Reply to Comment by Baloy Fox on "Revealing horizontal gravity force in geopotential coordinates via metric tensors"

10 May 2026

Thank you very much for your efforts and time to read and comment on my preprint. I would like to reply as follows.

Comment-1

"*The natural way to define the vertical is the direction where a plumb bob hangs (https://en.wikipedia.org/wiki/Plumb_bob). Carpenters have been using this trick to build strong homes for millennia. This combines all forces experienced by a body which is in static location relative to the surface of the earth, i.e., the vector sum of gravity and centrifugal forces*."

Response: Unit vector normal to a geopotential surface does align with the plumb line, i.e., the direction a plum bob points.

Comment-2:

'*If one insists on having "horizontal" components of gravity, then the hydrostatic balance becomes a partial differential equation rather than an ordinary differential equation, as now geopotential gradient has derivatives in both the "horizontal", latitude-like direction, as well as the "vertical". This engenders a spurious waste of computation*.'

Responses: Bumpy-geoid gradient is non-negligible in ocean dynamics.

(1) "Horizontal" Defined in Oceanographic Community

After high-resolution altimetric satellite into practice, the geoid undulation N (or called bumpy geoid) was first determined quantitatively by EGM96 (Fig. 1) in 1996. This surface is perpendicular to gravity with fluctuating $\pm 100\ m$ world-wide from the Earth reference ellipsoid.

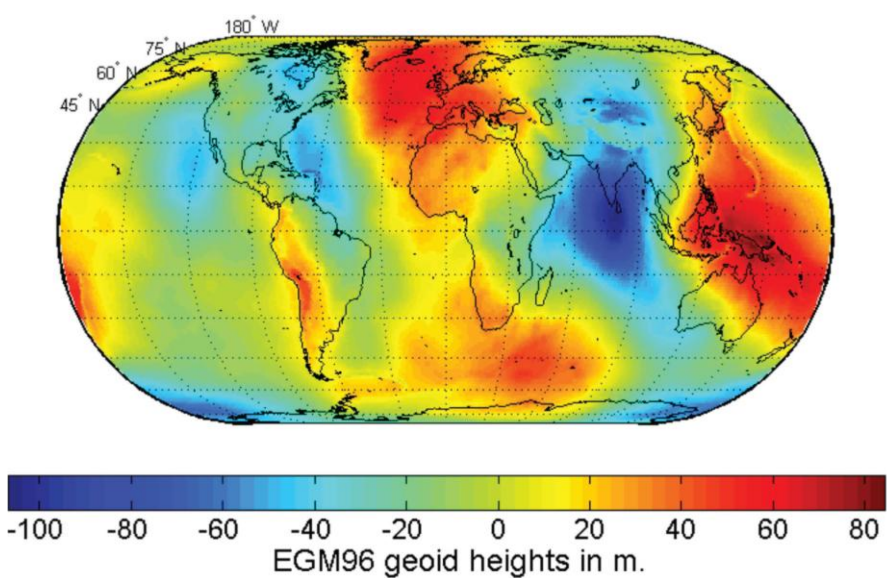

Fig. R1. Geoid undulation *N* from EGM96 (like Fig. 2 in the preprint).

However, the oceanographic community has never taken the bumpy geoid (*N*) as the "horizontal" but define the "horizontal" as tangential to the Earth spherical (or spheroidal) surface or "*x* increasing eastward, *y* increasing northward" local Cartesian coordinates. Table R1 shows the definition of horizontal in popular ocean models. With the commonly used "horizontal" in ocean models, the bumpy-geoid gradient, $g_0 \nabla_h N$, represents the horizontal gravity force and emerges in the horizontal equation of motion.

Table R1. Horizontal defined in popular ocean models.

