

Reply to Review by Chris W. Hughes of “Revealing horizontal gravity force in geopotential coordinates via metric tensors”

11 May 2026

Thank you very much for your efforts and time to review my paper. When you read my reply, would you please also read my reply to Reviewer #3 ‘s Comments especially the first section “Coordinate Invariance of Gravity-Pressure Gradient Forces and Emergence of Bumpy-Geoid Gradient in the Horizontal Momentum Equation.”

General Comments

“This paper purports to show that there is a, usually neglected, gravitational force acting along the horizontal direction in the ocean equations of motion when expressed in geopotential coordinates. My recommendation is that this paper be rejected, as its fundamental point is clearly incorrect.

Since “horizontal” is defined as the plane perpendicular to gravity, it is hard to see how there could ever be a horizontal component of gravity. There are exceptions to this, but they do not apply to the issue considered in the paper.”

If we considered “gravity” to be purely gravitational attraction to the earth’s mass, ignoring the centrifugal force, but defined “horizontal” in the usual manner in the earth’s rotating reference frame, thus including the centrifugal force in gravity when defining horizontal, then there could be a horizontal component of gravity. Alternatively, in tidal modelling, it is common to define the horizontal based on the time-averaged gravity field, and incorporate the time-dependent tidal forces as “horizontal” forces within the model. Neither of these interpretations is the issue in this paper.”

“The other possible interpretation is that “horizontal” is being used to represent some plane of the coordinate system which is informally referred to as “horizontal”, but is not actually perpendicular to the gravity vector. Since gravity is (plus or minus, depending on convention) the gradient of the geopotential, geopotential surfaces are horizontal by definition, so this cannot be an issue for the equations in geopotential coordinates.’

‘I believe that the correct interpretation is that “horizontal” in this paper turns out to mean “in the xy plane of an arbitrarily chosen local Cartesian coordinate system’.

Summarized Your General Comments: *My recommendation is to reject this paper because the “horizontal” is not defined as perpendicular to the gravity.*

Responses:

(a) “Horizontal” Defined in Oceanographic Community

After high-resolution altimetric satellite into practice, the geoid undulation N (or called bumpy geoid) was first determined quantitatively by EGM96 (Fig. R1) in 1996. This surface is perpendicular to gravity with fluctuating $\pm 100\text{ m}$ world-wide from the Earth reference ellipsoid.

However, the oceanographic community has never taken the bumpy geoid (N) as the “horizontal” but defines the “horizontal” as tangential to the Earth spherical (or spheroidal) surface or “ x increasing eastward, y increasing northward” local Cartesian coordinates. Table R1 shows the definition of horizontal in popular ocean models.

With the commonly used “horizontal” in ocean models, the bumpy-geoid gradient, $g_0 \nabla_h N$, represents the horizontal gravity force and emerges in the horizontal equation of motion.

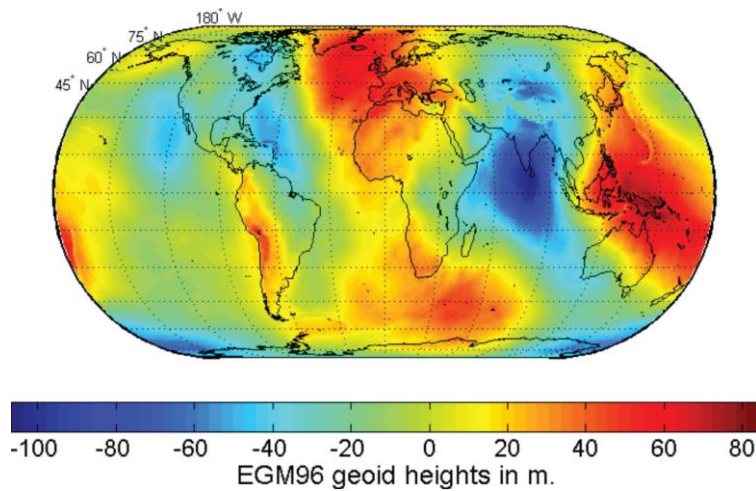


Fig. R1. Geoid undulation N from EGM96 (like Fig. 2 in the preprint).

Table R1. Horizontal defined in popular ocean models.

