Reply to Comment by Francis Poulin on "Revealing horizontal gravity force in geopotential coordinates via metric tensors"

10 May 2026

Thank you very much for your efforts and time to read and comment on my preprint. When you read my reply, would you please also read my reply to Reviewer #3 's Comments especially the first section "Coordinate Invariance of Gravity-Pressure Gradient Forces and Emergence of Bumpy-Geoid Gradient in the Horizontal Momentum Equation."

Comment

"*This manuscript has a lot mathematics where the author rewrites the equations of motion using covariant and contravariant metric tensors. I have not gone through the details of the mathematics but I am concerned more about the fundamental assumptions that go into this model. The focus is to look at the horizontal component of gravity and how that affects the dynamics, which the author seems to have studied previously. I don't believe this is appropriate for the oceans or atmosphere. It's true that the Earth's surface is not a geoid, but the oceans and atmosphere will naturally align themselves so that gravity is acting orthogonal to these thin fluids (at large scales). Rather than simply defining two non-dimensional numbers, I think there should be more evidence to support that this horizontal component of gravity contributes to the large scale flows before this is seriously considered for publication*."

Responses:

(1) "Horizontal" Defined in Oceanographic Community

After high-resolution altimetric satellite into practice, the geoid undulation N (or called bumpy geoid) was first determined quantitatively by EGM96 (Fig. 1) in 1996. This surface is perpendicular to gravity with fluctuating $\pm 100\ m$ world-wide from the Earth reference ellipsoid.

However, the oceanographic community has never taken the bumpy geoid (N) as the "horizontal" but define the "horizontal" as tangential to the Earth spherical (or spheroidal) surface or "x increasing eastward, y increasing northward" local Cartesian coordinates. Table R1 shows the definition of horizontal in popular ocean models. With the commonly used "horizontal" in ocean models, the bumpy-geoid gradient, $g_0 \nabla_h N$, represents the horizontal gravity force and emerges in the horizontal equation of motion.

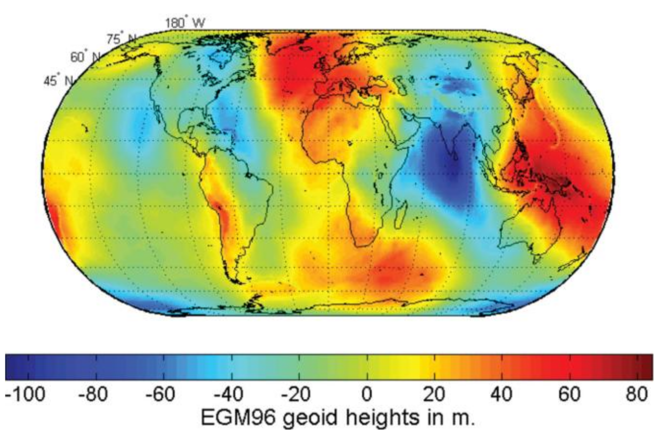

Fig. R1. Geoid undulation N from EGM96 (like Fig. 2 in the preprint).

Table R1. Horizontal defined in popular ocean models.

Model	Documentation (User 's Manual or Journal Paper)	Definition of Horizontal
Hybrid Coordinate Ocean Model (HYCOM)	https://www.hycom.org/attachments/063_hycom_users_guide.pdf In Section 3 The HYCOM Grid, Page 6	The HYCOM mesh was converted to standard **Cartesian coordinates**, with the *x*-axis pointing eastward and the *y*-axis pointing northward.
Nucleus for European Modelling of the Ocean (NEMO)	https://zenodo.org/records/19206664 Page 5	NEMO is written using locally orthogonal *horizontal coordinates*, such as the *familiar* ***spherical coordinates*** ….
MIT General Circulation Model (MITgcm)	http://app.readthedocs.org/projects/mitgcm/downloads/pdf/latest/ 2.11.4. Horizontal grid	The grid information is quite general and describes any of the available coordinate systems, ***Cartesian***, ***spherical-polar*** *or curvilinear*.
Modular Ocean Model (MOM4)	https://mom-ocean.github.io/pdf/MOM4_manual Page 56	MOM4 is written in generalized horizontal coordinates, where horizontal means coordinates within a locally defined tangent plane on the surface of a ***spherical*** earth
Parallel Ocean Program (POP)	https://files.cesm.ucar.edu/models/pop/2/POPRefManual.pdf Pages 7-8	***Spherical Surface*** → "… general horizontal coordinates (q_x, q_y, z) where q_x and q_y are arbitrary curvilinear coordinates in the horizontal directions, and z = *r* - *a*, is again the vertical coordinate ***normal to the surface of the sphere***"
Princeton Ocean Model (POM)	Blumberg, A.F. and G. Mellor, 1987: A description of a three-dimensional coastal ocean circulation model, AGU Coastal and Estuarine Science 4. Page 2	… with *x* increases eastward, *y* increases northward, and *z* increases upward
Regional Oceanic Modeling System (ROMS)	Kanarska, Y., A. Shchepetkin, and J.C. McWilliams, 2007: Algorithm for non-hydrostatic dynamics in the regional oceanic modeling system, *Ocean Modelling*, **18**, 143-174. https://data-croco.ifremer.fr/DOC/Roms_Agrif_manual/doc_roms_agrif_v2.1_19_07_2010.pdf Subsection 3.1. Model equations in curvilinear coordinates	For the case of a ***spherical coordinate system*** when … … where we use the same notation for the ***horizontal components*** (*u*, *v*) as in ***Cartesian coordinates*** …

