meteorology is represented by geopotential $\tilde{\Phi}$ [see equation (8) in Staniforth and White 2025, hereafter referred to to SW25]

$$\mathbf{g} = -\tilde{\nabla}\tilde{\Phi} \quad (1a)$$

or by Φ [see equation (1) in McWilliams 2024, hereafter referred to M24]

$$\mathbf{g} = \tilde{\nabla}\Phi \quad (1b)$$

Here, $\tilde{\nabla}$ is 3D gradient operator for any coordinates. Use of Eq.(1a) to define $\tilde{\Phi}$ implies that the values of iso $\tilde{\Phi}$-surfaces decrease towards the Earth interior, and use of Eq.(1b) suggests that the values of iso Φ-surfaces increase towards the Earth interior. The momentum equation for a rotating fluid without friction with the geopotential defined by (1b) is given by

$$\frac{D\mathbf{u}}{Dt} = -2\mathbf{\Omega} \times \mathbf{u} - \frac{1}{\rho}\tilde{\nabla}p + \tilde{\nabla}\Phi \quad (2)$$

where D/Dt is the material derivative; **u** is the velocity vector; p is the pressure; ρ is the density; and $\mathbf{\Omega}$ is the angular velocity vector.

Oceanographic and meteorological communities use two approaches to eliminate horizontal gravity force (i.e., the horizontal gradient of geopotential.) The first approach is to use the geopotential for uniform Earth mass density ($\Phi_{uniform}$) such as spherical, spheroidal, or ($\tilde{\Phi}_{uniform}$) such as recently developed Realistic, Ellipsoidal, Analytically Tractable (GREAT) [equation (8) in SW25]

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$$\begin{aligned}\tilde{\Phi}_{uniorm}(\chi, r) &= \tilde{\Phi}_{GREAT}(\chi, r) \\ &= -\frac{GM}{r} + \frac{GM}{R}\left\{1 + \left[\left(\frac{8\tilde{\varepsilon} - 5m}{2}\right) + \left(\frac{5m - 4\tilde{\varepsilon}}{2}\right)\frac{R^2}{a^2}\right]\sin^2\chi\right\}^{\frac{1}{2}} \\ &\quad - \frac{GM}{R}\left[1 + \left(\frac{8\tilde{\varepsilon} - 7m}{12}\right) + \left(\frac{11m - 4\tilde{\varepsilon}}{12}\right)\frac{R^2}{a^2}\right] \quad (3a)\end{aligned}$$

to represent $\tilde{\Phi}$. Perturbation due to geoid undulation [see equations (B2) and (B10) in SW25] is given by

$$\tilde{\Phi}(\lambda, \chi, r) = \tilde{\Phi}_{GREAT}(\chi, r) - \frac{GM}{R}\left[b\tilde{H}(\lambda, \chi)\right], \quad \tilde{H}(\lambda, \chi) = O(1), \quad b \sim 1.6 \times 10^{-5} \quad (3b)$$

where $G = 6.674\times10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$, is the gravitational constant; $M = 5.9736 \times 10^{24}$ kg, is the mass of the Earth; r is the radial distance; a is the Earth semi-major axis; R is the Earth Radius; λ is the longitude; χ is the geodetic latitude. The two small parameters ($\tilde{\varepsilon}$, m) are defined by [see equations (1) and (25) in SW25]

$$m \equiv \frac{\Omega^2 a^3}{GM} \approx \frac{centrifugal\ force}{gravitational\ attraction} \approx 0.003461, \qquad \tilde{\varepsilon} \equiv \frac{a^2 - c^2}{2c^2} \approx 0.003370, \quad (4)$$

where $\Omega = |\mathbf{\Omega}| = 2\pi/(86164\ \text{s})$, is Earth's (assumed constant) angular-rotation rate about its axis.

Eq.(3b) can be rewritten by

$$\tilde{\Phi}(\lambda, \chi, r) = \tilde{\Phi}_{GREAT}(\chi, r) + g_0 N(\lambda, \chi), \qquad g_0 \equiv \frac{GM}{R^2} = 9.81\text{m s}^{-2}, \quad (5)$$

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where N is the bumpy geoid undulation,

$$N \equiv \left[\tilde{\Phi}(\lambda,\chi,r) - \tilde{\Phi}_{GREAT}(\chi,r)\right]/\mathrm{g}_0 = -Rb\tilde{H}(\lambda,\chi) = O(100\ m) \tag{6}$$

For geopotential Φ defined by Eq.(1b), i.e., value of iso-Φ surfaces increase towards the Earth interior, we have

$$\Phi(\lambda,\chi,r) = \Phi_{uniform}(\chi,r) - \mathrm{g}_0 N(\lambda,\chi) \tag{7}$$

Both Eq.(5) and Eq.(7) have tiny relative errors of about 10^{-5} (SW25). Substitution of Eq.(5) into Eq.(1a) or Eq.(7) into Eq.(1b) and then into Eq.(2) leads to the horizontal momentum equation

$$\frac{D\mathbf{U}}{Dt} = -2\mathbf{\Omega}\times\mathbf{U} - \frac{1}{\rho}\tilde{\nabla}_2 p + \mathrm{g}_0\tilde{\nabla}_2 N \tag{8}$$

where $\tilde{\nabla}_2$ is the horizontal gradient operator; and $\mathbf{U}$ is the horizontal velocity vector.

Two necessary conditions should be satisfied to replace Φ by $\Phi_{uniform}$ or $\tilde{\Phi}$ by $\tilde{\Phi}_{GREAT}$. The first condition is the smallness of bumpy geoid undulation ($\mathrm{g}_0|N|$) compared to $|\Phi_{uniform}|$ or $|\Phi_{GREAT}|$

$$O(\mathrm{g}_0|N|)/O(|\Phi_{uniform}|) = O(\mathrm{g}_0|N|)/O(|\Phi_{GREAT}|) \sim 1.6\times 10^{-5} \ll 1 \tag{9a}$$

The second condition is the smallness of $\mathrm{g}_0|\tilde{\nabla}_2 N|$ compared to the other horizontal forces such as the Coriolis force and pressure gradient force,

$$O(\mathrm{g}_0|\tilde{\nabla}_2 N|)/O(|2\mathbf{\Omega}\times\mathbf{U}|) \ll 1\ (?),\ O(\mathrm{g}_0|\tilde{\nabla}_2 N|)/O\left(\left|\frac{1}{\rho}\tilde{\nabla}_2 p\right|\right) \ll 1\ (?) \tag{9b}$$

which cannot be satisfied (Chu, 2021a, b, c; 2023, 2024).

