Reply to Review by John Thuburn of “Revealing horizontal gravity force in geopotential coordinates via metric tensors”

11 May 2026

Thank you very much for your efforts and time to review my paper. When you read my reply, would you please also read my reply to Reviewer #3 ‘s Comments especially the first section “Coordinate Invariance of Gravity-Pressure Gradient Forces and Emergence of Bumpy-Geoid Gradient in the Horizontal Momentum Equation.”

1. Comments on “Horizontal”

“*The manuscript claims to show that, in geopotential coordinates, gravity has a horizontal component that is unjustifiably neglected in numerical models of the ocean and atmosphere. There is patently something illogical about the idea that the geopotential Φ should have a nonzero gradient along surfaces of constant Φ. This should have alerted the author to the likelihood of an error in the mathematics and, indeed, there are several (see specific points below). I am therefore recommending that the manuscript be rejected.*”

“*There is already an issue in the second line of the abstract, since standard usage is that ‘vertical’ refers to the direction of $\nabla\Phi$ and ‘horizontal’ refers to directions perpendicular to $\nabla\Phi$. But let us give the author the benefit of the doubt and suppose - at this point at least - ‘horizontal’ is defined in terms of some coordinate system such as spherical polars that is approximately aligned with gravity, since this meaning is also commonly used. Note, though, that it is important to be clear and precise, given the topic of the manuscript.*”

Responses:

(a) “Horizontal” Defined in Oceanographic Community

After high-resolution altimetric satellite into practice, the geoid undulation N (or called bumpy geoid) was first determined quantitatively by EGM96 (Fig. R1) in 1996. This surface is perpendicular to gravity with fluctuating $\pm 100\ m$ world-wide from the Earth reference ellipsoid.

However, the oceanographic community has never taken the bumpy geoid (N) as the “horizontal” but define the “horizontal” as tangential to the Earth spherical (or spheroidal) surface or “x increasing eastward, y increasing northward” local Cartesian coordinates. Table R1 shows the definition of horizontal in popular ocean models. With the commonly used “horizontal” in ocean models, the bumpy-geoid gradient, $g_0 \nabla_h N$, represents the horizontal gravity force and emerges in the horizontal equation of motion.

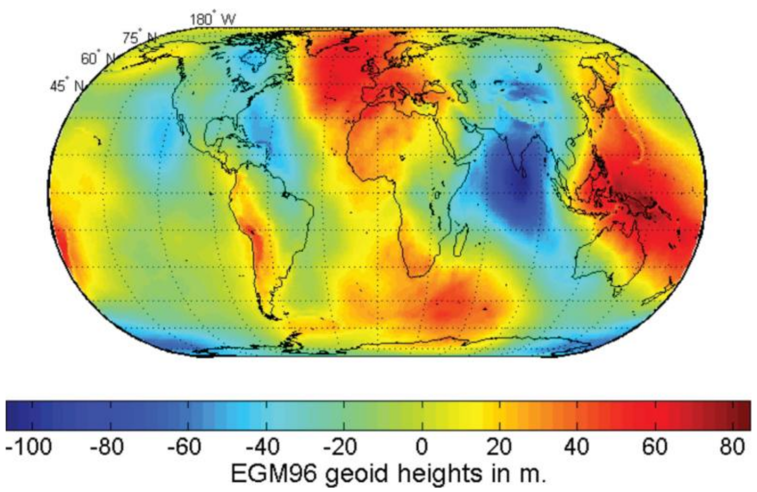

Fig. R1. Geoid undulation *N* from EGM96 (like Fig. 2 in the preprint).

Table R1. Horizontal defined in popular ocean models.