(2) Geopotential and Geopotential Coordinates

With geopotential surfaces as horizontal, the geopotential coordinates (x, y, Z) with unit vectors ($\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{Z}}$) are proposed to correspond to local Cartesian coordinates (ξ, η, ζ) with unit vectors ($\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}}$) by (McWilliams 2024)

$$x = \xi, \quad y = \eta, \qquad Z = -\frac{\Phi}{g_0}, \qquad \mathbf{g} = \nabla\Phi \qquad \text{(R1)}$$

and

$$\hat{\mathbf{x}} = \hat{\boldsymbol{\xi}}, \;\; \hat{\mathbf{y}} = \hat{\boldsymbol{\eta}}, \qquad \hat{\mathbf{Z}} = -\frac{\nabla\Phi}{|\nabla\Phi|} \tag{R2}$$

Gravity (**g**, shown as red arrows in Fig. 2) is perpendicular to geopotential (Φ) surface. For $\mathbf{g} = \nabla\Phi$, and $\Phi = -g_0 Z$, the bumpy geoid is defined by $Z = \zeta + N$, where ζ is the vertical Cartesian coordinate. There is no gravity component along the geopotential surface. With the hydrostatic equilibrium, gravity is balanced by the vertical pressure gradient force (PGF) but not the horizontal PGF, as shown as dashed arrows in Fig. 2. Let pressure be p_ζ at the Cartesian reference surface and be p_Z at the corresponding geopotential surface. The pressure on the geopotential surface is given by

$$p_Z = p_\zeta - g_0 \int_{\zeta}^{\zeta+N(x,y)} \rho dZ, \qquad Z = -\frac{\Phi}{g_0}, \qquad g_0 = 9.81\, m\, s^{-2} \tag{R3}$$

where density (ρ) is assumed horizontally uniform for simplicity without loss generality. Use of chain rules obtains the pressure gradient along the geopotential surface,

$$\partial p_Z/\partial x = \partial p_\zeta/\partial x - \rho\, g_0\, \partial N/\partial x, \;\; \partial p_Z/\partial y = \partial p_\zeta/\partial y - \rho\, g_0\, \partial N/\partial y \tag{R4}$$

which shows the emergence of bumpy-geoid gradients in the pressure gradient force along the geopotential surface.

*** Note that establishment of geopotential coordinates does not make the bumpy-geoid gradients vanish because they become part of the pressure gradient force along the geopotential surface.*

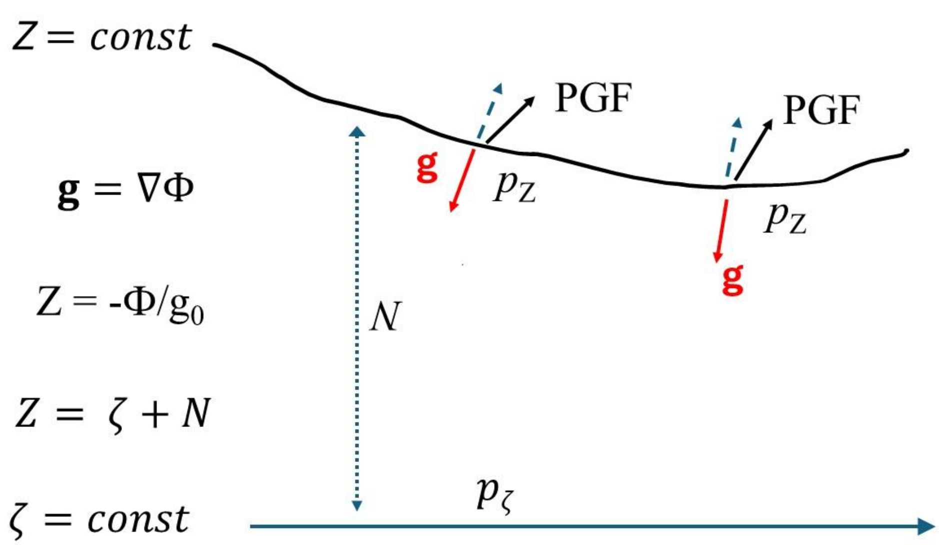

Fig. R2. Illustration of bumpy-geoid gradient as a part of the pressure gradient force along the geopotential surface.

