Let Φ be the geopotential (contributions from the bumpy geoid included). Construct an orthogonal curvilinear coordinate system such that the vertical unit vector is parallel to gravity at every point:

$$\hat{e}_3 = \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{\boldsymbol{g}}{g},$$

where $\boldsymbol{g} = \nabla\Phi$ and $g = |\boldsymbol{g}|$ may be functions of space. Choose the horizontal unit vectors $\hat{e}_1$ and $\hat{e}_2$ to be perpendicular to $\hat{e}_3$. The horizontal unit vectors thus lie within the geopotential surface.[1] The coordinates themselves are defined by parallel transport of the vectors $\boldsymbol{e}_i$. It is always possible in principle to define such a coordinate system—at least locally—as long as there are no points where $\nabla\Phi = 0$, although explicit construction may be difficult in practice (e.g., Staniforth 2022; Staniforth and White 2025).

The three-dimensional momentum equations (without rotation for simplicity) are

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p - \nabla\Phi, \tag{1}$$

where $\boldsymbol{v}$ is the 3D velocity with components

$$\boldsymbol{v} = v^i\hat{\boldsymbol{e}}_i = u\hat{\boldsymbol{e}}_1 + v\hat{\boldsymbol{e}}_2 + w\hat{\boldsymbol{e}}_3.$$

(with the Einstein summation convention). We will also define the 3D position vector as

$$\boldsymbol{x} = x^i\boldsymbol{e}_i = x\hat{\boldsymbol{e}}_1 + y\hat{\boldsymbol{e}}_2 + z\hat{\boldsymbol{e}}_3.$$

Expanded into components, the three terms in the momentum equations are

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}(v^i\hat{\boldsymbol{e}}_i) = \hat{\boldsymbol{e}}_i\frac{\mathrm{D}v^i}{\mathrm{D}t} + v^i\frac{\mathrm{D}\hat{\boldsymbol{e}}_i}{\mathrm{D}t}, \tag{2}$$

$$\nabla p = \hat{\boldsymbol{e}}_i\frac{\partial p}{\partial x^i}, \tag{3}$$

$$\nabla\Phi = g\hat{\boldsymbol{e}}_3, \tag{4}$$

where the last equation follows from the definition of $\hat{\boldsymbol{e}}_3$. The two horizontal components are found by projecting the vector equation in the $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$ directions:

$$\frac{\mathrm{D}u}{\mathrm{D}t} - v_2\hat{\boldsymbol{e}}_2\cdot\frac{\mathrm{D}\hat{\boldsymbol{e}}_1}{\mathrm{D}t} + v_3\hat{\boldsymbol{e}}_1\cdot\frac{\mathrm{D}\hat{\boldsymbol{e}}_3}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{5}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + v_1\hat{\boldsymbol{e}}_2\cdot\frac{\mathrm{D}\hat{\boldsymbol{e}}_1}{\mathrm{D}t} + v_3\hat{\boldsymbol{e}}_2\cdot\frac{\mathrm{D}\hat{\boldsymbol{e}}_3}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{6}$$

$$\tag{7}$$

where we have used the fact that

$$\hat{\boldsymbol{e}}_i\cdot\frac{\mathrm{D}\hat{\boldsymbol{e}}_j}{\mathrm{D}t} + \hat{\boldsymbol{e}}_j\cdot\frac{\mathrm{D}\hat{\boldsymbol{e}}_i}{\mathrm{D}t} = 0 \tag{8}$$

to put the extra terms into more familiar forms. These terms are simply the metric and curvature terms written in arbitrary curvilinear coordinates. For completeness, the vertical momentum equation is

$$\frac{\mathrm{D}w}{\mathrm{D}t} - v_1\hat{\boldsymbol{e}}_1\cdot\frac{\mathrm{D}\hat{\boldsymbol{e}}_3}{\mathrm{D}t} - v_2\hat{\boldsymbol{e}}_2\cdot\frac{\mathrm{D}\hat{\boldsymbol{e}}_3}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{9}$$

[1] Note that the contravariant and covariant basis vectors are identical since the basis vectors are orthonormal.

It is clear that there are no gradients of the geopotential appearing in the horizontal momentum equations, except implicitly in the metric terms through derivatives of the unit vectors. Importantly, the metric terms vanish if $v = 0$, so hydrostatic balance has the expected form

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial z} = -\rho g. \tag{10}$$

Not having an explicit form for the coordinate system is inconvenient, so in practice one replaces the true coordinate system with a simpler, more tractable system (e.g., spherical or the GREAT coordinates of Staniforth and White 2025). This leads to a small *geometrical* error in the metric terms rather than a *dynamical* error in the pressure gradient. Chang and Wolfe (2022) estimated the relative error associated with removing the wiggles of the geoid from the coordinate system to be less than 1% the standard spherical metric terms on scales larger than 100 km—equivalent to an acceleration error of less than 10^{-9} m s^{-2}. This is far smaller than other currently tolerated errors (such as not accounting for oblateness), so the community would be better suited by focusing on correcting these larger errors before accounting for the small deviations of the geoid from its reference ellipsoid.

References

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