The second approach is to use the geopotential coordinates to eliminate the horizontal gravity force (e.g., M24). The equation of motion without frictional force in the local Cartesian coordinates (ξ, η, ζ) with the corresponding unit vectors ($\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}}$) and traditional Coriolis approximation is given by

$$\left[\frac{D\mathbf{u}}{Dt}\right]_C + 2\mathbf{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla_C p + \nabla_C\Phi \tag{10}$$

where the (east, north, "up") vector components are (ξ, η, ζ) and (u, v, ω); and the subscript C denotes the Cartesian coordinates. The geopotential Φ represents gravity in the local Cartesian coordinates as,

$$\mathbf{g} = \nabla_C\Phi = \partial_\xi\Phi\hat{\boldsymbol{\xi}} + \partial_\eta\Phi\hat{\boldsymbol{\eta}} + \partial_\zeta\Phi\hat{\boldsymbol{\zeta}} \tag{11}$$

with

$$\nabla_C \equiv \hat{\boldsymbol{\xi}}\partial_\xi + \hat{\boldsymbol{\eta}}\partial_\eta + \hat{\boldsymbol{\zeta}}\partial_\zeta \tag{12}$$

the 3D gradient operator in the local Cartesian coordinates, and

$$\partial_\xi \equiv \frac{\partial}{\partial\xi},\qquad \partial_\eta \equiv \frac{\partial}{\partial\eta},\qquad \partial_\zeta \equiv \frac{\partial}{\partial\zeta} \tag{13}$$

The gravity in the direction of $\hat{\boldsymbol{\zeta}}$ would never be zero,

$$\partial_\zeta\Phi \neq 0 \tag{14}$$

The geopotential coordinates (x, y, Z) are well defined in oceanography and meteorology by [see equation (3) in M24]

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$$x = \xi,\ \ y = \eta, \qquad Z = -\frac{\Phi}{g_0} \tag{15}$$

The unit vectors of the geopotential coordinates $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{Z}})$ are determined by [see equation (18) in M24] (also shown in Figure 1),

$$\hat{\mathbf{x}} = \hat{\boldsymbol{\xi}},\ \ \hat{\mathbf{y}} = \hat{\boldsymbol{\eta}}, \qquad \hat{\mathbf{Z}} = -\frac{\nabla_C \Phi}{|\nabla_C \Phi|}, \tag{16}$$

When Z or Φ does not depend on (ξ, η), i.e., $\partial_\xi Z = 0, \partial_\eta Z = 0$, *the geopotential coordinates reduce to the local Cartesian coordinates.* With Eq.(15) we use standard covariant and contravariant metric tensors between the Cartesian and geopotential coordinates to prove the existence of horizontal gravity force.

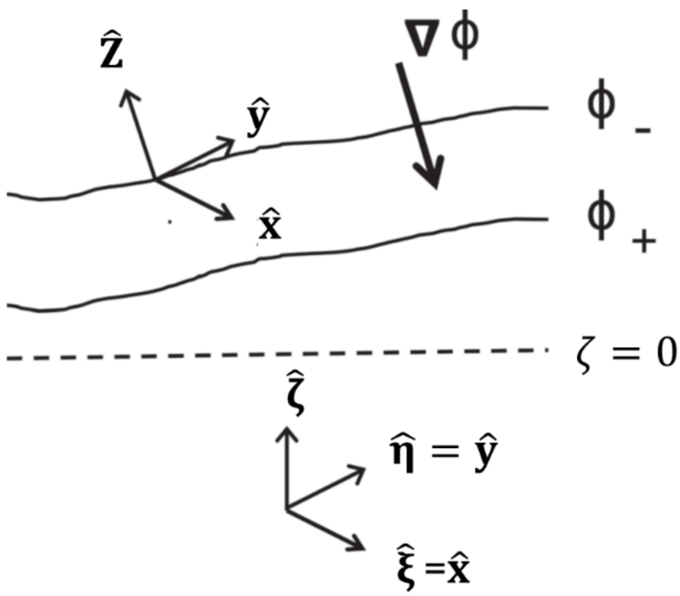

Figure 1. Illustration of local Cartesian coordinates (*ξ, η, ζ*) with the unit vectors $(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}})$ and geopotential coordinates (*x, y, Z*) with the unit vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{Z}})$ (like Figure 1. in M24).

2. Derivatives between Local Cartesian and Geopotential Coordinates

Compute the differentials,

$$dx = d\xi,\ dy = d\eta,\ dZ = Z_\xi d\xi + Z_\eta d\eta + Z_\zeta d\zeta \tag{17}$$

which can be transformed into

$$d\xi = dx,\ \ d\eta = dy, d\zeta = \left(dZ - Z_\xi d\xi - Z_\eta d\eta\right)/Z_\zeta \tag{18}$$

For any scalar field, the chain-rule relations lead to [see equation (17) in M24],

$$\partial_x = \partial_\xi - \left(Z_\xi/Z_\zeta\right)\partial_\zeta,\ \partial_y = \partial_\eta - \left(Z_\eta/Z_\zeta\right)\partial_\zeta,\ \partial_Z = \left(1/Z_\zeta\right)\partial_\zeta \tag{19}$$

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For any 2D variable depending on (ξ, η) only, such as sea level, bottom topography, and geoid undulation (N), we have

$$\partial_\xi N = \partial_x N,\ \ \partial_\eta N = \partial_y N \tag{20}$$

The uniform geopotential field $\Phi_{uniform}$ for the domain of interest is given by [see the line above equation (2) in M24],

$$\Phi_{uniform} = -g_0 \zeta \tag{21}$$

Substitution of Eq.(21) into Eq.(7) and then into Eq.(15) gives

$$Z = \zeta + N \tag{22}$$

which leads to

$$\partial_\zeta Z = 1 \tag{23}$$

which contains tiny relative error of about 10^{-5} (SW25.) Substitution of Eq.(23) into Eq.(19) gives

$$\partial_x Z = 0, \qquad \partial_y Z = 0 \tag{24}$$

3. Covariant and Contravariant Metric Tensors

Let [ξ^1, ξ^2, ξ^3] ≡ $[\xi, \eta, \zeta]$ and [x^1, x^2, x^3] ≡ [x, y, Z]. The standard approach is to use the metric tensors to identify the transformation between the local Cartesian and geopotential coordinates (Song and Hou, 2006). The covariant metric tensor (**G**) is the differentiation of the Cartesian coordinate variables versus the geopotential coordinate variables [see equation (46) in Staniforth (2014)]

$$\mathbf{G} = [g_{ij}], \qquad g_{ij} = \frac{\partial \xi^p}{\partial x^i}\frac{\partial \xi^q}{\partial x^j}\delta_{pq}, \ \ \delta_{pq} = \begin{cases} 1, & p = q \\ 0 & p \neq q \end{cases} \tag{25}$$

where repeated index denotes summation from 1 to 3 over the index (Einstein's summation convention); δ_{pq} is the Kronecker delta. The inverse of **G** is the contravariant metric tensor,

$$\mathbf{G}^{-1} = [g^{ij}], \qquad g^{ij} = \frac{\partial x^p}{\partial \xi^i}\frac{\partial x^q}{\partial \xi^j}\delta_{pq}, \ \ \delta_{pq} = \begin{cases} 1, & p = q \\ 0 & p \neq q \end{cases} \tag{26}$$

which is the differentiation of the geopotential coordinate variables versus the Cartesian coordinate variables. Substitution of Eq.(15) into Eq.(26) leads to,

$$\mathbf{G}^{-1} = [g^{ij}] = \begin{bmatrix} 1 + Z_\xi^2 & Z_\xi Z_\eta & Z_\xi Z_\zeta \\ Z_\xi Z_\eta & 1 + Z_\eta^2 & Z_\eta Z_\zeta \\ Z_\xi Z_\zeta & Z_\eta Z_\zeta & Z_\zeta^2 \end{bmatrix} \tag{27}$$