(3) Bumpy-Geoid Gradient

In Cartesian coordinates, one set of unit vectors works for everything because the basis vectors ($\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}}$) are orthogonal; they have unit length; dot products are zero. The gravity-pressure gradient forces, $-(\nabla p)/\rho + \nabla\Phi$, are major driving forces in ocean dynamics,

$$-(\nabla p)/\rho + \nabla\Phi = -\hat{\boldsymbol{\xi}}\left[(\partial_\xi p)/\rho - \partial_\xi\Phi\right] - \hat{\boldsymbol{\eta}}\left[(\partial_\eta p)/\rho - \partial_\eta\Phi\right] - \hat{\boldsymbol{\zeta}}\left[(\partial_\zeta p)/\rho - \partial_\zeta\Phi\right] \quad \text{(R5)}$$

where the gradient operator ∇ is given by

$$\nabla = \nabla_\zeta + \hat{\boldsymbol{\zeta}}\partial_\zeta,\ \nabla_\zeta \equiv \hat{\boldsymbol{\xi}}\partial_\xi + \hat{\boldsymbol{\eta}}\partial_\eta$$

The bumpy-geoid gradient

$$\nabla_\zeta\Phi = \mathrm{g}_0\nabla_\zeta N \quad \text{(R6)}$$

emerges in the horizontal gravity-pressure gradient forces.

Geopotential coordinates are "used" to eliminate $\nabla_\zeta N$ from the horizontal momentum equation (see McWilliams 2024). The basis vectors of the geopotential coordinates $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{Z}})$ are not orthogonal; their lengths vary with position; directions change from point to point. Because of this, a single set of basis vectors cannot simultaneously represent directions of coordinate lines and extract components of vectors cleanly. Therefore, geopotential coordinates have dual (paired) covariant $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_Z)$ and contravariant $(\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^Z)$, with corresponding gradient operators

$$\nabla = \mathbf{a}_x(\partial_x + N_x\partial_Z) + \mathbf{a}_y\left(\partial_y + N_y\partial_Z\right) + \mathbf{a}_Z\partial_Z;\ \mathbf{a}_x = \hat{\boldsymbol{\xi}} - N_x\hat{\boldsymbol{\zeta}},\ \mathbf{a}_y = \hat{\boldsymbol{\eta}} - N_y\hat{\boldsymbol{\zeta}},\ \mathbf{a}_Z = \hat{\boldsymbol{\zeta}} \quad \text{(R7)}$$

$$\nabla = \mathbf{a}^x\partial_x + \mathbf{a}^y\partial_y + \mathbf{a}^Z\partial_Z;\ \mathbf{a}^x = \hat{\boldsymbol{\xi}},\ \mathbf{a}^y = \hat{\boldsymbol{\eta}},\ \mathbf{a}^Z = N_x\hat{\boldsymbol{\xi}} + N_y\hat{\boldsymbol{\eta}} + \hat{\boldsymbol{\zeta}} \quad \text{(R8)}$$

Obviously, McWilliams (2024) geopotential coordinates use the contravariant basis vectors $(\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^Z)$ with the unit vectors

$$\hat{\mathbf{x}} = \hat{\boldsymbol{\xi}},\ \hat{\mathbf{y}} = \hat{\boldsymbol{\eta}},\ \hat{\mathbf{Z}} = \left[N_x\hat{\boldsymbol{\xi}} + N_y\hat{\boldsymbol{\eta}} + \hat{\boldsymbol{\zeta}}\right]/\left(1 + N_x^2 + N_y^2\right)^{1/2} = \nabla Z/|\nabla Z| = -\nabla\Phi/|\nabla\Phi| \quad \text{(R9)}$$

The gravity-pressure gradient forces are

$$-(\nabla p)/\rho + \nabla\Phi = -[(\partial_x p - \rho\mathrm{g}_0 N_x)/\rho]\mathbf{a}_x - \left[\left(\partial_y p - \rho\mathrm{g}_0 N_y\right)/\rho\right]\mathbf{a}_y \quad \text{(R10)}$$

with the covariant basis vectors and

$$-(\nabla p)/\rho + \nabla\Phi = -[(\partial_x p - \rho\mathrm{g}_0 N_x)/\rho]\mathbf{a}^x - \left[\left(\partial_y p - \rho\mathrm{g}_0 N_y\right)/\rho\right]\mathbf{a}^y \quad \text{(R11)}$$

with the contravariant basis vectors. Eq.(R10) and Eq.(R11) show the existence of $(\mathrm{g}_0\nabla_h N)$ on the $(\mathbf{a}_x, \mathbf{a}_y)$ and $(\mathbf{a}^x, \mathbf{a}^y)$ surfaces. Here,

$$\nabla_h \equiv \mathbf{a}_x\,\partial_x + \mathbf{a}_y\,\partial_y \ \text{(covariant) or}\ \nabla_h \equiv \mathbf{a}^x\,\partial_x + \mathbf{a}^y\,\partial_y \ \text{(contravariant)} \quad \text{(R12)}$$

Thus, the gravity-pressure gradient forces have $\mathrm{g}_0\nabla_h N$ in the horizontal momentum equation with Cartesian coordinates and geopotential coordinates using both covariant and contravariant basis vectors. You may read my responses to Review-3 for detail derivation

No matter which type of "*horizontal*" is defined, the bumpy geoidal forcing $\mathrm{g}_0\nabla N$ (i.e., horizontal gravity force) occurs in the horizontal equation of motion.