Model	Documentation (User 's Manual or Journal Paper)	Definition of Horizontal
Hybrid Coordinate Ocean Model (HYCOM)	https://www.hycom.org/attachments/063_hycom_users_guide.pdf In Section 3 The HYCOM Grid, Page 6	The HYCOM mesh was converted to standard **Cartesian coordinates**, with the *x*-axis pointing eastward and the *y*-axis pointing northward.
Nucleus for European Modelling of the Ocean (NEMO)	https://zenodo.org/records/19206664 Page 5	NEMO is written using locally orthogonal *horizontal coordinates*, such as the *familiar **spherical coordinates*** ….
MIT General Circulation Model (MITgcm)	http://app.readthedocs.org/projects/mitgcm/downloads/pdf/latest/ 2.11.4. Horizontal grid	The grid information is quite general and describes any of the available coordinate systems, ***Cartesian***, ***spherical-polar*** *or curvilinear*.
Modular Ocean Model (MOM4)	https://mom-ocean.github.io/pdf/MOM4_manual Page 56	MOM4 is written in generalized horizontal coordinates, where horizontal means coordinates within a locally defined tangent plane on the surface of a ***spherical*** earth
Parallel Ocean Program (POP)	https://files.cesm.ucar.edu/models/pop/2/POPRefManual.pdf Pages 7-8	***Spherical Surface*** → "… general horizontal coordinates (q_x, q_y, z) where q_x and q_y are arbitrary curvilinear coordinates in the horizontal directions, and z = *r* - *a*, is again the vertical coordinate ***normal to the surface of the sphere***"
Princeton Ocean Model (POM)	Blumberg, A.F. and G. Mellor, 1987: A description of a three-dimensional coastal ocean circulation model, AGU Coastal and Estuarine Science 4. Page 2	… with *x* increases eastward, *y* increases northward, and *z* increases upward
Regional Oceanic Modeling System (ROMS)	Kanarska, Y., A. Shchepetkin, and J.C. McWilliams, 2007: Algorithm for non-hydrostatic dynamics in the regional oceanic modeling system, *Ocean Modelling*, **18**, 143-174. https://data-croco.ifremer.fr/DOC/Roms_Agrif_manual/doc_roms_agrif_v2.1_19_07_2010.pdf Subsection 3.1. Model equations in curvilinear coordinates	For the case of a ***spherical coordinate system*** when … … where we use the same notation for the ***horizontal components*** (*u*, *v*) as in ***Cartesian coordinates*** …

(2) Geopotential and Geopotential Coordinates

With geopotential surfaces as horizontal, the geopotential coordinates (x, y, Z) with unit vectors ($\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{Z}}$) are proposed to correspond to local Cartesian coordinates (ξ, η, ζ) with unit vectors ($\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}}, \hat{\boldsymbol{\zeta}}$) by (McWilliams 2024)

$$x = \xi, \;\; y = \eta, \qquad Z = -\frac{\Phi}{g_0}, \qquad \mathbf{g} = \nabla\Phi \tag{R1}$$

and

$$\hat{\mathbf{x}} = \hat{\boldsymbol{\xi}}, \;\; \hat{\mathbf{y}} = \hat{\boldsymbol{\eta}}, \qquad \hat{\mathbf{Z}} = -\frac{\nabla\Phi}{|\nabla\Phi|} \tag{R2}$$

Gravity ($\mathbf{g}$, shown as red arrows in Fig. 2) is perpendicular to geopotential (Φ) surface. For $\mathbf{g} = \nabla\Phi$, and $\Phi = -g_0 Z$, the bumpy geoid is defined by $Z = \zeta + N$, where ζ is the vertical Cartesian coordinate. There is no gravity component along the geopotential surface. With the hydrostatic equilibrium, gravity is balanced by the vertical pressure gradient force (PGF) but not the horizontal PGF, as shown as dashed arrows in Fig. 2. Let pressure be p_ζ at the Cartesian reference surface and be p_Z at the corresponding geopotential surface. The pressure on the geopotential surface is given by

$$p_Z = p_\zeta - g_0 \int_{\zeta}^{\zeta+N(x,y)} \rho dZ, \qquad Z = -\frac{\Phi}{g_0}, \qquad g_0 = 9.81\, m\, s^{-2} \tag{R3}$$

where density (ρ) is assumed horizontally uniform for simplicity without loss generality. Use of chain rules obtains the pressure gradient along the geopotential surface,

$$\partial p_Z/\partial x = \partial p_\zeta/\partial x - \rho\, g_0\, \partial N/\partial x, \;\; \partial p_Z/\partial y = \partial p_\zeta/\partial y - \rho\, g_0\, \partial N/\partial y \tag{R4}$$

which shows the emergence of bumpy-geoid gradients in the pressure gradient force along the geopotential surface.

