

Review of  
'Revealing horizontal gravity force in  
geopotential coordinates via metric tensors'  
(egosphere-2026-2010)  
by Peter C. Chu

The manuscript claims to show that, in geopotential coordinates, gravity has a horizontal component that is unjustifiably neglected in numerical models of the ocean and atmosphere. There is patently something illogical about the idea that the geopotential  $\Phi$  should have a nonzero gradient along surfaces of constant  $\Phi$ . This should have alerted the author to the likelihood of an error in the mathematics and, indeed, there are several (see specific points below). I am therefore recommending that the manuscript be rejected.

### Specific points

1. There is already an issue in the second line of the abstract, since standard usage is that 'vertical' refers to the direction of  $\nabla\Phi$  and 'horizontal' refers to directions perpendicular to  $\nabla\Phi$ . But let us give the author the benefit of the doubt and suppose - at this point at least - 'horizontal' is defined in terms of some coordinate system such as spherical polars that is *approximately* aligned with gravity, since this meaning is also commonly used. Note, though, that it is important to be clear and precise, given the topic of the manuscript.

The author describes 'two approaches' supposedly used by modellers to eliminate horizontal gravity from their model equations. In reality the second approach is the one used, even if it is not always clearly explained as such. But let us address both.

#### First approach

2. Retaining the full 'bumpy' gravity field but writing the governing equations in some smooth coordinate system such as spherical polars or GREAT, we obtain an equation something like (8). (Actually, the author has made a couple of approximations to the terms involving the velocity, omitting a metric term and making the 'traditional' approximation, but that is not the most important thing here.)

For the present discussion, the important terms in (8) are the pressure gradient and geopotential gradient. In this approach we do indeed obtain a nonzero horizontal component to the geopotential gradient. However, the crucial point is that, to an excellent approximation, it is compensated by an (almost) equal and opposite horizontal pressure gradient term, because the atmosphere and ocean are very close to hydrostatic balance. In this approach, if we neglect the horizontal component of gravity, then the pressure gradient would also lose its horizontal hydrostatic component. Thus, we effectively omit two terms or contributions whose sum is virtually zero and so make very small error overall.

#### Second approach

The more usual approach is to adopt a geopotential coordinate so that, by construction, the horizontal component of gravity vanishes. The vanishing of the horizontal component is clear from (24). However, there are several errors in the subsequent mathematics that lead to an incorrect conclusion.

3. The expression (25) is correct and leads to the correct covariant metric tensor. However, the expression (26) is incorrect and leads to an incorrect expression for the contravariant metric tensor (27). One way to see this is to explicitly work out  $g_{ij}$  and multiply by the expression given for  $g^{ij}$  in (27); the answer is not the identity matrix.

Another way to see it is with a trivial 2D example. Suppose  $(\xi, \zeta)$  are Cartesian coordinates, and

$$x = \xi, \quad Z = \zeta + \alpha\xi,$$

where  $\alpha$  is constant. Then

$$dx = d\xi, \quad dZ = d\zeta + \alpha d\xi, \quad d\zeta = dZ - \alpha dx.$$

The metric is

$$ds^2 = d\xi^2 + d\zeta^2 = (1 + \alpha^2) dx^2 + dZ^2 - 2\alpha dx dZ,$$

showing that the metric tensor is

$$g_{ij} = \begin{pmatrix} 1 + \alpha^2 & -\alpha \\ -\alpha & 1 \end{pmatrix}$$

and its inverse is

$$g^{ij} = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 + \alpha^2 \end{pmatrix}.$$

Now

$$\frac{\partial \xi^p}{\partial x^i} = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \quad \text{and} \quad \frac{\partial x^p}{\partial \xi^i} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}.$$

Evaluating (25) then gives the above expression for  $g_{ij}$ ; however, evaluating (26) does not give the above expression for  $g^{ij}$ .

4. The unit vector  $\hat{\mathbf{Z}}$  (equation (16)) is related to the Cartesian coordinate unit vectors by

$$\hat{\mathbf{Z}} = \frac{1}{|\nabla Z|} (Z_\xi \hat{\boldsymbol{\xi}} + Z_\eta \hat{\boldsymbol{\eta}} + Z_\zeta \hat{\boldsymbol{\zeta}}).$$

where  $|\nabla Z| = (Z_\xi^2 + Z_\eta^2 + Z_\zeta^2)^{1/2}$ . The position vector  $\mathbf{r}$ , being a vector, is a coordinate-independent object. In terms of the Cartesian coordinates it is given by

$$\mathbf{r} = \xi \hat{\boldsymbol{\xi}} + \eta \hat{\boldsymbol{\eta}} + \zeta \hat{\boldsymbol{\zeta}}.$$

Using the above expression for  $\hat{\mathbf{Z}}$  to express  $\mathbf{r}$  in geopotential coordinates does not give the expression in the manuscript (between equations (27) and (28)).

5. As a result of the above errors, the expression (31) and its approximations (34a,b) for the gradient in geopotential coordinates are incorrect. We can obtain the correct expression as follows. For any function  $f$  of the spatial coordinates

$$\nabla f = f_\xi \hat{\boldsymbol{\xi}} + f_\eta \hat{\boldsymbol{\eta}} + f_\zeta \hat{\boldsymbol{\zeta}}.$$

Using the cited chain rule relations gives

$$\nabla f = (f_x + Z_\xi f_Z) \hat{\boldsymbol{\xi}} + (f_y + Z_\eta f_Z) \hat{\boldsymbol{\eta}} + Z_\zeta f_Z \hat{\boldsymbol{\zeta}},$$

and then using the above expression for  $\hat{\mathbf{Z}}$  leads, after some cancellation, to

$$\nabla f = f_x \hat{\mathbf{x}} + f_y \hat{\mathbf{y}} + |\nabla Z| f_Z \hat{\mathbf{Z}}.$$

This last expression agrees with equation (21) of M24. Thus, M24 is correct, and the last sentence of the abstract of the manuscript under review is incorrect.

I did not check the mathematics any further beyond this point. There are clearly significant problems with the manuscript that mean it is not publishable.