2026-05-12 Review by Chris W. Hughes of

Revealing horizontal gravity force in geopotential coordinates via metric tensors

by

Peter C. Chu

Submitted to Ocean Science.

General Comments

The author continues to insist that there are horizontal components of gravity, and simply contradicts without explanation (just providing more, unilluminating maths) the descriptions of errors in his analysis. My recommendation remains to reject the paper for publication.

More detailed analysis

It is hard to know how to explain the problem more clearly, so I will attempt a simple example. The equation of motion is given as (2):

$$\frac{D\boldsymbol{u}}{Dt} = -2\boldsymbol{\Omega} \times \boldsymbol{u} - \frac{1}{\rho}\nabla p + \nabla\Phi,$$

Where Φ increases downwards (the geodetic convention) and $g = \nabla\Phi$. A solution of this with $\boldsymbol{u} = 0$ and $D\boldsymbol{u}/Dt = 0$ is given by

$$p = p(\Phi); \quad \rho = \rho(\Phi)$$

with the condition $dp/d\Phi = \rho$.

This is simple to check, by using $\nabla p = \frac{dp}{d\Phi}\nabla\Phi$, which shows that $\nabla p = \rho\nabla\Phi$.

This describes a situation in which pressure and density are each constant along every geopotential surface: hydrostatic equilibrium. Such a physical balance holds independent of the coordinate system used to describe it.

The author subtracts off a background hydrostatic balance, replacing p and ρ with $\hat{p}$ and $\hat{\rho}$, but this plays no role in the derivation: we can interpret $\hat{p}$ and $\hat{\rho}$ as differences from any self-consistent background values, including zero, so they can be the same as p and ρ. Given this, (35) reads

$$\nabla_G p = (\partial_x p + \partial_x N \partial_Z p)\hat{\boldsymbol{x}} + (\partial_y p + \partial_y N \partial_Z p)\hat{\boldsymbol{y}} + (\partial_Z p)\hat{\boldsymbol{Z}}$$

in which ∂_x and ∂_y represent differentiation at constant geopotential (x, y and Z are non-orthogonal coordinates with Z being truly vertical, perpendicular to geopotential surfaces). For our hydrostatic equilibrium situation, $p = p(Z)$ and $\partial_Z p = \rho \partial_Z \Phi = -\rho g$, so the equation can be written:

$$\nabla_G p = (\partial_x p - \rho g \partial_x N)\hat{\boldsymbol{x}} + (\partial_y p - \rho g \partial_y N)\hat{\boldsymbol{y}} - \rho g \hat{\boldsymbol{Z}}.$$

Furthermore, since $p = p(\Phi)$ is constant along geopotentials, the $\partial_x p$ and $\partial_y p$ terms are both zero, leaving

$$\nabla_G p = -\rho g \partial_x N \hat{\boldsymbol{x}} - \rho g \partial_y N \hat{\boldsymbol{y}} - \rho g \hat{\boldsymbol{Z}}.$$

But we know from the full solution that the pressure gradient term is purely vertical in this case, so the $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ terms must be spurious, and (35) therefore incorrect.

On the other hand, if we interpret $\partial_x p$ and $\partial_y p$ as $\partial_\xi p$ and $\partial_\eta p$ (i.e. differentiation at constant Cartesian "vertical" coordinate ζ), then we find that

$$\partial_\xi p + \partial_\xi N \partial_Z p \approx \partial_x p$$

and the spurious lateral terms disappear (the approximation is due to the Z coordinate not being exactly perpendicular to ξ and η, and is second order in that small angle). In this case the interpretation of (35) is that the differentials are in the Cartesian frame, and not at constant geopotential, so there is a gravity force along this approximate "horizontal", but it disappears along the true horizontal.

This may explain the author's confusion, since his rebuttal focuses so strongly on the definitions of ocean model grids which are often described as spherical. It is true that an exactly spherical coordinate system applied to the real geometry of the earth would miss important lateral gravity terms, since constant radius would then not be constant geopotential. However, this is not the correct interpretation of such models. They are models in which the vertical coordinate is exactly aligned with gravity, and the gravity field is approximated as spherical. In these models, a more accurate representation of the gravity field would not add lateral gravity terms, but would add complications to the geometry of the coordinates. There is no missing force, only missing metric terms, which show up in the acceleration term.

The case of an exactly spherically symmetric gravity field illustrates this nicely. In this case, writing velocities u, v, w in the directions of longitude λ, latitude ϕ and radius r, with unit vectors in those three mutually perpendicular directions $\hat{\boldsymbol{\lambda}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{r}}$, the eastward component of the acceleration term is

$$(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \cdot \hat{\boldsymbol{\lambda}} = \left(\frac{u}{r \cos \phi} \partial_\lambda \boldsymbol{u} + \frac{v}{r} \partial_\phi \boldsymbol{u} + w \partial_r \boldsymbol{u} \right) \cdot \hat{\boldsymbol{\lambda}}.$$

Expanding $\boldsymbol{u} = u\hat{\boldsymbol{\lambda}} + v\hat{\boldsymbol{\phi}} + w\hat{\boldsymbol{r}}$, we see that we have to differentiate not only u, but also the unit vectors, which depend on longitude too.

The result is

$$(\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \cdot \hat{\boldsymbol{\lambda}} = \boldsymbol{u} \cdot \nabla u + \left(\frac{u}{r \cos \phi}\right)(v \sin \phi - w \cos \phi),$$

which looks like advection of u (as it would appear in a Cartesian representation), plus extra terms. There will be similar extra terms in the northward and radial components of the momentum equation.

These "metric terms", the ones proportional to uv and uw (and to other products of velocity components in the northward and radial equations), result from differentiating the unit vectors, and are therefore a result of the curvature of the coordinate system. When the coordinate system is more complex so that it follows non-spherical geopotentials, the metric terms will become more complicated, but there will still be no "horizontal" gravity term as long as the vertical coordinate is in the direction of gravity. The mass conservation equation may also acquire new metric terms if, as in the case of a rotating spheroidal earth, the strength of gravity varies with position.

The author is thus correct that there are terms resulting from the non-spherical nature of the true geoid which are neglected in ocean models, but he is wrong about where they occur. In a model for which gravity acts antiparallel to the vertical coordinate, the missing terms are metric terms in the acceleration.

It would be of interest to write these terms explicitly in order to be able to explore their size, but the expectation is that they are very small. As Staniforth and White (2025) argue, they should be much smaller than the already small spheroidal terms, and as Hughes (2002) noted, in the Traditional Approximation with hydrostatic balance in the vertical, the pressure torques on the earth (and, reciprocally, from the earth on the ocean) resulting from lack of rotational symmetry are balanced by gravitational torques due to the ocean mass acting on the earth's asymmetric mass distribution (and vice versa).

References

Hughes, C.W.: Torques exerted by a shallow fluid on a non-spherical, rotating planet. *Tellus* **54A**(1) 56-62, doi: 10.1034/j.1600-0870.2002.00326.x, 2002.

Staniforth, A., and White, A.: Almost everything you always wanted to know about representing gravity in global models but were afraid to ask. *J. Adv. Modeling Earth Syst.*, 17, e2024MS004271. https://doi.org/10.1029/2024MS004271, 2025.