*** Note that establishment of geopotential coordinates does not make the bumpy-geoid gradients vanish because they become part of the pressure gradient force along the geopotential surface.*

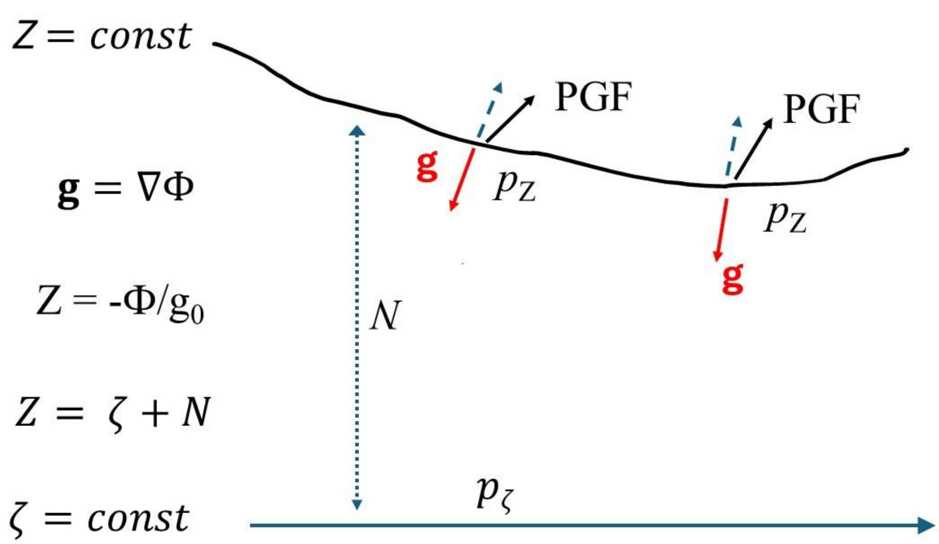

Fig. R2. Illustration of bumpy-geoid gradient as a part of the pressure gradient force along the geopotential surface.

Comment-3

"*In a coordinate-agnostic framework (see Joel Feske's recently defended PhD thesis and soon-to-be-submitted paper), the geopotential gradient direction can be implemented as vertical even in coordinate systems where no vertical coordinate exists (e.g., tracer coordinates: https://doi.org/10.1017/jfm.2012.638) by a simple projection operator ($k_i\, k^j$) where $k_i = \nabla_i$ (gravitational potential + centrifugal potential)*."

For $\mathbf{g} = \nabla\Phi$, and $\Phi = -\mathrm{g}_0 Z$, the bumpy geoid is defined by $Z = \zeta + N$. The unit vector, $\nabla\Phi/|\nabla\Phi| = \nabla Z/|\nabla Z|$, represents true vertical. The basis vectors of the geopotential coordinates ($\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{Z}}$) are not orthogonal; their lengths vary with position; directions change from point to point. Because of this, a single set of basis vectors cannot simultaneously represent directions of coordinate lines and extract components of vectors cleanly. Therefore, geopotential coordinates have dual (paired) covariant ($\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_Z$) and contravariant ($\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^Z$), with corresponding gradient operators

$$\nabla = \mathbf{a}_x(\partial_x + N_x\partial_Z) + \mathbf{a}_y(\partial_y + N_y\partial_Z) + \mathbf{a}_Z\partial_Z; \;\; \mathbf{a}_x = \hat{\boldsymbol{\xi}} - N_x\hat{\boldsymbol{\zeta}}, \;\; \mathbf{a}_y = \hat{\boldsymbol{\eta}} - N_y\hat{\boldsymbol{\zeta}}, \;\; \mathbf{a}_Z = \hat{\boldsymbol{\zeta}} \qquad \text{(R5)}$$

$$\nabla = \mathbf{a}^x\partial_x + \mathbf{a}^y\partial_y + \mathbf{a}^Z\partial_Z; \;\; \mathbf{a}^x = \hat{\boldsymbol{\xi}}, \;\; \mathbf{a}^y = \hat{\boldsymbol{\eta}}, \;\; \mathbf{a}^Z = N_x\hat{\boldsymbol{\xi}} + N_y\,\hat{\boldsymbol{\eta}} + \hat{\boldsymbol{\zeta}} \qquad \text{(R6)}$$