Differentiation of the position vector of the geopotential coordinates, $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + Z\hat{\mathbf{Z}}$, respect to [$\xi^1$, ξ^2, ξ^3] (≡ $[\xi, \eta, \zeta]$, local Cartesian coordinates) leads to the contravariant basis vectors

$$\hat{\mathbf{e}}_i = \partial \mathbf{r} / \partial \xi^i \qquad i=1,2,3 \tag{28}$$

The normalized contravariant basis vectors,

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$$\mathbf{e}_i = \hat{\mathbf{e}}_i / |\hat{\mathbf{e}}_i| \quad i=1,2,3 \tag{29}$$

are the unit vectors of the geopotential coordinates,

$$\mathbf{e}_1 \equiv \hat{\mathbf{x}},\ \mathbf{e}_2 = \hat{\mathbf{y}},\ \mathbf{e}_3 = \hat{\mathbf{Z}} \tag{30}$$

4. Gradient Operator in the Geopotential Coordinates

With the contravariant metric tensor [g^{ij}] [i.e., Eq.(27)], the gradient operator in the geopotential coordinates (∇_G) applied to variable Q such as pressure, temperature, and salinity, is given by

$$\nabla_G Q = g^{ij}\partial_j Q\mathbf{e}_i = \left[(1+Z_\xi^2)\partial_x Q + Z_\xi Z_\eta \partial_y Q + Z_\xi Z_\zeta \partial_Z Q\right]\hat{\mathbf{x}} + \left[Z_\xi Z_\eta \partial_x Q + (1+Z_\eta^2)\partial_y Q + Z_\xi Z_\zeta \partial_Z Q\right]\hat{\mathbf{y}} + \left(Z_\xi Z_\zeta \partial_x Q + Z_\eta Z_\zeta \partial_y Q + Z_\zeta^2 \partial_Z Q\right)\hat{\mathbf{Z}} \tag{31}$$

Note that

$$O\left(|Z_\xi|, |Z_\eta|\right) = 10^{-5}, \quad Z_\zeta = 1 \tag{32}$$

and

$$|\partial_x Q| \ll |\partial_Z Q|,\ |\partial_y Q| \ll |\partial_Z Q| \tag{33}$$

for large-scale process. Use of Eq.(32) and Eq.(33) for each component of $\nabla_G Q$ in Eq.(31) leads to the simplification

$$\nabla_G Q = (\partial_x Q + Z_\xi \partial_Z Q)\hat{\mathbf{x}} + (\partial_y Q + Z_\eta \partial_Z Q)\hat{\mathbf{y}} + (\partial_Z Q)\hat{\mathbf{Z}} \tag{34a}$$

Substitution of Eq.(22) into Eq.(34a) and use of Eq.(20), i.e., $\partial_\xi N = \partial_x N,\ \partial_\eta N = \partial_y N$, lead to

$$\nabla_G Q = (\partial_x Q + \partial_x N \partial_Z Q)\hat{\mathbf{x}} + (\partial_y Q + \partial_y N \partial_Z Q)\hat{\mathbf{y}} + (\partial_Z Q)\hat{\mathbf{Z}} \tag{34b}$$

For $Q = p$, we have

$$\nabla_G p = (\partial_x p + \partial_x N \partial_Z p)\hat{\mathbf{x}} + (\partial_y p + \partial_y N \partial_Z p)\hat{\mathbf{y}} + (\partial_Z p)\hat{\mathbf{Z}} \tag{35}$$

5. Reference Pressure

For motionless ocean with a constant density (i.e., reference value for total density), ρ_0 = 1,028 kg/m^3, the reference pressure (p_0) is defined by its balance with the gravity force,

$$\tilde{\nabla} p_0 = \rho_0 \tilde{\nabla}\Phi \tag{36}$$

In the geopotential coordinates, the gradient of geopotential, $\nabla_G \Phi_G$ (i.e., gravity force), is on the true vertical direction ($\hat{\mathbf{Z}}$),

$$\nabla_G \Phi_G = -g_0 \hat{\mathbf{Z}}, \tag{37}$$

which leads to

$$\nabla_G p_0 = \rho_0 \nabla_G \Phi_G = -\rho_0 g_0 \hat{\mathbf{Z}}. \tag{38}$$

Here, we use $\Phi_G = \Phi_G(x, y, Z)$ for the geopotential coordinates, to distinguish to $\Phi = \Phi(\xi, \eta, \zeta)$ for the local Cartesian coordinates. *The reference pressure field p_0 is associated with the geoid which is the ocean surface would take under the influence of gravity of Earth only.*

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6. Equation of Motion in the Geopotential Coordinates

For real density ρ, the density deviation from ρ_0, $\hat{\rho} = \rho - \rho_0$, leads to the dynamic pressure, $\hat{p} = p - p_0$, and causes ocean to move. The equation of motion in the geopotential coordinates is given by

$$\rho \frac{D\mathbf{u}}{Dt} + \rho(2\boldsymbol{\Omega} \times \mathbf{u}) = -\nabla_G \hat{p} + \hat{\rho}\nabla_G \Phi_G \tag{39}$$

Substitution of Eq.(37) into Eq.(39) leads to

$$\rho\left(\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u}\right) = -\nabla_G \hat{p} - \hat{\rho} g_0 \hat{\mathbf{Z}} \tag{40}$$

The hydrostatic equilibrium in the geopotential coordinates refers the pressure gradient force balanced by gravity force along the true vertical direction $\hat{\mathbf{Z}}$,

$$\partial_Z \hat{p} = -\hat{\rho} g_0 \tag{41}$$

Substitution of Eq.(41) into Eq.(35) leads to

$$\nabla_G \hat{p} = [\partial_x \hat{p} - \hat{\rho} g_0 N_x]\hat{\mathbf{x}} + [\partial_y \hat{p} - \hat{\rho} g_0 N_y]\hat{\mathbf{y}} - \hat{\rho} g_0 \hat{\mathbf{Z}} \tag{42}$$

