Reply to RC5 by John Thuburn – 2026-05-17

27 May 2026

(1) **General Comment**

"*The author makes a valid point in the table in several of the Author Comments: the community is perhaps not as careful as we should be in documenting the spherical geopotential approximation in our models. (I'm sure the same is true of atmospheric models.)*"

Response: Thanks. I agree.

(2) **Comment on Spherical Geopotential Approximation**

"*Unfortunately, the author continually misses the point that the approximation made is a geometrical one; we start by assuming a geopotential coordinate - with no horizontal component of gravity by definition - and then approximate the geometry of the coordinate surfaces by approximating the metric (see RC4). In approximating the metric we do not change the balance of forces such as hydrostatic balance. In reply to AC1 point 2: yes, the pressure gradient* can *balance the bumpy geoid geopotential gradient. For a fairly arbitrary geopotential, there exists a hydrostatically balanced pressure field with p = p(Φ) and ρ = ρ(Φ) chosen so that ρ ∇p = ∇Φ (should be ∇p/ ρ = ∇Φ) (again see RC4). As the author points out, this statement of hydrostatic balance is coordinate independent. As CC3 points out, the atmosphere and ocean tend to satisfy this balance to an excellent approximation, and that remains true in models that make the spherical geopotential approximation*."

Response: I disagree. Spherical/spheroidal geopotential approximation cannot be made.

Equation of motion without friction is given by

$$\frac{D\mathbf{u}}{Dt} = -2\boldsymbol{\Omega} \times \mathbf{u} - \frac{1}{\rho}\nabla p + \nabla\Phi \qquad \text{(R1)}$$

where $\boldsymbol{\Omega}$ is Earth angular velocity; $\mathbf{u}$ is velocity vector; ρ is density; p is pressure; Φ is geopotential; and ∇ is gradient operator. I use local Cartesian coordinates (ξ,η,ζ) with unit vectors ($\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}}$) representing (east, north, "up") for illustration. The local Cartesian coordinates are equivalent to the spherical/spheroidal coordinates. The hydrostatic equilibrium with $\mathbf{u}$ = 0 and D$\mathbf{u}$/Dt = 0 gives

$$-\frac{1}{\rho_B}\nabla p_B + \nabla\Phi_B = 0 \qquad \text{(R2)}$$

which shows the coincidence of the three constant surfaces of equilibrium density (ρ_B), pressure (p_B), and geopotential (Φ_B).

Gravity-pressure gradient forces are the only body forces and depend on three scalar fields: $\rho(\xi,\eta,\zeta)$, $p(\xi,\eta,\zeta)$, and $\Phi(\xi,\eta,\zeta)$. Both geopotential and pressure gradients are represented in the local Cartesian coordinates by

$$\nabla p = \frac{\partial p}{\partial \xi}\hat{\boldsymbol{\xi}} + \frac{\partial p}{\partial \eta}\hat{\boldsymbol{\eta}} + \frac{\partial p}{\partial \zeta}\hat{\boldsymbol{\zeta}}, \qquad \nabla \Phi = \frac{\partial \Phi}{\partial \xi}\hat{\boldsymbol{\xi}} + \frac{\partial \Phi}{\partial \eta}\hat{\boldsymbol{\eta}} + \frac{\partial \Phi}{\partial \zeta}\hat{\boldsymbol{\zeta}} \qquad \text{(R3)}$$

The spherical/spheroidal geopotential approximation is to use spherical/spheroidal surfaces to represent const Φ surfaces. For the local Cartesian coordinates, it is represented by

$$\frac{\partial \Phi}{\partial \xi} = 0, \qquad \frac{\partial \Phi}{\partial \eta} = 0 \qquad \text{(R4)}$$

Several prominent meteorologists and oceanographers used the following "justification"

$$[O(|\nabla_h \Phi|)]/O\left(\left|\frac{\partial \Phi}{\partial \zeta}\right|\right) = 10^{-5}, \qquad \nabla_h \equiv \hat{\boldsymbol{\xi}}\frac{\partial}{\partial \xi} + \hat{\boldsymbol{\eta}}\frac{\partial}{\partial \eta} \qquad \text{(R5)}$$

to make such a geometric approximation (see Staniforth and White 2025 for detail).

