These are great discussion comments. Given the interest, we intend to examine ways to show surface and bottom crevasse penetration depths without over cluttering existing figures.

The more weighty question posed—and this seems aligned with Roger Buck's question — are two fold. The first is whether forces in our numerical simulation balance and here the answer is definitely yes. The second is whether the stress is acting between crevasses in the unbuttressed case is given by the stress associated with ocean pressure. Here the answer is yes, but with the important caveat that only in a very specific depth integrated sense. We have had a wonderful dialogue with Yao offline which has helped clarify several assumptions in the horizontal force balance. Portions of this comment may end up as a supplementary note to our study, but there are enough open questions that it hints at a follow up study. Our response here is long and overly detailed because this has been a fun problem to work out.

As we eventually show, we can use our numerical simulations combined with theory to dispel once and for all the myth that the Nye model fails to satisfy a horizontal force balance. Doing so, however, does raise questions about the self-consistency of the assumption of closely spaced crevasses and the relationship between the horizontal force balance and closely spaced crevasses.

Before delving into the horizontal force balance, our model maintains a (numerically approximate) stress balance and it is worth stepping through how it does so. We numerically approximate conservation of momentum in both the horizontal and vertical directions:

(1) $$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0,$$

(2) $$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} = \rho_i g.$$

Crucially, in addition to traction (or displacement/velocity) boundary conditions at the right and left edges of our domain, we also impose traction boundary conditions associated with the ice-ocean and ice-atmosphere interface. These boundary conditions also apply to crevasse walls. A consequence of the traction boundary conditions is that tractions normal to surface crevasse walls and at surface crevasse tips must vanish and normal traction at bottom crevasse tips must be continuous with ocean water pressure.

The force (balance) awakens.

We have been reluctant to compare our results to the horizontal force balance, but we do here for clarity. Our understanding of the horizontal force balance is that the horizontal stress σ_{xx} in the region beneath surface crevasse and above bottom crevasses is written as (e.g., Equation 20 in Buck, 2023 with a slight change in notation):

(3) $\sigma_{xx} = R_{xx} - \rho_i g(s - z)$,

where s denotes the surface elevation of the ice shelf, ρ_i is the density of ice and g is the acceleration of gravity. Although never stated in Buck (2023), it seems as though an assumption required to obtain Equation (3) is that the vertical stress is given by

(4) $\sigma_{zz} = -\rho_i g(s - z)$,

such that the stress difference $\Delta\sigma = \sigma_{xx} - \sigma_{zz} = R_{xx}$. Now suppose a surface crevasse propagates to a depth d_s. We require that σ_{zz} must vanish at the crack tip. Evaluating Equation (4) at the tip of a surface crevasse located at $z = s - d_s$ results in $\sigma_{zz} = -\rho_i g d_s$, which is inconsistent with the traction boundary condition requirement that normal stress σ_{zz} vanishes at the bottom of surface crevasses. A discontinuity in the normal stress would require a vertical acceleration. A discontinuous stress, in addition to being unphysical, would render application of the divergence theorem in the horizontal force balance questionable.