Substitution of Eq.(42) into Eq.(40) leads to the horizontal momentum equation,

$$\rho\left(\frac{D\mathbf{U}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{U}\right) = -\nabla_h \hat{p} + \hat{\rho}\, g_0 \nabla_h N \tag{43}$$

where ∇_h is defined by

$$\nabla_h \equiv \hat{\mathbf{x}}\, \partial_x + \hat{\mathbf{y}}\, \partial_y \tag{44}$$

Eq.(43) and Eq.(41) are the basic ocean dynamic equations for large-scale motion in the geopotential coordinates. It clearly shows the existence of horizontal gravity force $[\hat{\rho} g_0 \nabla_h N]$. Vertical derivative of Eq.(43) gives

$$\frac{\partial}{\partial Z}\left[\frac{\rho}{\rho_0}\left(\frac{D\mathbf{U}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{U}\right)\right] = -\frac{\partial}{\partial Z}\left(\frac{1}{\rho_0}\nabla_h \hat{p}\right) + \frac{g_0}{\rho_0}\frac{\partial \hat{\rho}}{\partial Z}\nabla_h N \tag{45}$$

Substitution of Eq.(41) into Eq.(45) leads to

$$\frac{\partial}{\partial Z}\left[\frac{\rho}{\rho_0}\left(\frac{D\mathbf{U}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{U}\right)\right] = \left(\frac{g_0}{\rho_0}\right)\nabla_h \hat{\rho} - \Theta^2 \nabla_h N, \qquad \Theta^2 \equiv -\frac{g_0}{\rho_0}\frac{\partial \hat{\rho}}{\partial Z} \tag{46}$$

where the two forcing terms in the righthand side are the vertical gradients of horizontal pressure gradient and horizontal gravity force. Relative importance between the two forcing terms can be estimated quantitatively using publicly available data for sea-water density ρ and geoid undulation *N*.

7. Nondimensional D Number

A depth-dependent non-dimensional *D* number is defined as the ratio between the two forcing-terms on the righthand side of Eq.(46)

$$D(Z) = \frac{O\left(\left|\Theta^2 \nabla_h N\right|\right)}{O\left(\left(g_0/\rho_0\right)\left|\nabla_h \rho\right|\right)}, \quad \rho = \hat{\rho} + \rho_0 \tag{47}$$

to identify the relative importance on ocean dynamics due to the horizontal gravity force versus to the horizontal pressure gradient force. Hereafter, the mean values are used to represent the orders of magnitude,

$$D(Z) = \frac{\text{Mean}\left(\left|\Theta^2 \nabla_h N\right|\right)}{\text{Mean}\left[\left(g_0/\rho_0\right)\left|\nabla_h \rho\right|\right]} \qquad (48)$$

Two publicly available datasets are used to obtained depth-dependent nondimensional D-number: (a) the global static gravity field model EIGEN-6C4 for the geoid N (Figure 2, from http://icgem.gfz-potsdam.de/home,) (b) the climatological annual mean temperature and salinity from the NCEI WOA23 to get the sea water density (ρ) data (from https://www.ncei.noaa.gov/access/world-ocean-atlas-2023/.)

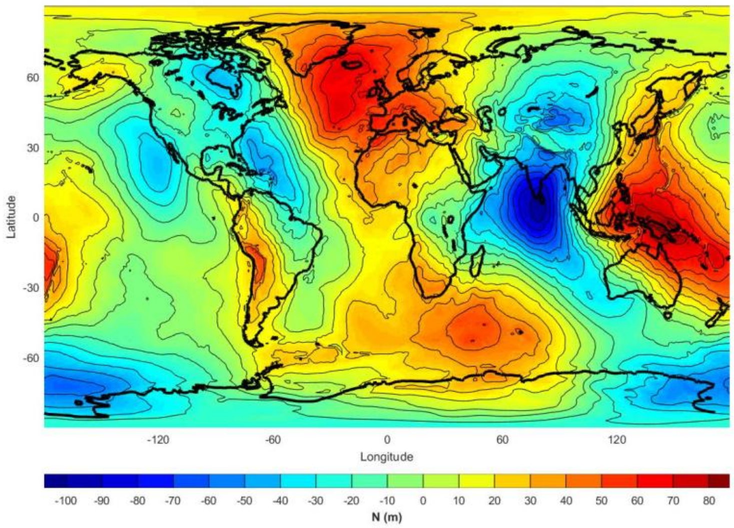

Figure 2. Digital data for EIGEN-6C4 geoid undulation (N) with 1°×1°, computed online at the website http://icgem.gfz-potsdam.de/home.

The WOA23 annual mean temperature and salinity data are used to compute ρ and Θ^2. The static gravity data EIGEN-6C4 is used to get N. With given (ρ, Θ^2, N), two vectors $(g_0/\rho_0)\nabla_h\rho$ and $\Theta^2\nabla_h N$ are computed at all grid points (1°×1°) and Z-levels (Z = 0 to -5,500 m) using the WOA23. Figures 3 and 4 show the contour plots and Figures 5 and 6 show the histograms of $(g_0/\rho_0)|\nabla_h\rho|$ and $|\Theta^2\nabla_h N|$ for the four levels, Z = 0, -500, -1,000, -2,000 m. The magnitude $(g_0/\rho_0)|\nabla_h\rho|$ has the mean of 13.45 Eotvos (1 Eotvos = $10^{-9}s^{-2}$) at z = 0, 2.154 Eotvos at Z = -500 m, 1.245 Eotvos at Z = -1,000 m, and 0.5615 Eotvos at Z = - 2,000 m (Figure 5). The magnitude $|\Theta^2\nabla_h N|$ has mean of 1.128 Eotvos at Z = 0, 0.4789 Eotvos at Z = -500 m, 0.4389 Eotvos at Z = -1,000 m, and 0.3894 Eotvos at Z = -2,000 m (Figure 6). The D number (Figure 7, Table 1) increases with depth monotonically from 8.4% at Z = 0, 22.2% at Z = -500 m, 35.3% at Z = -1,000 m, 69.3% at Z = -2,000 m, 81.4% at Z = -3,000 m, 108.7% at Z = -4,000 m, and 157.6% at Z = -5,000 m. These D numbers demonstrate the importance of the horizontal gravity force .

