

Reply to Reviewer 3 's Comments

6 May 2026

Thank you very much for your efforts and time to review my paper. My responses are summarized as follows.

1. Coordinate Invariance of Gravity-Pressure Gradient Forces and the Emergence of Bumpy-Geoid Gradient in the Horizontal Momentum Equation

1.1. Basic Physics

Let (ξ, η, ζ) be the local Cartesian coordinates with the basis vectors $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$. Gravity (\mathbf{g} , shown as red arrows in Fig. 1) is perpendicular to geopotential (Φ) surface. For $\mathbf{g} = \nabla\Phi$, and $\Phi = -g_0Z$, the bumpy geoid is defined by $Z = \zeta + N$, where ζ is the vertical Cartesian coordinate. There is no gravity component along the geopotential surface. With the hydrostatic equilibrium, gravity is balanced by the vertical pressure gradient force (PGF) but not the horizontal PGF, as shown as dashed arrows in Fig. 1. Let pressure be p_ζ at the Cartesian reference surface and be p_Z at the corresponding geopotential surface. The pressure on the geopotential surface is given by

$$p_Z = p_\zeta - g_0 \int_{\zeta}^{\zeta+N(x,y)} \rho dZ, \quad Z = -\frac{\Phi}{g_0}, \quad g_0 = 9.81 \text{ m s}^{-2} \quad (\text{R1})$$

where the density (ρ) is assumed uniform horizontally for simplicity without loss generality. Use of chain rules obtains the pressure gradient along the geopotential surface,

$$\partial p_Z / \partial x = \partial p_\zeta / \partial x - \rho g_0 \partial N / \partial x, \quad \partial p_Z / \partial y = \partial p_\zeta / \partial y - \rho g_0 \partial N / \partial y \quad (\text{R2})$$

which shows the emergence of bumpy-geoid gradients in the pressure gradient force along the geopotential surface.

*** Note that establishment of geopotential coordinates does not make the bumpy-geoid gradients vanish because they become part of the pressure gradient force along the geopotential surface.*

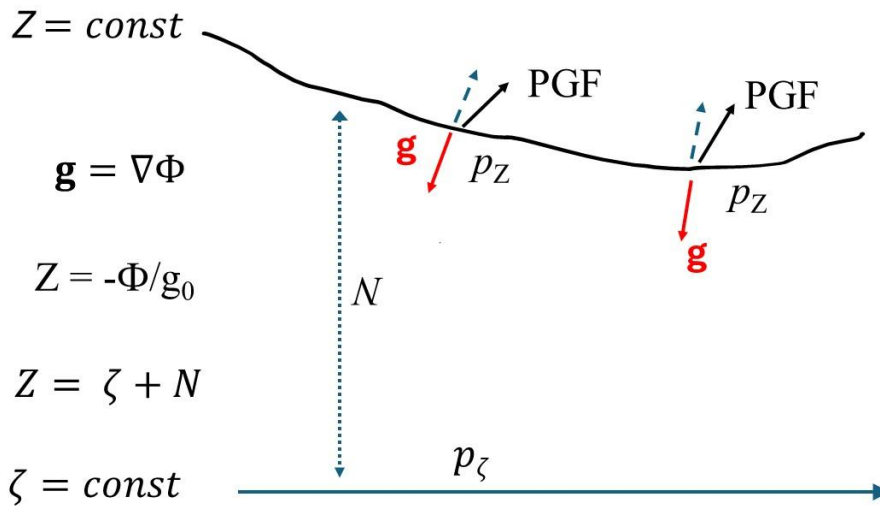


Fig. 1. Illustration of bumpy-geoid gradient as a part of the pressure gradient force along the geopotential surface.

1.2. Cartesian Coordinates

In Cartesian coordinates, one set of unit vectors works for everything because the basis vectors ($\hat{\xi}, \hat{\eta}, \hat{\zeta}$) are orthogonal; they have unit length; dot products are zero. The resultant gravity-pressure gradient forces, $-(\nabla p)/\rho + \nabla\Phi$, are major driving forces in ocean dynamics,

$$-(\nabla p)/\rho + \nabla\Phi = -\hat{\xi}[(\partial_{\xi} p)/\rho - \partial_{\xi}\Phi] - \hat{\eta}[(\partial_{\eta} p)/\rho - \partial_{\eta}\Phi] - \hat{\zeta}[(\partial_{\zeta} p)/\rho - \partial_{\zeta}\Phi] \quad (\text{R3})$$

where the gradient operator ∇ is given by

$$\nabla = \nabla_{\zeta} + \hat{\zeta}\partial_{\zeta}, \quad \nabla_{\zeta} \equiv \hat{\xi}\partial_{\xi} + \hat{\eta}\partial_{\eta} \quad (\text{R4})$$

The bumpy-geoid gradient

$$\nabla_{\zeta}\Phi = g_0\nabla_{\zeta}N \quad (\text{R5})$$

emerges in the horizontal resultant gravity-pressure gradient forces.

