Review of “Revealing horizontal gravity force in geopotential coordinates via metric tensors” by Peter C. Chu

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This manuscript aims to formulate equations of motion applicable to atmospheric and oceanic fluids in geopotential coordinates. It claims to show that horizontal (i.e. along-geopotential) components of the gravitational acceleration should be included in the equations of motion, and purports to explicitly refute a recent derivation by McWilliams (2024).

As another comment has already noted, the author’s pursuit is fundamentally ill-posed because gravitational acceleration is, by definition, perpendicular to geopotentials. Thus, the along-geopotential gravitational force is guaranteed to be zero. The author’s “horizontal gravity” terms are spurious, resulting from mathematical errors in the formulation of the inverse metric tensor, and from incorrect normalization of the basis vectors in formulating the gradient.

Below the these mathematical issues are detailed, in the hope of settling this issue firmly. This material draws on various standard calculus results in curvilinear coordinates (e.g. Arfken et al., 2013). The outcome is that, following mathematical corrections, the author’s derivation exactly reproduces the result of McWilliams (2024). This confirms the expectation (and physical necessity) that there be no horizontal gravitational force in geopotential coordinates.

1. Coordinate transformation

Let the reference Cartesian coordinates be

$$(\xi, \eta, \zeta), \quad (\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}}) \text{ orthonormal},$$

following the author’s convention. We also define geopotential coordinates following the author’s convention:

$$x = \xi, \quad y = \eta, \quad Z = \zeta + N(x, y),$$

so that

$$\zeta = Z - N(x, y).$$

Note that this implies that positive N corresponds to a depression of the geoid, in contrast with standard definitions. The position vector can then be written as

$$\boldsymbol{r}(x, y, Z) = x\,\hat{\boldsymbol{\xi}} + y\,\hat{\boldsymbol{\eta}} + \big(Z - N(x, y)\big)\hat{\boldsymbol{\zeta}}.$$

2. Metric tensor (Eq. 25)

The construction of the metric tensor, and of the gradient operator, leverage the covariant basis vectors $\boldsymbol{a}_i$. These vectors lie tangent to the coordinate curves, e.g. $\boldsymbol{a}_x$ lies tangent to the curve obtained by varying x while keeping y and Z held fixed:

$$\boldsymbol{a}_i = \frac{\partial \boldsymbol{r}}{\partial q^i}, \quad q^i = (x, y, Z),$$

so

$$\boldsymbol{a}_x = \hat{\boldsymbol{\xi}} - N_x\hat{\boldsymbol{\zeta}}, \quad \boldsymbol{a}_y = \hat{\boldsymbol{\eta}} - N_y\hat{\boldsymbol{\zeta}}, \quad \boldsymbol{a}_Z = \hat{\boldsymbol{\zeta}}.$$

Note that we use the notation $\boldsymbol{a}$, in contrast to the author's $\boldsymbol{e}$, to emphasize that these are not unit vectors. The metric tensor is defined by

$$g_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j = \frac{\partial \boldsymbol{r}}{\partial q^i} \cdot \frac{\partial \boldsymbol{r}}{\partial q^j}.$$

Thus

$$[g_{ij}] = \begin{pmatrix} 1 + N_x^2 & N_x N_y & -N_x \\ N_x N_y & 1 + N_y^2 & -N_y \\ -N_x & -N_y & 1 \end{pmatrix}.$$

3. Inverse metric (Eq. 26)

The author uses the inverse metric tensor to construct the gradient in geopotential coordinates. The inverse metric satisfies

$$g^{ik} g_{kj} = \delta^i_j,$$

and can also be written as

$$g^{ij} = \nabla q^i \cdot \nabla q^j.$$

Thus,

$$[g^{ij}] = \begin{pmatrix} 1 & 0 & N_x \\ 0 & 1 & N_y \\ N_x & N_y & 1 + N_x^2 + N_y^2 \end{pmatrix}.$$

This highlights a key error in the derivation presented in the manuscript: the inverse metric tensor $[g^{ij}]$ (Eq. (27)) is incorrect. The author's Eq. (27) is constructed from derivatives of the form $\partial x^i / \partial \xi^j$ and corresponds to the dot products of the vectors ∇q^i, but it is incorrectly identified as the inverse of the metric tensor defined in Eq. (25). As a result, it does not satisfy $g^{ik} g_{kj} = \delta^i_j$ when paired with the metric tensor defined in Eq. (25).