Model	Documentation (User ‘s Manual or Journal Paper)	Definition of Horizontal
Hybrid Coordinate Ocean Model (HYCOM)	https://www.hycom.org/attachments/063_hycom_users_guide.pdf In Section 3 The HYCOM Grid, Page 6	The HYCOM mesh was converted to standard Cartesian coordinates , with the x -axis pointing eastward and the y -axis pointing northward.
Nucleus for European Modelling of the Ocean (NEMO)	https://zenodo.org/records/19206664 Page 5	NEMO is written using locally orthogonal <i>horizontal coordinates</i> , such as the <i>familiar spherical coordinates</i>
MIT General Circulation Model (MITgcm)	http://app.readthedocs.org/projects/mitgcm/downloads/pdf/latest/ 2.11.4. Horizontal grid	The grid information is quite general and describes any of the available coordinate systems, <i>Cartesian, spherical-polar or curvilinear</i> .

Modular Ocean Model (MOM4)	https://mom-ocean.github.io/pdf/MOM4_manual Page 56	MOM4 is written in generalized horizontal coordinates, where horizontal means coordinates within a locally defined tangent plane on the surface of a <i>spherical</i> earth
Parallel Ocean Program (POP)	https://files.cesm.ucar.edu/models/pop/2/POPRefManual.pdf Pages 7-8	<i>Spherical Surface</i> → “... general horizontal coordinates (q_x, q_y, z) where q_x and q_y are arbitrary curvilinear coordinates in the horizontal directions, and $z = r - a$, is again the vertical coordinate <i>normal to the surface of the sphere</i> ”
Princeton Ocean Model (POM)	Blumberg, A.F. and G. Mellor, 1987: A description of a three-dimensional coastal ocean circulation model, AGU Coastal and Estuarine Science 4. Page 2	... with x increases eastward, y increases northward, and z increases upward
Regional Oceanic Modeling System (ROMS)	Kanarska, Y., A. Shchepetkin, and J.C. McWilliams, 2007: Algorithm for non-hydrostatic dynamics in the regional oceanic modeling system, <i>Ocean Modelling</i> , 18 , 143-174. https://data-croco.ifremer.fr/DOC/Roms_Agrif_manual/doc_roms_agrif_v2.1_19_07_2010.pdf Subsection 3.1. Model equations in curvilinear coordinates	For the case of a <i>spherical coordinate system</i> when where we use the same notation for the <i>horizontal components</i> (u, v) as in <i>Cartesian coordinates</i> ...

(b) Geopotential and Geopotential Coordinates

With geopotential surfaces as horizontal, the geopotential coordinates (x, y, Z) with unit vectors ($\hat{x}, \hat{y}, \hat{Z}$) are proposed to correspond to local Cartesian coordinates (ξ, η, ζ) with unit vectors ($\hat{\xi}, \hat{\eta}, \hat{\zeta}$) by (e.g., McWilliams 2024)

$$x = \xi, \quad y = \eta, \quad Z = -\frac{\Phi}{g_0}, \quad \mathbf{g} = \nabla\Phi \quad (\text{R1})$$

and

$$\hat{x} = \hat{\xi}, \quad \hat{y} = \hat{\eta}, \quad \hat{Z} = -\frac{\nabla\Phi}{|\nabla\Phi|} \quad (\text{R2})$$

Gravity (\mathbf{g} , shown as red arrows in Fig. R2) is perpendicular to geopotential (Φ) surface. For $\mathbf{g} = \nabla\Phi$, and $\Phi = -g_0Z$, the bumpy geoid is defined by $Z = \zeta + N$, where ζ is the vertical Cartesian coordinate. There is no gravity component along the geopotential surface. With the hydrostatic equilibrium, gravity is balanced by the vertical pressure gradient force (PGF) but not the horizontal PGF, as shown as dashed arrows in Fig. R2. Let pressure be p_ζ at the Cartesian reference surface and be p_Z at the corresponding geopotential surface. The pressure on the geopotential surface is given by

$$p_Z = p_\zeta - g_0 \int_{\zeta}^{\zeta+N(x,y)} \rho dZ, \quad Z = -\frac{\Phi}{g_0}, \quad g_0 = 9.81 \text{ m s}^{-2} \quad (\text{R3})$$

where density (ρ) is assumed horizontally uniform for simplicity without loss generality. Use of chain rules obtains the pressure gradient along the geopotential surface,

$$\partial p_Z / \partial x = \partial p_\zeta / \partial x - \rho g_0 \partial N / \partial x, \quad \partial p_Z / \partial y = \partial p_\zeta / \partial y - \rho g_0 \partial N / \partial y \quad (\text{R4})$$

which shows the emergence of bumpy-geoid gradients in the pressure gradient force along the geopotential surface.

