Reply to CC4 by Michael Montgomery – 2026-06-09

09 June 2026

(1) Firm Support on M24 's Correctness

"*I have examined the C26 preprint, the paper by McWilliams (2024, PNAS; hereafter M24) and, to some degree, the predecessor papers by McWilliams and colleagues on the subject of the existence and relevance of the horizontal component of gravity force in Geophysical Fluid Dynamics. Aside from an orientational error (Eq. (4) in M24) and a typo in M24 using lower case 'z' instead of upper case 'Z' as a partial derivative subscript in his Eq.(11) [both of which have been noted and corrected by McWilliams (2026, PNAS; hereafter M26)], I can state for the record that I have determined independently that all key numbered equations in M24 (i.e., Eqs. (1) - (3), (5) – (22)) are correct. C26's claim that M24 is in error because tensor calculus was not used to derive the transformed pressure gradient force per unit mass is mistaken. Moreover, tensor calculus is a red herring as M24's demonstration is a direct demonstration of the lack of the horizontal gravity force in the governing fluid dynamical equations in geopotential coordinates. I am surprised by the still ongoing controversy on this matter and the discussion whether there is an important horizontal force of gravity that has been neglected in the many decades since the birth of Geophysical Fluid Dynamics*."

Response: M24 itself shows the emergence of bumpy-geoid gradient in horizontal momentum equation in geopotential coordinates, which is opposite to the conclusion that McWilliams claimed (*the lack of the horizontal gravity force in the governing fluid dynamical equations in geopotential coordinates*).

K**ey equation** to reach the conclusion that McWilliams claimed in M24 is given by

$$\nabla p = \partial_\xi p \hat{\boldsymbol{\xi}} + \partial_\eta p \hat{\boldsymbol{\eta}} + \partial_\zeta p \hat{\boldsymbol{\zeta}} = \partial_x p \hat{\mathbf{x}} + \partial_y p \hat{\mathbf{y}} + \partial_Z p \hat{\mathbf{Z}} \qquad \text{Eq. (21) in M24} \qquad \text{(R1)}$$

which indicates that the gradient operator is

$$\nabla = \hat{\boldsymbol{\xi}}\partial_\xi + \hat{\boldsymbol{\eta}}\partial_\eta + \hat{\boldsymbol{\zeta}}\partial_\zeta \qquad \text{(R2)}$$

for local Cartesian coordinates, and

$$\nabla_G = \hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}}\partial_y + \hat{\mathbf{Z}}\partial_Z \qquad \text{(R3)}$$

for geopotential coordinates. The gradient of geopotential (Φ) is

$$\nabla\Phi = \hat{\boldsymbol{\xi}}\partial_\xi\Phi + \hat{\boldsymbol{\eta}}\partial_\eta\Phi + \hat{\boldsymbol{\zeta}}\partial_\zeta\Phi \qquad \text{(R4)}$$

in local Cartesian coordinates, and

$$\nabla_G\Phi = \hat{\mathbf{x}}\partial_x\Phi + \hat{\mathbf{y}}\partial_y\Phi + \hat{\mathbf{Z}}\partial_Z\Phi \qquad \text{(R5)}$$

in geopotential coordinates.

The combined gravity and pressure gradient forces are given by

$$-\frac{1}{\rho}\nabla p + \nabla\Phi = -\left[\left(\partial_\xi p + \partial_\xi N \partial_\zeta p\right)/\rho\right]\hat{\boldsymbol{\xi}} - \left[\left(\partial_\eta p + \partial_\eta N \partial_\zeta p\right)/\rho\right]\hat{\boldsymbol{\eta}} - \left(\partial_\zeta p/\rho + \mathrm{g}\right)\hat{\boldsymbol{\zeta}} \quad \text{(R6)}$$

in local Cartesian coordinates after using (R1) and (R4), and by

$$-\frac{1}{\rho}\nabla_G p + \nabla_G\Phi = -[(\partial_x p + \partial_x N \partial_Z p)/\rho]\hat{\mathbf{x}} - \left[\left(\partial_y p + \partial_y N \partial_Z p\right)/\rho\right]\hat{\mathbf{y}} - (\partial_Z p/\rho + \mathrm{g})\hat{\mathbf{Z}} \quad \text{(R7)}$$

in geopotential coordinates after using (R1) and (R5).

With the hydrostatic balance in both coordinate systems,

$$\partial_\zeta p/\rho + \mathrm{g} = 0, \quad \partial_Z p/\rho + \mathrm{g} = 0 \quad \text{(R8)}$$

Eq.(R6) and Eq.(R7) reduce to

$$-\frac{1}{\rho}\nabla p + \nabla\Phi = -\left[\left(\partial_\xi p\right)/\rho - \mathrm{g}\,\partial_\xi N\right]\hat{\boldsymbol{\xi}} - \left[\left(\partial_\eta p\right)/\rho - \mathrm{g}\partial_\eta N\right]\hat{\boldsymbol{\eta}} \quad \text{(R9)}$$

in local Cartesian coordinates, and

$$-\frac{1}{\rho}\nabla_G p + \nabla_G\Phi = -[(\partial_x p)/\rho - \mathrm{g}\,\partial_x N]\hat{\mathbf{x}} - \left[\left(\partial_y p\right)/\rho - \mathrm{g}\partial_y N\right]\hat{\mathbf{y}} \quad \text{(R10)}$$

in geopotential coordinates. Both Eq.(R9) and Eq.(R10) show the emergence of bumpy-geoid gradient in horizontal momentum equation in both local Cartesian coordinates and geopotential coordinates. The conclusion of M24 is wrong.

(2) Comment on Nonzero Horizontal Geopotential Gradient

"*I will close my comment by re-stating a few conclusions from two reviewers, which are public. Reviewer J. Thuburn succinctly describes the matter at hand. "For the present discussion, the important terms in (8) are the pressure gradient and geopotential gradient. In this approach we do indeed obtain a nonzero horizontal component to the geopotential gradient*."

Response: I agree with you.

(3) Comment on Approximation of Compensation

"*However, the crucial point is that, to an excellent approximation, it is compensated by an (almost) equal and opposite horizontal pressure gradient term, because the atmosphere and ocean are very close to hydrostatic balance (emphasis added by me). In this approach, if we neglect the horizontal component of gravity, then the pressure gradient would also lose its horizontal hydrostatic* *component. Thus, we effectively omit two terms or contributions whose sum is virtually zero and so make very small error overall*."

Response: Such approximation has never been verified. Now I have run out of time to respond (9 June deadline for closing discussion). I will show the invalidity of this approximation in revision.