4. Dual basis vectors (Eqs. 28–30)

Following McWilliams (2024), the author aims to write the gradient in terms of the contravariant basis vectors. Each contravariant basis vector $\boldsymbol{a}^i$ lies perpendicular to isosurfaces of the coordinate q^i:

$$\boldsymbol{a}^i = \nabla q^i,$$

with

$$\boldsymbol{a}^x = \hat{\boldsymbol{\xi}}, \quad \boldsymbol{a}^y = \hat{\boldsymbol{\eta}}, \quad \boldsymbol{a}^Z = N_x\hat{\boldsymbol{\xi}} + N_y\hat{\boldsymbol{\eta}} + \hat{\boldsymbol{\zeta}}.$$

Note again that these are not unit vectors.

5. Gradient operator (Eq. 31)

One way to formulate the gradient of a scalar field, as noted by the author, is

$$\nabla f = g^{ij} (\partial_j f)\, \boldsymbol{a}_i.$$

In formulating the pressure gradient, the author makes two errors: (i) they use an incorrect form of the inverse metric tensor; (ii) they replace the basis vectors with normalized vectors, and then use these in the metric-based gradient formula. The latter step is not valid, because the metric identities apply only to the unnormalized basis vectors.

Using the correct forms of g^{ij} and $\boldsymbol{a}_i$ yields

$$\nabla p = (p_x + N_x p_Z)\hat{\boldsymbol{\xi}} + (p_y + N_y p_Z)\hat{\boldsymbol{\eta}} + p_Z\hat{\boldsymbol{\zeta}}.$$

This can be equivalently written in terms of the reciprocal basis as

$$\nabla p = p_x \boldsymbol{a}^x + p_y \boldsymbol{a}^y + p_Z \boldsymbol{a}^Z = (\partial_j p)\boldsymbol{a}^j,$$

which is an alternative standard formulation of the gradient operator.

6. Reconciliation with McWilliams (2024)

McWilliams (2024) defines the following unit vectors for a geopotential coordinate system:

$$\boldsymbol{b}_x = \hat{\boldsymbol{\xi}}, \quad \boldsymbol{b}_y = \hat{\boldsymbol{\eta}}, \quad \boldsymbol{b}_Z = \frac{\nabla Z}{|\nabla Z|}.$$

These basis vectors are normalized equivalents of the contravariant basis vectors. For example, from the relations above it is immediately clear that $\boldsymbol{b}_x = \boldsymbol{a}^x$ and $\boldsymbol{b}_y = \boldsymbol{a}^y$, and that

$$\nabla Z = \boldsymbol{a}^Z.$$

Thus $\boldsymbol{b}_Z$ is related to the quasi-vertical contravariant basis vector $\boldsymbol{a}^Z$ via

$$\boldsymbol{a}^Z = |\nabla Z|\, \boldsymbol{b}_Z.$$

Therefore, the pressure gradient can be rewritten exactly as

$$\nabla p = p_x \boldsymbol{b}_x + p_y \boldsymbol{b}_y + |\nabla Z| p_Z \boldsymbol{b}_Z,$$

which matches McWilliams (2024).

References

- Arfken, G. B., H. J. Weber, and F. E. Harris (2013), *Mathematical Methods for Physicists*, 7th ed., Academic Press.
- McWilliams, J. C. (2024), There is no horizontal gravity force in geopotential coordinates, *PNAS*.