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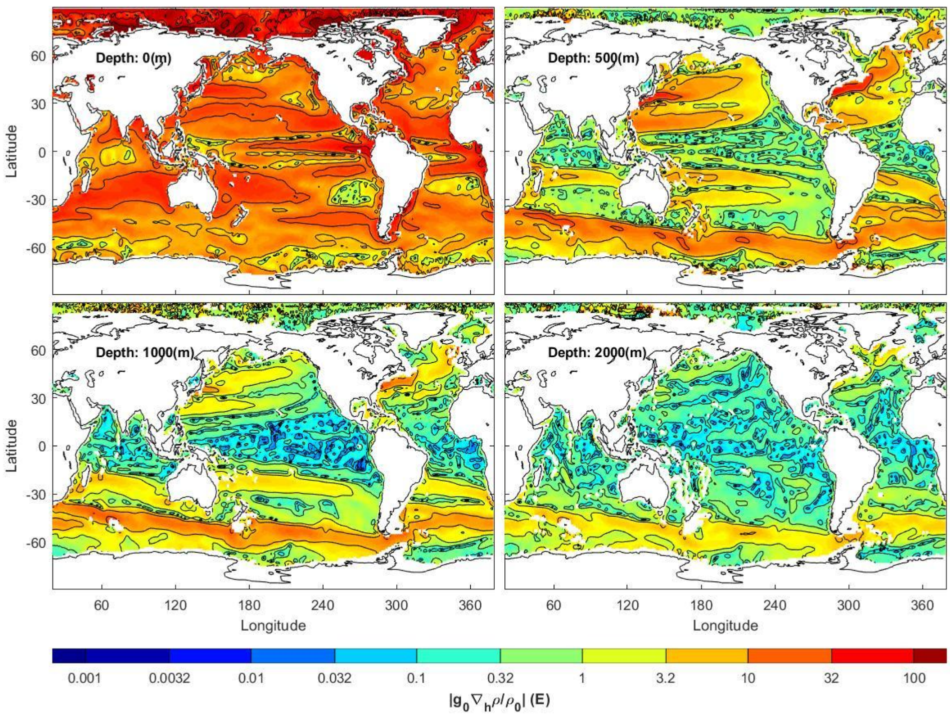

Figure 3. Horizontal contour plots of the magnitudes $|(g_0/\rho_0)\nabla_h \rho|$ in the unit of Eotvos (E) (1 E = 10^{-9} s^{-2}) at the four levels (Z = 0, -500 m, -1,000 m, and -2,000 m).

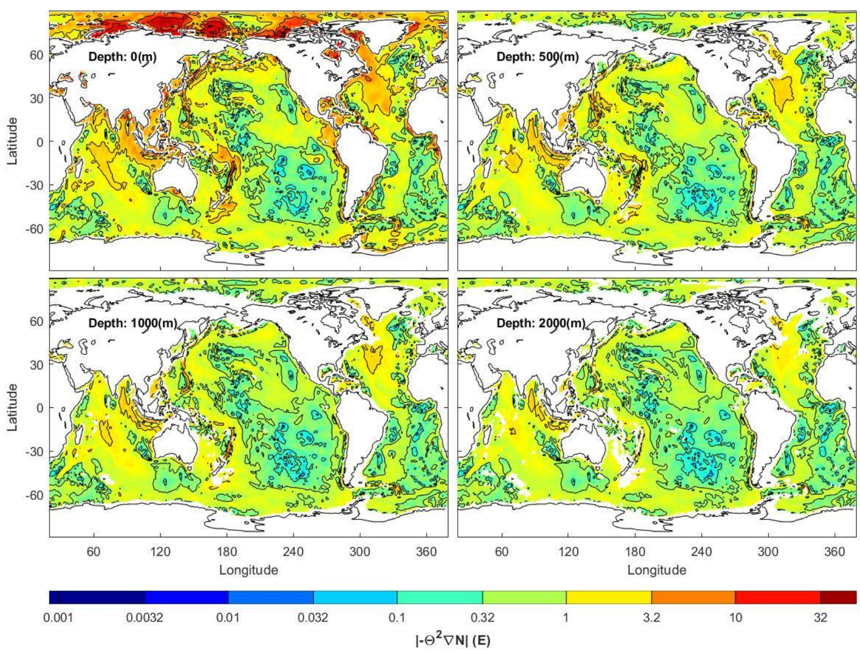

Figure 4. Horizontal contour plots of the magnitudes $|\Theta^2 \nabla_h N|$ in the unit of Eotvos (E) (1 E = 10^{-9} s^{-2}) at the four levels (Z = 0, -500 m, -1,000 m, and -2,000 m).

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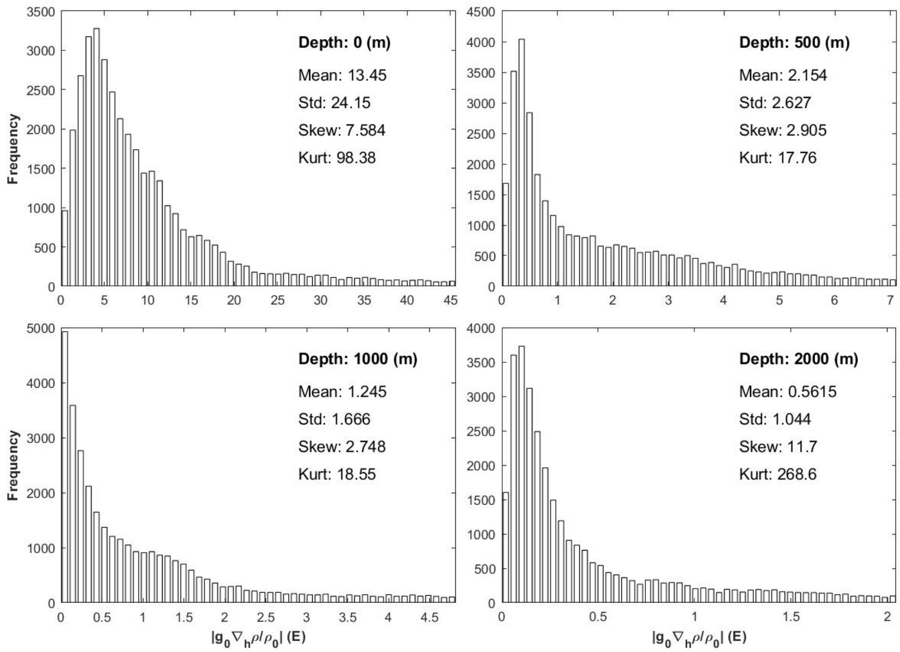

Figure 5. Histograms of the magnitudes $|(g_0/\rho_0)\nabla_h \rho|$ in the unit of Eotvos (E) (1 E = 10^{-9} s^{-2}) at the four levels (Z = 0, -500 m, -1,000 m, and -2,000 m).