*** Note that establishment of geopotential coordinates does not make the bumpy-geoid gradients vanish because they become part of the pressure gradient force along the geopotential surface.*

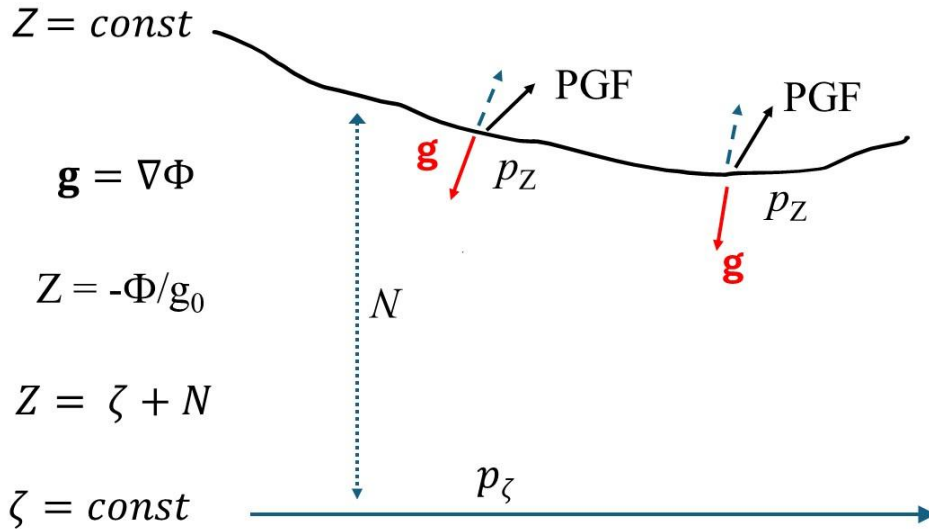


Fig. R2. Illustration of bumpy-geoid gradient as a part of the pressure gradient force along the geopotential surface.

(c) Coordinate Invariance of Bumby-Geoid Gradient

In Cartesian coordinates, one set of unit vectors works for everything because the basis vectors ($\hat{\xi}, \hat{\eta}, \hat{\zeta}$) are orthogonal; they have unit length; dot products are zero. The gravity-pressure gradient forces, $-(\nabla p)/\rho + \nabla \Phi$, are major driving forces in ocean dynamics,

$$-(\nabla p)/\rho + \nabla \Phi = -\hat{\xi}[(\partial_\xi p)/\rho - \partial_\xi \Phi] - \hat{\eta}[(\partial_\eta p)/\rho - \partial_\eta \Phi] - \hat{\zeta}[(\partial_\zeta p)/\rho - \partial_\zeta \Phi] \quad (\text{R5})$$

where the gradient operator ∇ is given by

$$\nabla = \nabla_\zeta + \hat{\zeta}\partial_\zeta, \quad \nabla_\zeta \equiv \hat{\xi}\partial_\xi + \hat{\eta}\partial_\eta \quad (\text{R6})$$

The bumpy-geoid gradient $\nabla_\zeta\Phi = \hat{\xi}\partial_\xi\Phi + \hat{\eta}\partial_\eta\Phi$, emerges in the horizontal gravity-pressure gradient forces.

The purpose of establishing geopotential coordinates is to eliminate $\nabla_\zeta\Phi$ from the horizontal momentum equation (e.g., McWilliams 2024). The basis vectors of the geopotential coordinates $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$ are not orthogonal; their lengths vary with position; directions change from point to point. Because of this, a single set of basis vectors cannot simultaneously represent directions of coordinate lines and extract components of vectors cleanly. Therefore, geopotential coordinates have dual (paired) covariant $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z)$ and contravariant $(\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^z)$, with corresponding gradient operators

$$\nabla = \mathbf{a}_x(\partial_x + N_x\partial_z) + \mathbf{a}_y(\partial_y + N_y\partial_z) + \mathbf{a}_z\partial_z; \quad \mathbf{a}_x = \hat{\xi} - N_x\hat{\zeta}, \quad \mathbf{a}_y = \hat{\eta} - N_y\hat{\zeta}, \quad \mathbf{a}_z = \hat{\zeta} \quad (\text{R7})$$

$$\nabla = \mathbf{a}^x\partial_x + \mathbf{a}^y\partial_y + \mathbf{a}^z\partial_z; \quad \mathbf{a}^x = \hat{\xi}, \quad \mathbf{a}^y = \hat{\eta}, \quad \mathbf{a}^z = N_x\hat{\xi} + N_y\hat{\eta} + \hat{\zeta} \quad (\text{R8})$$