Model	Documentation (User 's Manual or Journal Paper)	Definition of Horizontal
Hybrid Coordinate Ocean Model (HYCOM)	https://www.hycom.org/attachments/063_hycom_users_guide.pdf In Section 3 The HYCOM Grid, Page 6	The HYCOM mesh was converted to standard **Cartesian coordinates**, with the *x*-axis pointing eastward and the *y*-axis pointing northward.
Nucleus for European Modelling of the Ocean (NEMO)	https://zenodo.org/records/19206664 Page 5	NEMO is written using locally orthogonal *horizontal coordinates*, such as the *familiar* ***spherical coordinates*** ….
MIT General Circulation Model (MITgcm)	http://app.readthedocs.org/projects/mitgcm/downloads/pdf/latest/ 2.11.4. Horizontal grid	The grid information is quite general and describes any of the available coordinate systems, ***Cartesian***, ***spherical-polar*** *or curvilinear*.
Modular Ocean Model (MOM4)	https://mom-ocean.github.io/pdf/MOM4_manual Page 56	MOM4 is written in generalized horizontal coordinates, where horizontal means coordinates within a locally defined tangent plane on the surface of a ***spherical*** earth
Parallel Ocean Program (POP)	https://files.cesm.ucar.edu/models/pop/2/POPRefManual.pdf Pages 7-8	***Spherical Surface*** → "… general horizontal coordinates (q_x, q_y, z) where q_x and q_y are arbitrary curvilinear coordinates in the horizontal directions, and z = *r* - *a*, is again the vertical coordinate ***normal to the surface of the sphere***"

Princeton Ocean Model (POM)	Blumberg, A.F. and G. Mellor, 1987: A description of a three-dimensional coastal ocean circulation model, AGU Coastal and Estuarine Science 4. Page 2	… with *x* increases eastward, *y* increases northward, and *z* increases upward
Regional Oceanic Modeling System (ROMS)	Kanarska, Y., A. Shchepetkin, and J.C. McWilliams, 2007: Algorithm for non-hydrostatic dynamics in the regional oceanic modeling system, *Ocean Modelling*, **18**, 143-174. https://data-croco.ifremer.fr/DOC/Roms_Agrif_manual/doc_roms_agrif_v2.1_19_07_2010.pdf Subsection 3.1. Model equations in curvilinear coordinates	For the case of a ***spherical coordinate system*** when … … where we use the same notation for the ***horizontal components*** (*u*, *v*) as in ***Cartesian coordinates*** …

(b) Geopotential and Geopotential Coordinates

With geopotential surfaces as horizontal, the geopotential coordinates (x, y, Z) with unit vectors ($\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{Z}}$) are proposed to correspond to local Cartesian coordinates (ξ, η, ζ) with unit vectors ($\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}}$) by (e.g., McWilliams 2024)

$$x = \xi,\ \ y = \eta, \qquad Z = -\frac{\Phi}{g_0}, \qquad \mathbf{g} = \nabla\Phi \tag{R1}$$

and

$$\hat{\mathbf{x}} = \hat{\boldsymbol{\xi}},\ \ \hat{\mathbf{y}} = \hat{\boldsymbol{\eta}}, \qquad \hat{\mathbf{Z}} = -\frac{\nabla\Phi}{|\nabla\Phi|} \tag{R2}$$

Gravity ($\mathbf{g}$, shown as red arrows in Fig. R2) is perpendicular to geopotential (Φ) surface. For $\mathbf{g} = \nabla\Phi$, and $\Phi = -g_0 Z$, the bumpy geoid is defined by $Z = \zeta + N$, where ζ is the vertical Cartesian coordinate. There is no gravity component along the geopotential surface. With the hydrostatic equilibrium, gravity is balanced by the vertical pressure gradient force (PGF) but not the horizontal PGF, as shown as dashed arrows in Fig. 2. Let pressure be p_ζ at the Cartesian reference surface and be p_Z at the corresponding geopotential surface. The pressure on the geopotential surface is given by

$$p_Z = p_\zeta - g_0 \int_{\zeta}^{\zeta+N(x,y)} \rho dZ, \qquad Z = -\frac{\Phi}{g_0}, \qquad g_0 = 9.81\ m\ s^{-2} \tag{R3}$$

where density (ρ) is assumed horizontally uniform for simplicity without loss generality. Use of chain rules obtains the pressure gradient along the geopotential surface,

$$\partial p_Z/\partial x = \partial p_\zeta/\partial x - \rho\, g_0\, \partial N/\partial x,\ \ \partial p_Z/\partial y = \partial p_\zeta/\partial y - \rho\, g_0\, \partial N/\partial y \tag{R4}$$

which shows the emergence of bumpy-geoid gradients in the pressure gradient force along the geopotential surface.