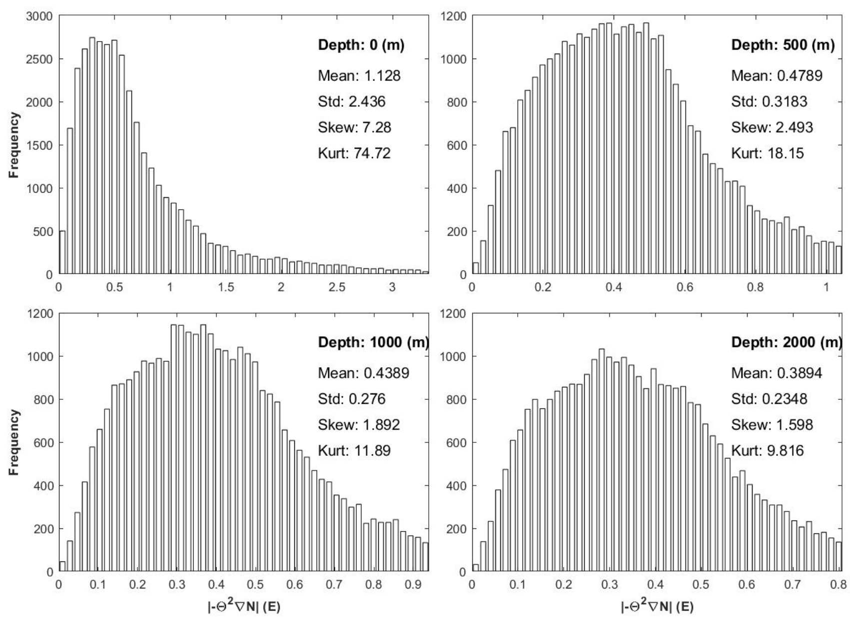

Figure 6. Histograms of the magnitudes $|\Theta^2\nabla_h N|$ in the unit of Eotvos (E) (1 E = 10^{-9} s^{-2}) at the four levels (Z = 0, -500 m, -1,000 m, and -2,000 m).

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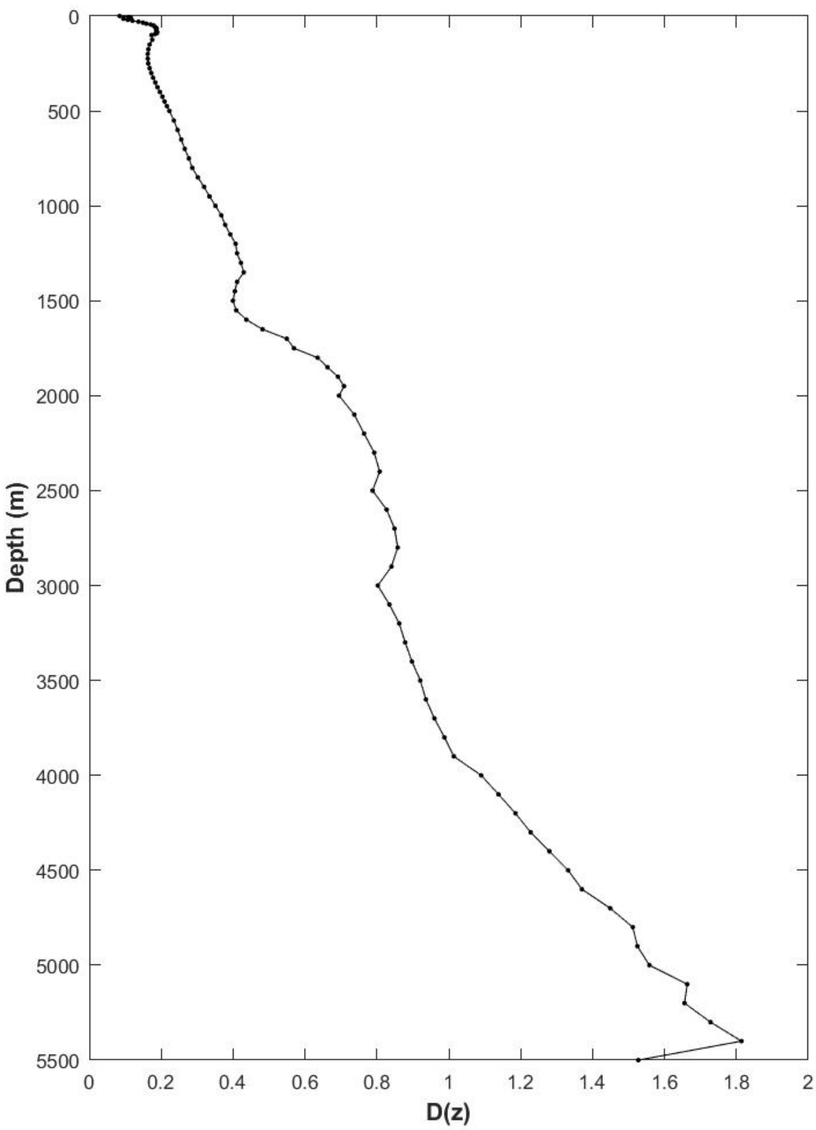

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Figure 7. Depth dependent *D*-number calculated from the EIGEN-6C4 and WOA23 datasets.

Table 1. Depth-dependent D number calculated from the EIGEN-6C4 and WOA23 datasets.

Depth (m)	0	500	1,000	2,000	3,000	4,000	5,000
D-Number	0.084	0.222	0.353	0.693	0.814	1.087	1.576

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As a summary, we list the comparison between the local Cartesian and geopotential coordinates in Table 2.