Obviously, McWilliams (2024) geopotential coordinates use the contravariant basis vectors $(\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^z)$ with the unit vectors

$$\hat{\mathbf{x}} = \hat{\xi}, \quad \hat{\mathbf{y}} = \hat{\eta}, \quad \hat{\mathbf{z}} = [N_x\hat{\xi} + N_y\hat{\eta} + \hat{\zeta}]/(1 + N_x^2 + N_y^2)^{1/2} = \nabla Z/|\nabla Z| = -\nabla\Phi/|\nabla\Phi| \quad (\text{R9})$$

The gravity-pressure gradient forces are

$$-(\nabla p)/\rho + \nabla\Phi = -[(\partial_x p - \rho g_0 N_x)/\rho]\mathbf{a}_x - [(\partial_y p - \rho g_0 N_y)/\rho]\mathbf{a}_y - [(\partial_z p + \rho g_0)/\rho]\mathbf{a}_z$$

with the covariant basis vectors and

$$-(\nabla p)/\rho + \nabla\Phi = -[(\partial_x p - \rho g_0 N_x)/\rho]\mathbf{a}^x - [(\partial_y p - \rho g_0 N_y)/\rho]\mathbf{a}^y - [(\partial_z p + \rho g_0)/\rho]\mathbf{a}^z$$

with the contravariant basis vectors. With the hydrostatic balance the above two equations are

$$-(\nabla p)/\rho + \nabla\Phi = -[(\partial_x p - \rho g_0 N_x)/\rho]\mathbf{a}_x - [(\partial_y p - \rho g_0 N_y)/\rho]\mathbf{a}_y = -(\nabla_h p)/\rho + g_0 \nabla_h N \quad (\text{R10})$$

$$-(\nabla p)/\rho + \nabla\Phi = -[(\partial_x p - \rho g_0 N_x)/\rho]\mathbf{a}^x - [(\partial_y p - \rho g_0 N_y)/\rho]\mathbf{a}^y = -(\nabla_h p)/\rho + g_0 \nabla_h N \quad (\text{R11})$$

where,

$$\nabla_h \equiv \mathbf{a}_x \partial_x + \mathbf{a}_y \partial_y \quad (\text{covariant}) \quad \text{or} \quad \nabla_h \equiv \mathbf{a}^x \partial_x + \mathbf{a}^y \partial_y \quad (\text{contravariant}) \quad (\text{R12})$$

Note that Eq.(R10) and Eq.(R11) show the existence of $(g_0 \nabla_h N)$ on the $(\mathbf{a}_x, \mathbf{a}_y)$ and $(\mathbf{a}^x, \mathbf{a}^y)$ surfaces. Thus, the gravity-pressure gradient forces have $g_0 \nabla_h N$ in the horizontal momentum equation with Cartesian coordinates and geopotential coordinates using both covariant and contravariant basis vectors. You may read my responses to Review-3 for detail derivation. No matter which type of “horizontal” is defined, the bumpy geoidal forcing $g_0 \nabla_h N$ (i.e., horizontal gravity force) occurs in the horizontal equation of motion.

Thus, your rejection based on definition of horizontal might not be right.

Specific Comments

(1) “The central result is equation (43): $LHS = -\nabla_h \hat{p} + \hat{\rho} g_0 \nabla_h N$, where ∇_h is referred to as “horizontal”, but is defined as $\nabla_h \equiv \hat{x} \partial_x + \hat{y} \partial_y$.”

Response:

$\nabla_h \equiv \hat{x} \partial_x + \hat{y} \partial_y$ is referred as “horizontal” in both Cartesian and geopotential coordinates. See Eqs. (R5)-(R12).

(2) “Going back to the definitions around Figure 1, we see that $(\hat{x}, \hat{y}, \zeta)$ are a set of mutually orthogonal unit vectors, in which ζ is not perpendicular to geopotentials, and therefore not vertical (later approximations show that it is assumed to be close enough to vertical for quadratic, but not linear terms to be neglected). That means the \hat{x} and \hat{y} directions are not actually horizontal, but this doesn’t seem to be the biggest issue: **since ∂_x and ∂_y represent derivatives at constant geopotential**, there should be no force acting along the geopotential surface if pressure is constant along that surface, meaning that the gravity term should not appear. The issue is that the right-hand side of (43) is clearly incorrect.”