*** Note that establishment of geopotential coordinates does not make the bumpy-geoid gradients vanish because they become part of the pressure gradient force along the geopotential surface.*

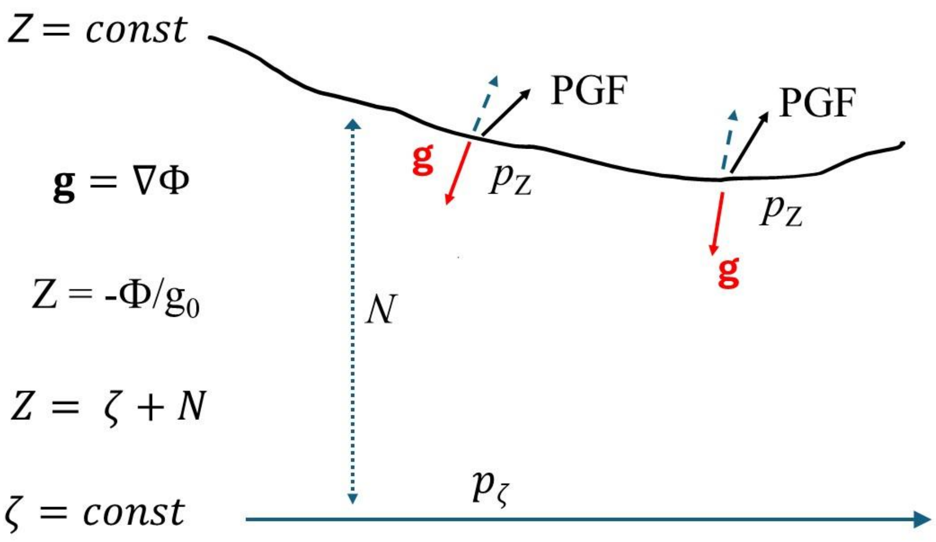

Fig. R2. Illustration of bumpy-geoid gradient as a part of the pressure gradient force along the geopotential surface.

Thus, your rejection based on definition of horizontal might not be right.

2. Comments on First Approach

"*Retaining the full 'bumpy' gravity field but writing the governing equations in some smooth coordinate system such as spherical polars or GREAT, we obtain an equation something like (8). (Actually, the author has made a couple of approximations to the terms involving the velocity, omitting a metric term and making the 'traditional' approximation, but that is not the most important thing here.) For the present discussion, the important terms in (8) are the pressure gradient and geoptential gradient. In this approach we do indeed obtain a nonzero horizontal component to the geopotential gradient. However, the crucial point is that, to an excellent approximation, it is compensated by an (almost) equal and opposite horizontal pressure gradient term, because the atmosphere and ocean are very close to hydrostatic balance. In this approach, if we neglect the horizontal component of gravity, then the pressure gradient would also lose its horizontal hydrostatic component. Thus, we effectively omit two terms or contributions whose sum is virtually zero and so make very small error overall.*"

Response: No. The bumpy-geoid gradient cannot be balanced by pressure gradient force.

