Continuing Reply on RC4 by Chris Hughes – 2026-05-13

(1) Comment on My Rebuttal

"*This may explain the author's confusion, since his rebuttal focuses so strongly on the definitions of ocean model grids which are often described as spherical.*"

Response: I disagree because I also show the coordinate invariance of gravity-pressure gradient forces which is the evidence to include bumpy-geoid gradient in horizontal momentum equation.

(2) Comment on an Exactly Spherical Coordinate System

"It is true that an exactly spherical coordinate system applied to the real geometry of the earth would miss important lateral gravity terms, since constant radius would then not be constant geopotential."

Response: I agree.

(3) Comment on Spherical Geopotential Approximation

"*However, this is not the correct interpretation of such models. They are models in which the vertical coordinate is exactly aligned with gravity, and the gravity field is approximated as spherical. In these models, a more accurate representation of the gravity field would not add lateral gravity terms, but would add complications to the geometry of the coordinates. There is no missing force, only missing metric terms, which show up in the acceleration term*."

Response: I disagree. Spherical/spheroidal geopotential approximation cannot be made.

Equation of motion without friction is given by

$$\frac{D\mathbf{u}}{Dt} = -2\mathbf{\Omega} \times \mathbf{u} - \frac{1}{\rho}\nabla p + \nabla\Phi \tag{R1}$$

where $\mathbf{\Omega}$ is Earth angular velocity; $\mathbf{u}$ is velocity vector; ρ is density; p is pressure; Φ is geopotential; and ∇ is gradient operator. I use local Cartesian coordinates (ξ,η,ζ) with unit vectors $(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}})$ representing (east, north, "up") for illustration. The local Cartesian coordinates are equivalent to the spherical/spheroidal coordinates. As you mentioned, the hydrostatic equilibrium with $\mathbf{u} = 0$ and $D\mathbf{u}/Dt = 0$ gives

$$-\frac{1}{\rho_B}\nabla p_B + \nabla\Phi_B = 0 \tag{R2}$$

which shows the coincidence of the three constant surfaces of equilibrium density (ρ_B), pressure (p_B), and geopotential (Φ_B).

Gravity-pressure gradient forces are the only body forces and depend on three scalar fields: $\rho(\xi,\eta,\zeta)$, $p(\xi,\eta,\zeta)$, and $\Phi(\xi,\eta,\zeta)$. Both geopotential and pressure gradients are represented in the local Cartesian coordinates by

$$\nabla p = \frac{\partial p}{\partial \xi}\hat{\boldsymbol{\xi}} + \frac{\partial p}{\partial \eta}\hat{\boldsymbol{\eta}} + \frac{\partial p}{\partial \zeta}\hat{\boldsymbol{\zeta}}, \qquad \nabla \Phi = \frac{\partial \Phi}{\partial \xi}\hat{\boldsymbol{\xi}} + \frac{\partial \Phi}{\partial \eta}\hat{\boldsymbol{\eta}} + \frac{\partial \Phi}{\partial \zeta}\hat{\boldsymbol{\zeta}} \qquad \text{(R3)}$$

The spherical/spheroidal geopotential approximation is to use spherical/spheroidal surfaces to represent const Φ surfaces. For the local Cartesian coordinates, it is represented by

$$\frac{\partial \Phi}{\partial \xi} = 0, \qquad \frac{\partial \Phi}{\partial \eta} = 0 \qquad \text{(R4)}$$

Several prominent meteorologists and oceanographers used the following "justification" to make such a geometric approximation (see Staniforth and White 2025 for detail)

$$[\mathrm{O}(|\nabla_h \Phi|)]/\mathrm{O}\left(\left|\frac{\partial \Phi}{\partial \zeta}\right|\right) = 10^{-5}, \qquad \nabla_h \equiv \hat{\boldsymbol{\xi}}\frac{\partial}{\partial \xi} + \hat{\boldsymbol{\eta}}\frac{\partial}{\partial \eta} \qquad \text{(R5)}$$

The equilibrium pressure (p_B) surfaces coincide with the equilibrium geopotential (Φ_B) surfaces, and the pressure field has similar characteristics for large-scale motion,

$$[\mathrm{O}(|\nabla_h p|)]/\mathrm{O}\left(\left|\frac{\partial p}{\partial \zeta}\right|\right) = 10^{-5} - 10^{-4} \qquad \text{(R6)}$$

Following this "justification", can we make spherical/spheroidal pressure approximation? Of course NOT. Because comparison between ζ-component and (ξ, η)-components of the same gradient vector to neglect (ξ, η)-components is MEANINGLESS.

The validity of spherical/spheroidal geopotential approximation must be verified by the comparison between corresponding components of geopotential and pressure gradients as well as between geopotential gradient and Coriolis acceleration:

$$\mathrm{O}(\rho|\nabla_h \Phi|)/\mathrm{O}(|\nabla_h p|) \Rightarrow small\,(?), \quad \mathrm{O}(|\nabla_h \Phi|)/\mathrm{O}(|f\mathbf{U}|) \Rightarrow small\,(?) \qquad \text{(R7)}$$

where $\mathbf{U} = u\hat{\boldsymbol{\xi}} + v\hat{\boldsymbol{\eta}}$. Datasets for verifying Eq.(R7) are publicly available. My early work shows that these ratios are not small (Chu 2021, 2024).

(4) Comment on an Exactly Spherically Symmetric Gravity

"The case of an exactly spherically symmetric gravity field illustrates this nicely. In this case, writing velocities u, v, w in the directions of longitude λ, latitude ϕ and radius r, ..., to the end"

Response: The comment is incorrect since the spherical/spheroidal geopotential approximation cannot be made.