(3) **Response:** No. The right-hand side of (43) is clearly correct.

Eq.(R5), Eq.(R10), and Eq.(R11) show that the gravity-pressure gradient forces are coordinate invariant. If pressure is constant on geopotential surface,

$$\partial_x p = 0, \quad \partial_y p = 0$$

the gravity-pressure gradient forces become

$$-(\nabla p)/\rho + \nabla \Phi = g_0 N_x \mathbf{a}_x + g_0 N_y \mathbf{a}_y$$

with the covariant basis vectors and

$$-(\nabla p)/\rho + \nabla \Phi = g_0 N_x \mathbf{a}^x + g_0 N_y \mathbf{a}^y$$

with the contravariant basis vectors.

(4) “In fact, the N term in (43) appears to perform the role of (approximately, ignoring terms of order δ^2 and higher) “correcting” the horizontal pressure gradient so that it represents a gradient at constant ζ .”

Response: Not only at constant ζ but also at constant geopotential surface. This is because **the gravity-pressure gradient forces are coordinate invariant**.

(5) “The form is very similar to the conventional relationship between the gradient of the 2D ocean bottom pressure field projected onto the horizontal, and the horizontal component of the 3D pressure gradient at the bottom: $\nabla_h p_b = (\nabla_h p)_b - \rho_b g \nabla_h z_b$, where subscript b means ‘evaluated at the bottom’.”

Response: Yes. There is some similarity.

(6) *“Thus, it appears that the “horizontal” gravity force in (43) is actually the component of gravity acting along the non-horizontal ζ surface along which the gradient of pressure is being evaluated. It is not an unexpected new force, but a known force which does not act along geopotential surfaces.”*

Response: No. It does act along geopotential surfaces too due to the coordinate invariance of gravity-pressure gradient forces.

(7) *“We can also see that there is a problem from the fact that the orientation of the Cartesian “vertical” is arbitrary (although it must be close to the true vertical given the approximations made). A small angular change δ in its orientation would make little difference to $-\nabla_h \hat{p}$, if the derivative is at constant Z , since it would then be a derivative at constant geopotential and the geopotential is not changing (there is a second order change in the derivative, proportional to $1-\cos\delta$ because distances along the geopotential would slightly change with respect to the changing x and y distances).”*

Response: No. Please see my responses to the general comments, especially the coordinate invariance of gravity-pressure gradient forces.

(8) *“On the other hand, there would be a first order change in $\nabla_h N$, which represents the tilt of geopotential surfaces relative to the Cartesian “horizontal”. In fact, in an extreme case for which \hat{p} happens to be constant along geopotential surfaces, $-\nabla_h \hat{p}$ would be zero for any choice of Cartesian coordinate if ∇_h is truly evaluated along geopotentials. However, although $\nabla_h N$ could be zero if the Cartesian and geopotential coordinates happen to coincide, it would be nonzero for any relative tilt. The correct form of this equation must produce cancelling changes in these two terms for a small tilt of the Cartesian reference.”*

Response: No.

The gravity-pressure gradient forces for $-\nabla_h \hat{p} = 0$, are given by [see Eq.(R10), Eq.(R11)],

$$-(\nabla p)/\rho + \nabla \Phi = g_0 \nabla_h N \tag{R13}$$

Your statement of *“the Cartesian and geopotential coordinates happen to coincide”* is confusing because Cartesian coordinates are fixed, independent on the bumpy-geoid N , and exist no matter if there is bumpy-geoid or not.

If your statement of *“the Cartesian and geopotential coordinates happen to coincide”* means no bumpy geoid, i.e.,

$$\nabla_h N = 0$$

which leads to zero gravity-pressure gradient forces [see Eq.(R13)].

(9) *“My tensor calculus is not up to identifying the error clearly, but I think the issue is that, in (35), $\nabla_G \hat{p}$ should not simply be interpreted as $\nabla \hat{p}$ expressed in geopotential coordinates, but as a contravariant vector, which is a different kind of object. This can be seen in (35) by*

considering $p=p(Z)$, a purely hydrostatic balance with $\partial_x p = \partial_y p = 0$, in which it is clear that the pressure gradient is parallel to Z , but (35) produces nonzero \hat{x} and \hat{y} terms if N depends on x and y . It is therefore incorrect in (39) to equate a conventional vector on the left-hand side with a contravariant vector on the right.”

Response: No. Eq.(39) is correct.