The equilibrium pressure (p_B) surfaces coincide with the equilibrium geopotential (Φ_B) surfaces, and the pressure field has similar characteristics for large-scale motion,

$$[O(|\nabla_h p|)]/O\left(\left|\frac{\partial p}{\partial \zeta}\right|\right) = 10^{-5} - 10^{-4} \qquad \text{(R6)}$$

Following this "justification", we may also make similar spherical/spheroidal pressure approximation. But we do not. Because comparison between ζ-component and (ξ, η)-components of the same gradient vector to neglect (ξ, η)-components is MEANINGLESS.

The validity of spherical/spheroidal geopotential approximation must be verified by the comparison between corresponding components of geopotential and pressure gradients as well as between geopotential gradient and Coriolis acceleration:

$$O(\rho|\nabla_h \Phi|)/O(|\nabla_h p|) \Rightarrow small\ (?), \quad O(|\nabla_h \Phi|)/O(|f\mathbf{U}|) \Rightarrow small\ (?) \qquad \text{(R7)}$$

where $\mathbf{U} = u\hat{\boldsymbol{\xi}} + v\hat{\boldsymbol{\eta}}$. Datasets for verifying Eq.(R7) are publicly available. My early work shows that these ratios are not small (Chu 2021, 2024).

(3) **Comment on Mathematics**

"*I don't have time to carefully check all the mathematics in the author comments. However, there remain errors*."

Response: I followed the mathematical procedures presented in RC3 without careful check. Thank you very much for your comments. I will correct the errors in revision.

General Response to RC2 and RC5:

My manuscript deals with an untouched area as mentioned by Staniforth and White (2025) "Almost everything you always wanted to know about representing gravity in global models but were afraid to ask." After the geopotential (Φ) field is treated similarly as other scalar fileds such as pressure (p) and density (ρ), everything is very easy to understand.

My rebuttal sufficiently shows that my manuscript deserves "major revision" not "rejection."

Following is my sketch of revised manuscript with title change. In the revised manuscript, I will use as little mathematics as possible.

Title: **Coordinate Invariance of Gravity-Pressure Gradient Forces and Rejection of Spherical/Spheroidal Geopotential Approximation**

Abstract: Gravity-pressure gradient forces, $-(\nabla p)/\rho + \nabla\Phi$, are the only body forces for frictionless flow in ocean dynamics and depend on three scalar fields, density ρ, pressure p, and geopotential Φ. Geopotential Φ field should be treated similarly as pressure p and density ρ fields. The gravity-pressure gradient forces, $-(\nabla p)/\rho + \nabla\Phi$, are coordinate invariant. Such a feature makes the emergence of the bumpy-geoid (N) gradient ($g_0 \nabla_h N$) in the horizontal momentum equation in all ocean models with various coordinates. Use of publicly available ocean and geoidal (N) datasets lead to the rejection of popularly used spherical/spheroidal geopotential approximation.

1. **Introduction**
2. **True Horizontal and Horizontal Defined in Ocean Models**
3. **Gravity-Pressure Gradient Forces in Various Coordinates**
   3.1. Local Cartesian Coordinates
   3.2. Geopotential Coordinates
   3.3. Pressure Coordinates
   3.4. Terrain-Following Coordinates
   3.5. Isopycnal Coordinates
   3.6. Hybrid Coordinates
4. **Publicly Available Datasets**
5. **Invalid Spherical/Spheroidal Geopotential Approximation**
   5.1. Geostrophic Flow and Thermal Wind Relation
   5.2. 1½ Layered Model with Rigid Lid
   5.3. Sverdrup-Stommel-Munk Wind-Driven Circulation Models
   5.4. Global Sverdrup and Stommel Volume Transport
6. **Conclusions**