Table 2. Comparison between the local Cartesian and geopotential coordinates

	Local Cartesian Coordinates	Geopotential Coordinates
Independent Variables	(ξ, η, ζ)	(x, y, Z): $x = \xi,\ y = \eta,\ Z = -\Phi/g_0$
Unit Vectors	$(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}})$ - (east, north, "up")	$\hat{\mathbf{x}} = \hat{\boldsymbol{\xi}},\ \hat{\mathbf{y}} = \hat{\boldsymbol{\eta}},\ \hat{\mathbf{Z}} = -\nabla_C\Phi/\lvert\nabla_C\Phi\rvert$
Gradient Operator	$\nabla_C = \hat{\boldsymbol{\xi}}\partial_\xi + \hat{\boldsymbol{\eta}}\partial_\eta + \hat{\boldsymbol{\zeta}}\partial_\zeta$	$\nabla_G = \hat{\mathbf{x}}(\partial_x + \partial_x N\partial_Z) + \hat{\mathbf{y}}\left(\partial_y + \partial_y N\partial_Z\right) + \hat{\mathbf{Z}}\,\partial_Z$
Gravity	$\mathbf{g} = \nabla_C\Phi = \partial_\xi\Phi\hat{\boldsymbol{\xi}} + \partial_\eta\Phi\hat{\boldsymbol{\eta}} + \partial_\zeta\Phi\hat{\boldsymbol{\zeta}}$	$\mathbf{g} = \nabla_G\Phi_G = -g_0\hat{\mathbf{Z}}$
Pressure Gradient	$\nabla_C\hat{p} = \hat{\boldsymbol{\xi}}\partial_\xi\hat{p} + \hat{\boldsymbol{\eta}}\partial_\eta\hat{p} + \hat{\boldsymbol{\zeta}}\partial_\zeta\hat{p}$	$\nabla_G\hat{p} = \hat{\mathbf{x}}(\partial_x p + \partial_x N\partial_Z\hat{p}) + \hat{\mathbf{y}}\left(\partial_y p + \partial_y N\partial_Z\hat{p}\right) + \hat{\mathbf{Z}}(\partial_Z\hat{p})$
Hydrostatic Equilibrium	$\partial_\zeta\hat{p} = -\hat{\rho}\ g_0$	$\partial_Z\hat{p} = -\hat{\rho}\ g_0$
Horizontal Gravity Force	$\hat{\rho}\left(\partial_\xi\Phi\hat{\boldsymbol{\xi}} + \partial_\eta\Phi\hat{\boldsymbol{\eta}}\right)$	$\hat{\rho}g_0\left(\partial_x N\hat{\mathbf{x}} + \partial_y N\hat{\mathbf{y}}\right)$
Derivatives	$\partial_x = \partial_\xi - \left(Z_\xi/Z_\zeta\right)\partial_\zeta,\quad \partial_y = \partial_\eta - \left(Z_\eta/Z_\zeta\right)\partial_\zeta,\quad \partial_Z = \left(1/Z_\zeta\right)\partial_\zeta,\quad \partial_\xi N = \partial_x N,\quad \partial_\eta N = \partial_y N,\quad \partial_\zeta Z = 1 + O(10^{-5})$	

7. Conclusions

Existence of horizontal gravity force in the geopotential coordinates is proved via standard approach using the covariant and contravariant metric tensors between the local Cartesian and geopotential coordinates. Two publicly available datasets are used to evaluate the relative importance of horizontal gravity force (i.e., due to the geoid undulation) versus density on large-scale ocean dynamics through the non-dimensional D-number: (a) the global static gravity field model EIGEN-6C4 for the geoid undulation N and (b) the climatological annual mean temperature and salinity from the NCEI WOA23 for sea water density (ρ). Relative effect of horizontal gravity force versus density on ocean dynamics is weak (8.4%) at Z = 0, enhances monotonically with depth to 69.3% at Z = -2,000 m, and to 108.7% at Z = -4,000 m. Five concerns with M24 are presented in Appendix. Its title "there is no horizontal gravity force in geopotential coordinates" should be corrected into "there is no horizontal gravity force in geopotential coordinates for the gravity with no horizontal component."

Appendix A. Five Concerns with M24

The preprint was sent to Dr. McWilliams on 6 January 2026 for review and comments but no responses from him. There are five major concerns with M24.

First, the author of M24 subjectively writes the horizontal momentum equations in the geopotential coordinates without any justification,

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$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\partial_x p, \;\; \frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\partial_y p$ [questionable equation (6) in M24]

Second, the author of M24 did not use the standard covariant and contravariant metric tensors depicted in Section-3 to establish the relationship between the geopotential and local Cartesian coordinates, but just used $Z = -\Phi/g_0$ to get

$$\hat{\mathbf{Z}} = \frac{1}{|\nabla\Phi|}\left(\Phi_\xi\hat{\boldsymbol{\xi}} + \Phi_\eta\hat{\boldsymbol{\eta}} + \Phi_\zeta\hat{\boldsymbol{\zeta}}\right) = \frac{1}{|\nabla Z|}\left(Z_\xi\hat{\boldsymbol{\xi}} + Z_\eta\hat{\boldsymbol{\eta}} + Z_\zeta\hat{\boldsymbol{\zeta}}\right) \quad (A1)$$

Then he defined the unit vector $\hat{\boldsymbol{\zeta}}$ by [see equation (4) in M24],

$$\hat{\boldsymbol{\zeta}} \equiv -\langle\frac{\nabla\Phi}{|\nabla\Phi|}\rangle = \langle\hat{\mathbf{Z}}\rangle$$

where the angle brackets denote a volume average over the domain of interest. Later, the author of SM24 changed it into

$$\hat{\boldsymbol{\zeta}} \equiv \frac{\langle\hat{\mathbf{Z}}\rangle}{|\langle\hat{\mathbf{Z}}\rangle|} \quad (A2)$$

because $\langle\hat{\mathbf{Z}}\rangle$ is not a unit vector.

Since there is no spatial variation in the unit vectors $(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}})$ of the local Cartesian coordinates, the volume average of Eq.(A1) leads to

$$\langle\hat{\mathbf{Z}}\rangle = \langle\frac{Z_\xi}{|\nabla Z|}\rangle\hat{\boldsymbol{\xi}} + \langle\frac{Z_\eta}{|\nabla Z|}\rangle\hat{\boldsymbol{\eta}} + \langle\frac{Z_\zeta}{|\nabla Z|}\rangle\hat{\boldsymbol{\zeta}} \quad (A3)$$

Relationship between $(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}})$ and $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \langle\hat{\mathbf{Z}}\rangle)$ is given by

$$\hat{\boldsymbol{\xi}} = \hat{\mathbf{x}}, \qquad \hat{\boldsymbol{\eta}} = \hat{\mathbf{y}}, \qquad \hat{\boldsymbol{\zeta}} = \left(\langle\hat{\mathbf{Z}}\rangle - \langle\frac{Z_\xi}{|\nabla Z|}\rangle\hat{\mathbf{x}} - \langle\frac{Z_\eta}{|\nabla Z|}\rangle\hat{\mathbf{y}}\right)\Big/\langle\frac{Z_\zeta}{|\nabla Z|}\rangle \quad (A4)$$

However, (A4) is mistakenly replaced in M24 by

$$\hat{\boldsymbol{\xi}} = \hat{\mathbf{x}}, \qquad \hat{\boldsymbol{\eta}} = \hat{\mathbf{y}}, \qquad \hat{\boldsymbol{\zeta}} = \frac{1}{Z_\zeta}\left(|\nabla Z|\hat{\mathbf{Z}} - Z_\xi\hat{\mathbf{x}} - Z_\eta\hat{\mathbf{y}}\right) \quad \text{[see equation (20) in M24]}$$

Third, incorrect equation (20) in M24 is used to derive the transformation of pressure gradient between the local Cartesian and geopotential coordinates,

$$\nabla p = \partial_\xi p\hat{\boldsymbol{\xi}} + \partial_\eta p\hat{\boldsymbol{\eta}} + \partial_\zeta p\hat{\boldsymbol{\zeta}} = \partial_x p\hat{\mathbf{x}} + \partial_y p\hat{\mathbf{y}} + |\nabla Z|\partial_Z p\hat{\mathbf{Z}} \quad \text{[see equation (21)in M24]}$$

which is the key equation to support the conclusion of no horizontal gravity force in geopotential coordinates in M24, but it is incorrect.