Horizontal component of $\nabla_G p$ in Eq.(35) is interpreted as the gravity-pressure gradient forces with the covariant basis vectors [see (R7) and Eq.(35)],

$$[-\nabla_G p]|_{horizontal} = -(\nabla_h p)/\rho + g_0 \nabla_h N \quad (\text{R14})$$

For $p = p(Z)$,

$$\nabla_h p = 0, \text{ but } [-\nabla_G p]|_{horizontal} = g_0 \nabla_h N \neq 0$$

Thus, Eq.(39) is correct.

(10) “Since the right-hand side of the momentum equation is what is in question here, that covers the most important issue. However, it is worth noting that there is also a problem with the left-hand side. To extract the horizontal component of the acceleration it is not sufficient to replace \mathbf{u} with its horizontal component \mathbf{U} , as is done in (43), either for the Coriolis or the nonlinear term. In fact, the nonlinear term takes quite a complex form when evaluated in curvilinear (even spherical) coordinates, and it is here, and in the mass conservation equation, that any effect of the irregular geoid would appear in the equations when written in geopotential coordinates. Staniforth and White (2024) argue convincingly that the size of such perturbations would be very small, but no explicit derivation of the relevant equations has been attempted, to my knowledge”

Response: Excellent comments.

(a) Invariant Total Acceleration in Coordinate Transformation (Nonlinear Term)

Let $\mathbf{u} = (u, v, \omega)$ be the velocity in local Cartesian coordinates with ω the vertical velocity and $\mathbf{u} = (u, v, w)$ in geopotential coordinates with w the vertical velocity [equation (8) in McWilliams (2024)],

$$w = \omega Z_\zeta + u Z_\xi + v Z_\eta \quad (\text{R15})$$

The total acceleration in Cartesian coordinates is given by [equation (2) in McWilliams (2024)],

$$\left[\frac{D\mathbf{u}}{Dt} \right]_c = \partial_t \mathbf{u} + u \partial_\xi \mathbf{u} + v \partial_\eta \mathbf{u} + \omega \partial_\zeta \mathbf{u} \quad (\text{R16})$$

Substitution of the following equations in the preprint,

$$\partial_x = \partial_\xi - (Z_\xi/Z_\zeta) \partial_\zeta, \quad \partial_y = \partial_\eta - (Z_\eta/Z_\zeta) \partial_\zeta, \quad \partial_z = (1/Z_\zeta) \partial_\zeta \quad (19)$$

into (R16) gives

$$\left[\frac{D\mathbf{u}}{Dt}\right]_C = \partial_t \mathbf{u} + u[\partial_x + Z_\xi \partial_Z] \mathbf{u} + v[\partial_y + Z_\eta \partial_Z] \mathbf{u} + \omega Z_\zeta \partial_Z \mathbf{u} = \left[\frac{D\mathbf{u}}{Dt}\right]_G \quad (\text{R17})$$

where

$$\left[\frac{D\mathbf{u}}{Dt}\right]_G = \partial_t \mathbf{u} + u \partial_x \mathbf{u} + v \partial_y \mathbf{u} + w \partial_z \mathbf{u} \quad (\text{R18})$$

which is equation (7) in McWilliams (2024).

(b) Coriolis Acceleration

The Coriolis acceleration (CA) in local Cartesian coordinates is represented by

$$[\mathbf{CA}]_C = f \hat{\boldsymbol{\zeta}} \times \mathbf{U} = -fv \hat{\boldsymbol{\xi}} + fu \hat{\boldsymbol{\eta}}, \quad f = 2\Omega \sin \varphi, \quad \mathbf{U} = u \hat{\boldsymbol{\xi}} + v \hat{\boldsymbol{\eta}}, \quad \varphi = \text{latitude} \quad (\text{R19})$$

The Coriolis acceleration (CA) in the geopotential coordinates is represented by

$$[\mathbf{CA}]_G = f \hat{\boldsymbol{Z}} \times \mathbf{U} = f(\hat{\boldsymbol{\zeta}} + N_x \hat{\boldsymbol{\xi}} + N_y \hat{\boldsymbol{\eta}}) \times \mathbf{U} = [\mathbf{CA}]_C + f(N_x v - N_y u) \hat{\boldsymbol{\zeta}} \quad (\text{R20})$$

where Eq.(R9) is used. Difference of Coriolis acceleration between local Cartesian and geopotential coordinates is

$$[\mathbf{CA}]_G - [\mathbf{CA}]_C = f(N_x v - N_y u) \hat{\boldsymbol{\zeta}}$$