Obviously, McWilliams (2024) geopotential coordinates use the contravariant basis vectors ($\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^Z$) with the unit vectors

$$\hat{\mathbf{x}} = \hat{\boldsymbol{\xi}}, \;\; \hat{\mathbf{y}} = \hat{\boldsymbol{\eta}}, \;\; \hat{\mathbf{Z}} = \left[N_x\hat{\boldsymbol{\xi}} + N_y\,\hat{\boldsymbol{\eta}} + \hat{\boldsymbol{\zeta}}\right]/\left(1 + N_x^2 + N_y^2\right)^{1/2} = \nabla Z/|\nabla Z| = -\nabla\Phi/|\nabla\Phi| \qquad \text{(R7)}$$

a. Gravity-Pressure Gradient Forces with the Covariant Basis Vectors

The geopotential gradient is given by

$$\nabla\Phi = (\partial_x\Phi + N_x\partial_Z\Phi)\mathbf{a}_x + (\partial_y\Phi + N_y\partial_Z\Phi)\mathbf{a}_y + \partial_Z\Phi\mathbf{a}_Z = -\mathrm{g}_0\mathbf{a}_Z \qquad \text{(R8)}$$

where Eq.(R5) is used. The pressure gradient is given by

$$\nabla p = (\partial_x p + N_x\partial_Z p)\mathbf{a}_x + (\partial_y p + N_y\partial_Z p)\mathbf{a}_y + \partial_Z p\mathbf{a}_Z \qquad \text{(R9)}$$

The gravity-pressure gradient forces are

$$-(\nabla p)/\rho + \nabla\Phi = -[(\partial_x p + N_x\partial_Z p)/\rho]\mathbf{a}_x - [(\partial_y p + N_y\partial_Z p)/\rho]\mathbf{a}_y - (\partial_Z p/\rho + \mathrm{g}_0)\mathbf{a}_Z \qquad \text{(R10)}$$

Use of hydrostatic balance,

$$\partial_Z p/\rho + \mathrm{g}_0 = 0 \qquad \text{(R11)}$$

leads to

$$-(\nabla p)/\rho + \nabla\Phi = -[(\partial_x p - \rho\mathrm{g}_0 N_x)/\rho]\mathbf{a}_x - [(\partial_y p - \rho\mathrm{g}_0 N_y)/\rho]\mathbf{a}_y \qquad \text{(R12)}$$

which shows the existence of ($\mathrm{g}_0\nabla_h N$) on the ($\mathbf{a}_x, \mathbf{a}_y$) surface.

b. Gravity-Pressure Gradient Forces with the Contravariant Basis Vectors

With the contravariant basis vectors, use of Eq.(R6) for Φ leads to the geopotential gradient

$$\nabla\Phi = (\partial_x\Phi)\mathbf{a}^x + (\partial_y\Phi)\mathbf{a}^y + (\partial_Z\Phi)\mathbf{a}^Z \qquad \text{(R13)}$$

and for p leads to the pressure gradient

$$\nabla p = \partial_x p \mathbf{a}^x + \partial_y p \mathbf{a}^y + \partial_Z p \mathbf{a}^Z \quad \text{(R14)}$$

The gravity-pressure gradient forces are

$$-(\nabla p)/\rho + \nabla \Phi = -[(\partial_x p - \rho \partial_x \Phi)/\rho]\mathbf{a}^x - \left[\left(\partial_y p - \rho \partial_y \Phi\right)/\rho\right]\mathbf{a}^y - [(\partial_Z p - \rho \partial_Z \Phi)/\rho]\mathbf{a}^Z \quad \text{(R15)}$$

which shows the emergence of bumpy-geoid gradient

$$\mathrm{g}_0 \nabla_h N = \nabla_h \Phi, \quad \nabla_h \equiv \partial_x \mathbf{a}^x + \partial_y \mathbf{a}^y \quad \text{(R16)}$$

on the $(\mathbf{a}^x, \mathbf{a}^y)$ surface.

Thus, the gravity-pressure gradient forces have $\mathrm{g}_0 \nabla_h N$ in the horizontal momentum equation with geopotential coordinates using either covariant or contravariant basis vectors.

No matter which coordinates (Cartesian, geopotential, …) are used, the bumpy geoidal forcing $\mathrm{g}_0 \nabla N$ (i.e., horizontal gravity force) occurs in the horizontal equation of motion.