1.3. Geopotential Coordinates

The geopotential coordinates are defined by

$$x = \xi, \quad y = \eta, \quad Z = -\Phi/g_0, \quad Z = \zeta + N(x, y), \quad \mathbf{g} = \nabla\Phi \quad (\text{R6})$$

to try elimination of $\nabla_{\zeta}N$ from the horizontal momentum equation. The basis vectors of the geopotential coordinates are not orthogonal; their lengths vary with position; directions change from point to point. Because of this, a single set of basis vectors cannot simultaneously represent directions of coordinate lines and extract components of vectors cleanly. Therefore, the geopotential coordinates need dual (paired) basis vectors: one is tied to coordinate lines and the other is tied to gradient; or one describes how space is built (geometry) and the other describes how fields vary such as gradients.

1.4. Covariant and Contravariant (Reciprocal) Basis Vectors

The position vector \mathbf{r} in the geopotential coordinates is given by

$$\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + Z\mathbf{a}_z \quad (\text{R7})$$

Partial derivatives of \mathbf{r} with respect to (x, y, Z) give covariant basis vectors ($\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$) (first equation in your Item-2 Metric tensor)

$$\mathbf{a}_i = \frac{\partial \mathbf{r}}{\partial q^i}, \quad q^i = (x, y, Z)$$

to represent tangent to coordinate curves, to point in the direction of increasing coordinate (x, y, Z) , and to form the natural basis for displacement. So, any small displacement is:

$$d\mathbf{r} = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dZ \quad (\text{R8})$$

which means that we use the covariant basis vectors ($\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$) to describe geometry or motion along geopotential coordinate direction.

The contravariant (reciprocal) basis vectors ($\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^z$) are defined by gradient of coordinates (first equation in your Item-4 Dual basis vectors)

$$\mathbf{a}^i = \nabla q^i$$

The gradient operator ∇ is given by (the last equation in your Comments Item 5),

$$\nabla = \partial_x \mathbf{a}^x + \partial_y \mathbf{a}^y + \partial_z \mathbf{a}^z \quad (\text{R9})$$

Use of covariant metric tensor $[g_{ij}]$ leads to the gradient operator with the covariant basis vectors $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z)$,

$$\nabla = \mathbf{a}_x(\partial_x + N_x \partial_z) + \mathbf{a}_y(\partial_y + N_y \partial_z) + \mathbf{a}_z \partial_z \quad (\text{R10})$$

which likes the second last equation in your “Comment-Item 5 Gradient operator”. Note that yours is incorrect since the covariant basis vectors are $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z)$, not $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$.

1.5. Gravity-Pressure Gradient Forces with the Covariant Basis Vectors

The geopotential gradient is given by

$$\nabla \Phi = (\partial_x \Phi + N_x \partial_z \Phi) \mathbf{a}_x + (\partial_y \Phi + N_y \partial_z \Phi) \mathbf{a}_y + \partial_z \Phi \mathbf{a}_z = -g_0 \mathbf{a}_z \quad (\text{R11})$$

where Eq.(R5) is used. The pressure gradient is given by

$$\nabla p = (\partial_x p + N_x \partial_z p) \mathbf{a}_x + (\partial_y p + N_y \partial_z p) \mathbf{a}_y + \partial_z p \mathbf{a}_z \quad (\text{R12})$$

The resultant gravity-pressure gradient forces are

$$-(\nabla p)/\rho + \nabla \Phi = -[(\partial_x p + N_x \partial_z p)/\rho] \mathbf{a}_x - [(\partial_y p + N_y \partial_z p)/\rho] \mathbf{a}_y - (\partial_z p/\rho + g_0) \mathbf{a}_z \quad (\text{R13})$$

Use of hydrostatic balance,

$$\partial_z p/\rho + g_0 = 0 \quad (\text{R14})$$

leads to

$$-(\nabla p)/\rho + \nabla \Phi = -[(\partial_x p - \rho g_0 N_x)/\rho] \mathbf{a}_x - [(\partial_y p - \rho g_0 N_y)/\rho] \mathbf{a}_y \quad (\text{R15})$$

which shows the existence of $(g_0 \nabla_h N)$ on the $(\mathbf{a}_x, \mathbf{a}_y)$ surface.

