

On the theoretical limitations of joint inversion for basal slipperiness and effective viscosity in ice-flow models

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Reply to Anonymous Referee #1

We are very grateful to both reviewers for their helpful advice and positive comments. We have incorporated their suggestions into a revised version of the manuscript. We have addressed the comments of Anonymous Referee #1 (highlighted in blue) as follows:

Below are two points that could be considered for a minor revision of the paper:

- **as the proposed experiment involves out-of-phase perturbations, it could be interesting to show if the results still holds for perturbations that would be in phase or not completely out of phase.**

Thank you for this suggestion - it raises an interesting question of whether the current experiment is a unique point in the parameter space, and so could give deceptively good or bad results. The alignment of the viscosity and slipperiness perturbations themselves is perhaps not what is important - what matters is the alignment of the resulting perturbations in the surface fields. The transfer functions are imaginary and so introduce a phase component of their own into the resulting surface field perturbations. The functional form of the Eta and C transfer functions are distinct, which is why the phase difference won't be preserved in the effect on the surface fields. The combined effect on the surface fields is frequency dependent and not necessarily in-phase or out-of-phase. Given this, we prefer not to include an additional experiment which might confuse the readers, but instead have included this point in the new section "5. Limitations" (see below) to highlight that the space of different frequencies and different phase differences for the perturbations hasn't been explored.

- **it would be interesting to add a discussion section to summarise the main assumptions and how this could affect inversions for real settings; For example, the proposed experiment assume a perfect model, linear friction and flow laws, uncorrelated observations errors, perfectly known surface and bed elevations, well constrained prior.**

Thank you for this suggestion, and we have now added a new section "5. Limitations" ahead of the Conclusions:

"5. LIMITATIONS

The results presented in this paper have made a number of necessary simplifying assumptions, so that the inversion procedure could be broken down and understood from a theoretical stand point. In this section we discuss these assumptions and their potential impact.

All experiments have been done with respect to idealised flow down a uniform, laterally-confined channel. Different regions of the flow experience stresses from bed sliding and lateral drag, each to a varying degree, and while the overall flow is simplified, these different regions should be representative of the stresses found within localised regions of a real ice-sheets. The transfer functions derived in this study have distinct frequency profiles, indicating that in general there is

not mixing between the viscosity and slipperiness fields. Our expectation is that more complicated stress patterns observed in real ice-sheets should differentiate the transference to the surface of perturbations in slipperiness and viscosity even further.

We have also simplified the flow by assuming linear bed sliding and rheology, when in practice non-linear flow is supported by observations. Higher values of m in the bed sliding law will generally lead to faster velocities, and this will be the dominant impact on the transference to the surface of perturbations in slipperiness. Higher values of n will concentrate higher strain rates into a narrow band close to the lateral margin, and so the gradients in velocity -- which are necessary for the transference of viscosity perturbations to the surface -- will occur in a narrower region of the domain. Our expectation is that, if anything, values of $n > 1$ and $m > 1$ will likely differentiate the transference to the surface of perturbations in the slipperiness and viscosity even further. Studies by Gudmundsson and Raymond [1] performed a numerical analysis of the transfer functions for higher order n and m , although applied to the full Stokes equations and with a different background state, which showed limited impact on the amplitude and phase of the transfer functions. Nonetheless, an exploration of the effect of non-linear bed friction and rheology on the inversion could be the subject of a further numerical study exploring the wider parameter space.

Another key assumption feeding into these experiments is that the SSA equations are a true representation of the ice-sheet physics. This approximation is appropriate for shallow ice-sheets where the thickness is small compared to the horizontal span, but will breakdown for thicker tributaries. In this study we have only considered measurement errors in the surface velocities, and assumed perfect knowledge of the surface elevation, while considering only uncorrelated measurement errors. Relaxing these assumptions could lead to more difficulty in separating the effects of slipperiness and viscosity in the inversion.

Finally, we would like to refer the reader to the transfer functions in Figure #2. These illustrate the strength of transference to the surface across a range of perturbation frequencies, while the experiments in Figures #3 to #5 are for a particular choice of perturbation frequency, and for a particular phase shift between the viscosity and slipperiness perturbations. The space of different frequencies and different phase differences for the perturbations hasn't been explored in the experimental results.

[1] Gudmundsson & Raymond, On the relationship between surface and basal properties on glaciers, ice sheets, and ice streams, JoGR (2005). "

minor comments:

- **Section 2.1 : define the notation with bar on top for the background fields. This is defined only in the Appendix.**

Thank you for catching this. This definition has now been added to the manuscript body as well.

- **Line 203 "prior modelled using the Matern covariance function" and Appendix: equivalence between Tikhonov regularisation and covariance functions can be derived analytically for some special cases (see also e.g. Barth et al. (2014)) and helps to understand the role of the regularisation. But can be worth to mention that in general it will not be explicit. The effect of a prior error covariance matrix parameterised using Matern correlation functions can be reproduced using a diffusion operator, as it is done**

in e.g. FeniCs (Recinos et al., 2023).

Thank you for these references and extra details, we have included them in the revised manuscript line 204:

“...maximising the a-posteriori distribution in the Bayesian MAP framework with uncorrelated Gaussian measurement errors and a prior distribution modelled using the Matérn covariance function (e.g. Lindgren et al., 2011; Barth et al., 2014). In general, the relationship between the covariance of the prior distribution and regularisation terms in a cost function is not explicit, although it is possible to reproduce a covariance parameterised with Matern correlation functions using a diffusion operator, as done in FeniCs (Recinos et al, 2023).

[2]Barth, A., Beckers, J.-M., Troupin, C., Alvera-Azcárate, A., Vandenbulcke, L., 2014. divand-1.0: n-dimensional variational data analysis for ocean observations. *Geoscientific Model Development* 7, 225–241. <https://doi.org/10.5194/gmd-7-225-2014>

[3]Recinos, B., Goldberg, D., Maddison, J.R., Todd, J., 2023. A framework for time-dependent Ice Sheet Uncertainty Quantification, applied to three West Antarctic ice streams (preprint). *Ice sheets/Data Assimilation*. <https://doi.org/10.5194/tc-2023-27>”

- **Line 227 "T*" is an m x m" matrix; m is defined in the Appendix not in the main text and m is already used for the exponent of the friction law.**

Thank you for catching this. We have now added this definition to the manuscript body as well, and switched to dimension “d” everywhere.