In Cartesian coordinates, one set of unit vectors works for everything because the basis vectors $(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}})$ are orthogonal; they have unit length; dot products are zero. The gravity-pressure gradient forces, $-(\nabla p)/\rho + \nabla\Phi$, are major driving forces in ocean dynamics,

$$-(\nabla p)/\rho + \nabla\Phi = -\hat{\boldsymbol{\xi}}\left[(\partial_\xi p)/\rho - \partial_\xi\Phi\right] - \hat{\boldsymbol{\eta}}\left[(\partial_\eta p)/\rho - \partial_\eta\Phi\right] - \hat{\boldsymbol{\zeta}}\left[(\partial_\zeta p)/\rho - \partial_\zeta\Phi\right] \quad \text{(R5)}$$

where the gradient operator ∇ is given by

$$\nabla = \nabla_\zeta + \hat{\boldsymbol{\zeta}}\partial_\zeta, \ \nabla_\zeta \equiv \hat{\boldsymbol{\xi}}\partial_\xi + \hat{\boldsymbol{\eta}}\partial_\eta$$

The bumpy-geoid gradient

$$\nabla_\zeta \Phi = \mathrm{g}_0 \nabla_\zeta N \tag{R6}$$

emerges in the horizontal gravity-pressure gradient forces.

Geopotential coordinates are "used" to eliminate $\nabla_\zeta N$ from the horizontal momentum equation (e.g., McWilliams 2024). The basis vectors of the geopotential coordinates $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{Z}})$ are not orthogonal; their lengths vary with position; directions change from point to point. Because of this, a single set of basis vectors cannot simultaneously represent directions of coordinate lines and extract components of vectors cleanly. Therefore, geopotential coordinates have dual (paired) covariant $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_Z)$ and contravariant $(\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^Z)$, with corresponding gradient operators

$$\nabla = \mathbf{a}_x(\partial_x + N_x\partial_Z) + \mathbf{a}_y(\partial_y + N_y\partial_Z) + \mathbf{a}_Z\partial_Z; \ \mathbf{a}_x = \hat{\boldsymbol{\xi}} - N_x\hat{\boldsymbol{\zeta}}, \ \mathbf{a}_y = \hat{\boldsymbol{\eta}} - N_y\hat{\boldsymbol{\zeta}}, \ \mathbf{a}_Z = \hat{\boldsymbol{\zeta}} \tag{R7}$$

$$\nabla = \mathbf{a}^x\partial_x + \mathbf{a}^y\partial_y + \mathbf{a}^Z\partial_Z; \ \mathbf{a}^x = \hat{\boldsymbol{\xi}}, \ \mathbf{a}^y = \hat{\boldsymbol{\eta}}, \ \mathbf{a}^Z = N_x\hat{\boldsymbol{\xi}} + N_y\hat{\boldsymbol{\eta}} + \hat{\boldsymbol{\zeta}} \tag{R8}$$

Obviously, McWilliams (2024) geopotential coordinates use the contravariant basis vectors $(\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^Z)$ with the unit vectors

$$\hat{\mathbf{x}} = \hat{\boldsymbol{\xi}}, \ \hat{\mathbf{y}} = \hat{\boldsymbol{\eta}}, \ \hat{\mathbf{Z}} = \left[N_x\hat{\boldsymbol{\xi}} + N_y\hat{\boldsymbol{\eta}} + \hat{\boldsymbol{\zeta}}\right]/\left(1 + N_x^2 + N_y^2\right)^{1/2} = \nabla Z/|\nabla Z| = -\nabla\Phi/|\nabla\Phi| \tag{R9}$$

The gravity-pressure gradient forces are

$$-(\nabla p)/\rho + \nabla\Phi = -[(\partial_x p - \rho \mathrm{g}_0 N_x)/\rho]\mathbf{a}_x - \left[\left(\partial_y p - \rho \mathrm{g}_0 N_y\right)/\rho\right]\mathbf{a}_y \tag{R10}$$

with the covariant basis vectors and

$$-(\nabla p)/\rho + \nabla\Phi = -[(\partial_x p - \rho \mathrm{g}_0 N_x)/\rho]\mathbf{a}^x - \left[\left(\partial_y p - \rho \mathrm{g}_0 N_y\right)/\rho\right]\mathbf{a}^y \tag{R11}$$