1.6. Gravity-Pressure Gradient Forces with the Contravariant Basis Vectors

With the contravariant basis vectors, use of Eq.(R9) for Φ leads to the geopotential gradient

$$\nabla \Phi = (\partial_x \Phi) \mathbf{a}^x + (\partial_y \Phi) \mathbf{a}^y + (\partial_z \Phi) \mathbf{a}^z \quad (\text{R16})$$

and for p leads to the pressure gradient

$$\nabla p = \partial_x p \mathbf{a}^x + \partial_y p \mathbf{a}^y + \partial_z p \mathbf{a}^z \quad (\text{R17})$$

The resultant gravity-pressure gradient forces are

$$-(\nabla p)/\rho + \nabla \Phi = -[(\partial_x p - \rho \partial_x \Phi)/\rho] \mathbf{a}^x - [(\partial_y p - \rho \partial_y \Phi)/\rho] \mathbf{a}^y - [(\partial_z p - \rho \partial_z \Phi)/\rho] \mathbf{a}^z \quad (\text{R18})$$

which shows the emergence of bumpy-geoid gradient

$$g_0 \nabla_h N = \nabla_h \Phi, \quad \nabla_h \equiv \partial_x \mathbf{a}^x + \partial_y \mathbf{a}^y \quad (\text{R19})$$

on the $(\mathbf{a}^x, \mathbf{a}^y)$ surface.

Thus, the resultant gravity and pressure gradient forces have $\mathbf{g}_0 \nabla_h N$ in the horizontal momentum equation with the Cartesian coordinates, and the geopotential coordinates with both covariant and contravariant basis vectors. This term is currently missing in ocean dynamics and needs to be included in the horizontal momentum equation [i.e., Eq.(43) in the preprint]

$$\rho \left(\frac{D\mathbf{U}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{U} \right) = -\nabla_h \hat{p} + \hat{p} \mathbf{g}_0 \nabla_h N, \quad \nabla_h \equiv \mathbf{a}^x \partial_x + \mathbf{a}^y \partial_y \quad (\text{R20})$$

where ∇_h is defined by

$$\nabla_h \equiv \mathbf{a}_x \partial_x + \mathbf{a}_y \partial_y \quad (\text{covariant}) \text{ or } \nabla_h \equiv \mathbf{a}^x \partial_x + \mathbf{a}^y \partial_y \quad (\text{contravariant}) \quad (\text{R21})$$

2. Response to Comments on Eq. 25 and Eq. 26

There was a typo in Eq.(26), which should be

$$\mathbf{G}^{-1} = [g^{ij}], \quad g^{ij} = \frac{\partial x^i}{\partial \xi^p} \frac{\partial x^j}{\partial \xi^q} \delta_{pq}, \quad \delta_{pq} = \begin{cases} 1, & p = q \\ 0 & p \neq q \end{cases} \quad (\text{R22})$$

which was used to derive Eq. (27). It does not affect the derivation of Eq.(27). In Eq.(26), (ξ^1, ξ^2, ξ^3) are the Cartesian coordinates (ξ, η, ζ) and (x^1, x^2, x^3) are the geopotential coordinates (x, y, Z) . I used

$$\begin{aligned} \mathbf{r} &= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + Z\hat{\mathbf{z}} \\ x = \xi, \quad y = \eta, \quad Z = \zeta + N &\Rightarrow Z_\xi = N_\xi, Z_\eta = N_\eta, \quad Z_\zeta = 1 \end{aligned} \quad (\text{R23})$$

for (R22) [i.e., Eq.(26) after typo being corrected] to get Eq.(27)

$$\mathbf{G}^{-1} = [g^{ij}] = \begin{bmatrix} 1 + Z_\xi^2 & Z_\xi Z_\eta & Z_\xi Z_\zeta \\ Z_\xi Z_\eta & 1 + Z_\eta^2 & Z_\eta Z_\zeta \\ Z_\xi Z_\zeta & Z_\eta Z_\zeta & Z_\zeta^2 \end{bmatrix} = \begin{bmatrix} 1 + N_\xi^2 & N_\xi N_\eta & N_\xi \\ N_\xi N_\eta & 1 + N_\eta^2 & N_\eta \\ N_\xi & N_\eta & 1 \end{bmatrix}$$

which is like the last equation of Item-2 Metric tensor.

3. Response to Comments on Gradient Operator (Eq. 31)

I admit that Eq.(31) is too messy and not well derived. The basis vectors should not be replaced by normalized vectors (my mistake). The correct gradient operator is given by Eq.(R10) in this reply.

4. Response to Comments on Reconciliation with McWilliams (2024)

Since the resultant gravity-pressure gradient forces are coordinates invariant, use of geopotential coordinates does not make the bumpy-geoid gradient, $\mathbf{g}_0 \nabla_h N$, disappear in the horizontal momentum equation. This term ($\mathbf{g}_0 \nabla_h N$) appears in the pressure gradient force and disappears from gravity with the covariant basis vectors $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z)$, and disappears from the pressure gradient force and appears in gravity with the contravariant basis vectors $(\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^z)$. McWilliams (2024) mistakenly used the disappearance of $\mathbf{g}_0 \nabla_h N$ from the pressure gradient force but disregarded the appearance of $\mathbf{g}_0 \nabla_h N$ in gravity with the contravariant basis vectors $(\mathbf{a}^x, \mathbf{a}^y, \mathbf{a}^z)$ to claim the vanish of $\mathbf{g}_0 \nabla_h N$ in the horizontal momentum equation.