2026-04-23 Review by Chris W. Hughes of

Revealing horizontal gravity force in geopotential coordinates via metric tensors

by

Peter C. Chu

Submitted to Ocean Science.

General Comments

This paper purports to show that there is a, usually neglected, gravitational force acting along the horizontal direction in the ocean equations of motion when expressed in geopotential coordinates. My recommendation is that this paper be rejected, as its fundamental point is clearly incorrect.

Since "horizontal" is defined as the plane perpendicular to gravity, it is hard to see how there could ever be a horizontal component of gravity. There are exceptions to this, but they do not apply to the issue considered in the paper. If we considered "gravity" to be purely gravitational attraction to the earth's mass, ignoring the centrifugal force, but defined "horizontal" in the usual manner in the earth's rotating reference frame, thus including the centrifugal force in gravity when defining horizontal, then there could be a horizontal component of gravity. Alternatively, in tidal modelling, it is common to define the horizontal based on the time-averaged gravity field, and incorporate the time-dependent tidal forces as "horizontal" forces within the model. Neither of these interpretations is the issue in this paper.

The other possible interpretation is that "horizontal" is being used to represent some plane of the coordinate system which is informally referred to as "horizontal", but is not actually perpendicular to the gravity vector. Since gravity is (plus or minus, depending on convention) the gradient of the geopotential, geopotential surfaces are horizontal by definition, so this cannot be an issue for the equations in geopotential coordinates.

I believe that the correct interpretation is that "horizontal" in this paper turns out to mean "in the xy plane of an arbitrarily chosen local Cartesian coordinate system".

Specific Comments

The central result is equation (43):

$$LHS = -\nabla_h \hat{p} + \hat{\rho} g_0 \nabla_h N,$$

where $\nabla_h$is referred to as "horizontal", but is actually defined as

$$\nabla_h \equiv \hat{\boldsymbol{x}} \partial_x + \hat{\boldsymbol{y}} \partial_y.$$

Going back to the definitions around Figure 1, we see that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{\zeta}})$ are a set of mutually orthogonal unit vectors, in which $\hat{\boldsymbol{\zeta}}$ is not perpendicular to geopotentials, and therefore not vertical (later approximations show that it is assumed to be close enough to vertical for quadratic, but not linear terms to be neglected). That means the $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ directions are not actually horizontal, but this doesn't seem to be the biggest issue: since ∂_x and ∂_y represent derivatives at constant geopotential, there should be no force acting along the geopotential surface if pressure is constant along that surface, meaning that the gravity term should not appear. The issue is that the right-hand side of (43) is clearly incorrect.

In fact, the N term in (43) appears to perform the role of (approximately, ignoring terms of order δ^2 and higher) "correcting" the horizontal pressure gradient so that it represents a gradient at constant ζ . The form is very similar to the conventional relationship between the gradient of the 2D ocean bottom pressure field projected onto the horizontal, and the horizontal component of the 3D pressure gradient at the bottom: $\nabla_h p_b = (\nabla_h p)_b - \rho_b g \nabla_h z_b$, where subscript b means "evaluated at the bottom".

Thus, it appears that the "horizontal" gravity force in (43) is actually the component of gravity acting along the non-horizontal ζ surface along which the gradient of pressure is being evaluated. It is not an unexpected new force, but a known force which does not act along geopotential surfaces.

We can also see that there is a problem from the fact that the orientation of the Cartesian "vertical" is arbitrary (although it must be close to the true vertical given the approximations made). A small angular change δ in its orientation would make little difference to $-\nabla_h \hat{p}$, if the derivative is at constant Z, since it would then be a derivative at constant geopotential and the geopotential is not changing (there is a second order change in the derivative, proportional to $1 - \cos\delta$ because distances along the geopotential would slightly change with respect to the changing x and y distances). On the other hand, there would be a first order change in $\nabla_h N$, which represents the tilt of geopotential surfaces relative to the Cartesian "horizontal". In fact, in an extreme case for which $\hat{p}$ happens to be constant along geopotential surfaces, $-\nabla_h \hat{p}$ would be zero for any choice of Cartesian coordinate if ∇_h is truly evaluated along geopotentials. However, although $\nabla_h N$ could be zero if the Cartesian and geopotential coordinates happen to coincide, it would be nonzero for any relative tilt. The correct form of this equation must produce cancelling changes in these two terms for a small tilt of the Cartesian reference.

My tensor calculus is not up to identifying the error clearly, but I think the issue is that, in (35), $\nabla_G p$ should not simply be interpreted as ∇p expressed in geopotential coordinates, but as a contravariant vector, which is a different kind of object. This can be seen in (35) by considering $p = p(Z)$, a purely hydrostatic balance with $\partial_x p = \partial_y p = 0$, in which it is clear that the pressure gradient is parallel to $\hat{Z}$, but (35) produces nonzero $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ terms if N depends on x and y. It is therefore incorrect in (39) to equate a conventional vector on the left-hand side with a contravariant vector on the right.

Since the right-hand side of the momentum equation is what is in question here, that covers the most important issue. However, it is worth noting that there is also a problem with the left-hand side. To extract the horizontal component of the acceleration it is not sufficient to replace $\boldsymbol{u}$ with its horizontal component $\boldsymbol{U}$, as is done in (43), either for the Coriolis or the nonlinear term. In fact, the nonlinear term takes quite a complex form when evaluated in curvilinear (even spherical) coordinates, and it is here, and in the mass conservation equation, that any effect of the irregular geoid would appear in the equations when written in geopotential coordinates. Staniforth and White (2024) argue convincingly that the size of such perturbations would be very small, but no explicit derivation of the relevant equations has been attempted, to my knowledge.