Fourth, use Eq.(A4) for the transformation of the two sets of vectors $(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}})$ and $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \langle\hat{\mathbf{Z}}\rangle)$. Take similar procedure as in M24 from equation (20) to equation (21) (both in M24), i.e., substitute Eq.(A4) and Eq.(19) in the main text into the pressure gradient in the local Cartesian coordinates (∇p),

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$$\nabla p = \partial_\xi p\hat{\boldsymbol{\xi}} + \partial_\eta p\hat{\boldsymbol{\eta}} + \partial_\zeta p\hat{\boldsymbol{\zeta}}$$
$$= \left(\partial_x p + Z_\xi \partial_Z p - Z_\zeta \partial_Z p \left\langle\frac{Z_\xi}{|\nabla Z|}\right\rangle \Big/ \left\langle\frac{Z_\zeta}{|\nabla Z|}\right\rangle\right)\hat{\mathbf{x}}$$
$$+\left(\partial_y p + Z_\eta \partial_Z p - Z_\zeta \partial_Z p \left\langle\frac{Z_\eta}{|\nabla Z|}\right\rangle \Big/ \left\langle\frac{Z_\zeta}{|\nabla Z|}\right\rangle\right)\hat{\mathbf{y}} + \left(Z_\zeta \Big/ \left\langle\frac{Z_\zeta}{|\nabla Z|}\right\rangle\right)\partial_Z p\langle\hat{\mathbf{Z}}\rangle \quad \text{(A5)}$$

Then substitute Eq.(A2) and ($\hat{\mathbf{x}} = \hat{\boldsymbol{\xi}}$, $\hat{\mathbf{y}} = \hat{\boldsymbol{\eta}}$) in Eq.(A5),

$$\nabla p = \partial_\xi p\hat{\boldsymbol{\xi}} + \partial_\eta p\hat{\boldsymbol{\eta}} + \partial_\zeta p\hat{\boldsymbol{\zeta}}$$
$$= \left(\partial_x p + Z_\xi \partial_Z p - Z_\zeta \partial_Z p \left\langle\frac{Z_\xi}{|\nabla Z|}\right\rangle \Big/ \left\langle\frac{Z_\zeta}{|\nabla Z|}\right\rangle\right)\hat{\boldsymbol{\xi}} + \left(\partial_y p + Z_\eta \partial_Z p - Z_\zeta \partial_Z p \left\langle\frac{Z_\eta}{|\nabla Z|}\right\rangle \Big/ \left\langle\frac{Z_\zeta}{|\nabla Z|}\right\rangle\right)\hat{\boldsymbol{\eta}}$$
$$+ |\langle\hat{\mathbf{Z}}\rangle| \left(Z_\zeta \Big/ \left\langle\frac{Z_\zeta}{|\nabla Z|}\right\rangle\right)\partial_Z p\hat{\boldsymbol{\zeta}} \quad \text{(A6)}$$

which is NOT the transformation of pressure gradient from the local Cartesian to geopotential coordinates, but from $(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}})$ to $(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}})$ with no sense.

Fifth, if equation (20) in M24 is assumed right to represent the relationship between $(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}})$ and $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{Z}})$, we have

$$\hat{\boldsymbol{\zeta}} = \frac{1}{Z_\zeta}\left(|\nabla Z|\hat{\mathbf{Z}} - Z_\xi\hat{\mathbf{x}} - Z_\eta\hat{\mathbf{y}}\right) = \langle\hat{\boldsymbol{\zeta}}\rangle = \left\langle\frac{1}{Z_\zeta}\left(|\nabla Z|\hat{\mathbf{Z}} - Z_\xi\hat{\mathbf{x}} - Z_\eta\hat{\mathbf{y}}\right)\right\rangle \quad \text{(A7)}$$

due to no spatial variability of $\hat{\boldsymbol{\zeta}}$. Equation (A7) can be rewritten by

$$\frac{|\nabla Z|}{Z_\zeta}\hat{\mathbf{Z}} = \left\langle\frac{|\nabla Z|}{Z_\zeta}\hat{\mathbf{Z}}\right\rangle, \quad \frac{Z_\xi}{Z_\zeta} = \left\langle\frac{Z_\xi}{Z_\zeta}\right\rangle \equiv \alpha = const, \qquad \frac{Z_\eta}{Z_\zeta} = \left\langle\frac{Z_\eta}{Z_\zeta}\right\rangle \equiv \beta = const \quad \text{(A8)}$$

which is equivalent to

$$Z(\xi,\eta,\zeta) = F(\kappa), \;\; \kappa \equiv \zeta + \alpha\xi + \beta\eta, \;\; F(\kappa) \to any\ differentiable\ function \quad \text{(A9)}$$

It clearly requires that the geopotential Φ has only one independent variable κ for its spatial variability,

$$\Phi(\xi,\eta,\zeta) = -gF(\kappa) \quad \text{(A10)}$$

which cannot be satisfied for any physically realistic geopotential Φ except for $\alpha = 0$, $\beta = 0$, i.e.,

$$Z_\xi = 0, \; Z_\eta = 0 \quad \text{(A11)}$$

which represents the geopotential Φ with no horizontal variation, i.e., no horizontal gravity force, and the geopotential coordinates coincident with the local Cartesian coordinates. Eq.(A11) is the necessary condition for the validity of equation (21) in M24

$$\nabla p = \partial_\xi p\hat{\boldsymbol{\xi}} + \partial_\eta p\hat{\boldsymbol{\eta}} + \partial_\zeta p\hat{\boldsymbol{\zeta}} = \partial_x p\hat{\mathbf{x}} + \partial_y p\hat{\mathbf{y}} + |\nabla Z|\partial_Z p\hat{\mathbf{Z}}$$

Thus, M24 needs to assume no horizontal gravity force to prove "there is no horizontal gravity force in geopotential coordinates". The title of M24 "There is no horizontal gravity force in geopotential coordinates" should be "There is no horizontal gravity force in geopotential coordinates for gravity with no horizontal component".

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Code, data, or code and data availability

The data used in this study are publicly available with the climatological annual mean temperature and salinity from NOAA/NCEP website https://www.ncei.noaa.gov/products/world-ocean-atlas and the geoid undulation (*N*) data from the International Centre for Global Earth Models (ICGEM) website https://icgem.gfz.de/home

Supplement link

There is no supplement link.

Author contributions

PCC conducted the research and prepared the manuscript.

Competing interests

There are no competing interests.

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