with the contravariant basis vectors. Eq.(R10) and Eq.(R11) show the existence of $(\mathrm{g}_0\nabla_h N)$ on the $(\mathbf{a}_x, \mathbf{a}_y)$ and $(\mathbf{a}^x, \mathbf{a}^y)$ surfaces. Here,

$$\nabla_h \equiv \mathbf{a}_x\,\partial_x + \mathbf{a}_y\,\partial_y \quad \text{(covariant) or} \quad \nabla_h \equiv \mathbf{a}^x\,\partial_x + \mathbf{a}^y\,\partial_y \quad \text{(contravariant)} \tag{R12}$$

Thus, the gravity-pressure gradient forces have $\mathrm{g}_0\nabla_h N$ in the horizontal momentum equation with Cartesian coordinates and geopotential coordinates using both covariant and contravariant basis vectors. No matter which type of "*horizontal*" is defined, the bumpy geoidal forcing $\mathrm{g}_0\nabla N$ (i.e., horizontal gravity force) occurs in the horizontal equation of motion.

3. Incorrect Equation (26) leads to Incorrect Equation (27)

Response: There was a typo in Eq.(26), which should be

$$\mathbf{G}^{-1} = [g^{ij}], \qquad g^{ij} = \frac{\partial x^i}{\partial \xi^p}\frac{\partial x^j}{\partial \xi^q}\delta_{pq}, \ \ \delta_{pq} = \begin{cases} 1, & p = q \\ 0 & p \neq q \end{cases} \tag{R13}$$

which was used to derive Eq. (27). It does not affect the derivation of Eq.(27). In Eq.(26), (ξ^1, ξ^2, ξ^3) are the Cartesian coordinates (ξ, η, ζ) and (x^1, x^2, x^3) are the geopotential coordinates (x, y, Z). I used

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + Z\hat{\mathbf{Z}}$$
$$x = \xi,\ \ y = \eta,\ \ Z = \zeta + N\ \Rightarrow Z_\xi = N_\xi, Z_\eta = N_\eta,\ Z_\zeta = 1 \tag{R14}$$

for (R13) [i.e., Eq.(26) after typo being corrected] to get Eq.(27)

$$\mathbf{G}^{-1} = [g^{ij}] = \begin{bmatrix} 1 + Z_\xi^2 & Z_\xi Z_\eta & Z_\xi Z_\zeta \\ Z_\xi Z_\eta & 1 + Z_\eta^2 & Z_\eta Z_\zeta \\ Z_\xi Z_\zeta & Z_\eta Z_\zeta & Z_\zeta^2 \end{bmatrix} = \begin{bmatrix} 1 + N_\xi^2 & N_\xi N_\eta & N_\xi \\ N_\xi N_\eta & 1 + N_\eta^2 & N_\eta \\ N_\xi & N_\eta & 1 \end{bmatrix}$$

4. McWilliams (2024) ⇒ Right/Wrong?

Response:

Since the resultant gravity-pressure gradient forces are coordinates invariant, use of geopotential coordinates does not make the bumpy-geoid gradient, $\mathrm{g}_0\nabla_\mathrm{h}N$, disappear in the horizonal momentum equation. This term ($\mathrm{g}_0\nabla_\mathrm{h}N$) appears in the pressure gradient force and disappears from gravity with the covariant basis vectors $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_Z)$, and disappears from the pressure gradient force and appears in gravity with the contravariant basis vectors $(\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^Z)$. McWilliams (2024) mistakenly used the disappearance of $\mathrm{g}_0\nabla_h N$ from the pressure gradient force but disregarded the appearance of $\mathrm{g}_0\nabla_h N$ in gravity with the contravariant basis vectors $(\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^Z)$ to claim the vanish of $\mathrm{g}_0\nabla_h N$ in the horizontal momentum equation.