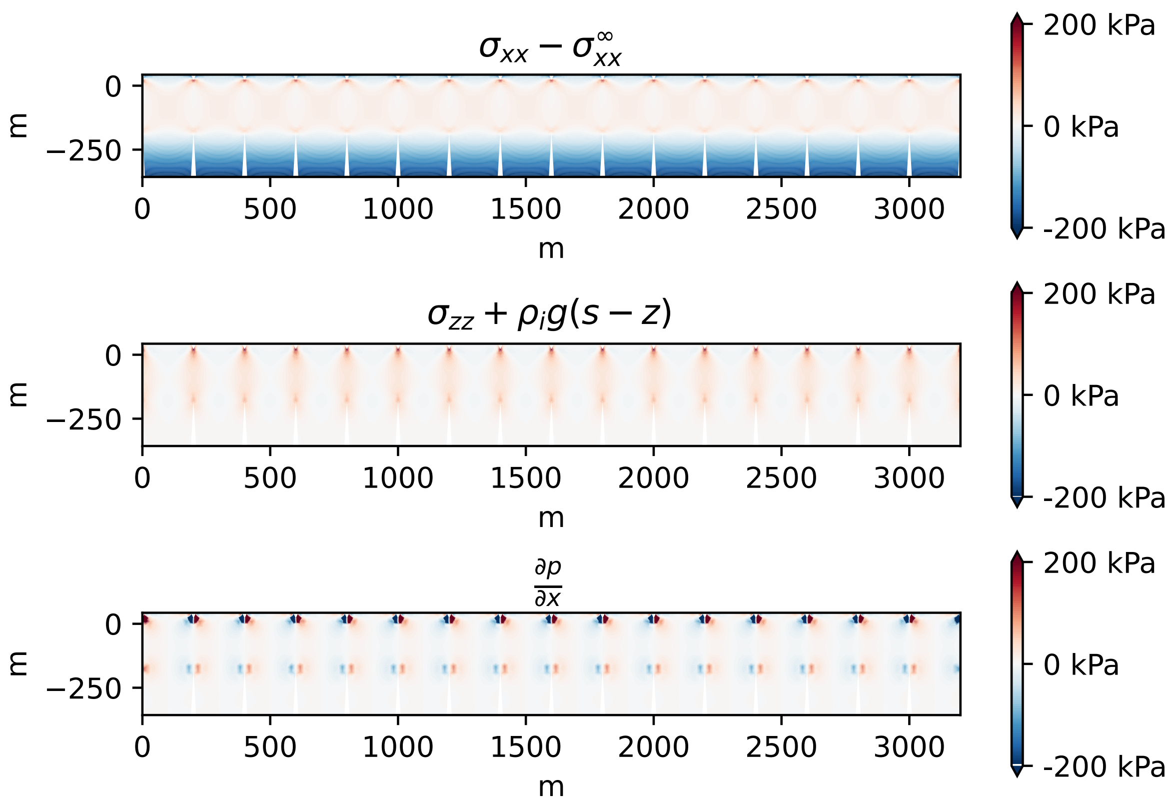

Figure R4. Top panel: Difference between far field stress applied as a traction boundary condition and simulated horizontal stress. Middle panel: Difference between the simulated vertical stress and hydrostatic stress assumed by the horizontal force balance (e.g., Equation 4). Bottom panel: Horizontal pressure gradient in our simulation.

By comparison, Figure R4 shows results from our simulations for a crevasse spacing equal to half of the ice thickness with initially 20 m wide crevasses (crevasse width/thickness = 0.05<<1). The middle panel of Figure R4, shows that Equation (4) might be approximately accurate away from crevasses, but fails close to crevasses, where the inability to satisfy the traction boundary condition becomes apparent. A consequence is that our numerical simulations predict a pressure gradient force directed into crevasses that is not accounted for in the horizontal force balance and a horizontal stress that differs markedly from that assumed in Buck (2023). To make progress we need to better identify the conditions when Equations (4) and (5) are expected to be valid.

The last force balance.
We now show that the horizontal force balance is simply a restatement of the Nye model, although with several important caveats. To do this, we shall use a similar argument to that used in Buck (2023), but applied to the geometry considered in our numerical simulations.

The geometry we consider is one in which there is an evenly spaced array of co-aligned surface crevasses and bottom crevasses with crevasse tips separated by distance w with crevasses centered at $x = 0, \pm w, \pm 2w, \ldots$. Surface and bottom crevasses penetrate to depths d_s and d_b and we assume that the maximum width of crevasses after they open is of the order of the crevasse spacing w. Similar to the horizontal force balance, we approximate the horizontal stress in the portion of ice in a crevassed column by:

(5a) $\sigma_{xx}^c = 0, \quad s - d_s < z \leq s$

(5b) $\sigma_{xx}^c \approx R_{xx}^c - \rho_i g(s - z), \quad b + d_b \leq z \leq s - d_s,$

(5c) $\sigma_{xx}^c = \rho_w g z, \quad b \leq z < b + d_b.$

Although not necessary, we assume that R_{xx}^c in Equation (5b) is constant. The fact that Equation (5) must be an approximation has already been made clear because the vertical stress fails to satisfy the traction boundary conditions and this will result in a stress concentration near the tip of bottom and surface crevasses that is not accounted for by Equation (5) (see, e.g., Figure R4).

Defining the dimensionless parameters $\epsilon_s = w/d_s$ and $\epsilon_b = w/d_b$, we anticipate corrections to Equations (4) and (5) must emerge in a region around the crack tips to satisfy the traction conditions. The size of these regions scale with ϵ_s and ϵ_b to a power (with the power determine by the rheology). Nonetheless, if crevasses are narrow, then the closely spaced crevasse approximation assumes the limit in which $\epsilon_s \to 0$ and $\epsilon_b \to 0$ and for this limit, we anticipate that Equation (5) and hence, Equation (4) may be approximately valid. (Our numerical simulations confirm this, but the horizontal pressure gradient term remains.)

We next seek to estimate the horizontal and vertical stress at the mid-point between crevasses, located a distance $w/2$ from the crevasse. To do this, we invert the procedure of the horizontal force balance and assume that the horizontal stress in this completely intact column of ice is given by:

(6) $$\sigma_{xx}^{intact} = R_{xx}^{intact}(z) - \rho_i g(s - z).$$

For simplicity, we denote the total fraction of ice thickness penetrated by crevasses with the symbol r and, although not necessary, further assume that the surface and bottom crevasses occur in hydrostatic proportions such that $r_s = d_s/H = r(1 - \rho_i/\rho_w)$ and $r_b = d_b/H = r\rho_i/\rho_w$. Denoting the depth-averaged resistive stress as $\bar{R}_{xx}^{intact}$ setting $\bar{R}_{xx}^{intact} = cR_{xx}^{c}$ and depth-integrating Equations (5) and (6) using an analogous procedure to Buck (2023), we to find:

(7) $$c = \frac{1}{2}r^2 S0 - r + 1.$$

It is worth emphasizing that we did not apply a far-field traction or displacement boundary condition in deriving Equation (7); it is sufficient that the horizontal stress is approximated by Equation (5) irrespective of the manner in which the stress is created. Clearly, the procedure we have applied to determine c is the inverse of the horizontal force balance performed by Buck (2013)—instead of estimating the stress in the crevassed portion from the stress in a fully intact column, we estimated the stress in an intact column from the stress in the crevasses column. Equation (7) is, however, valid for any crevasse penetration ratio.

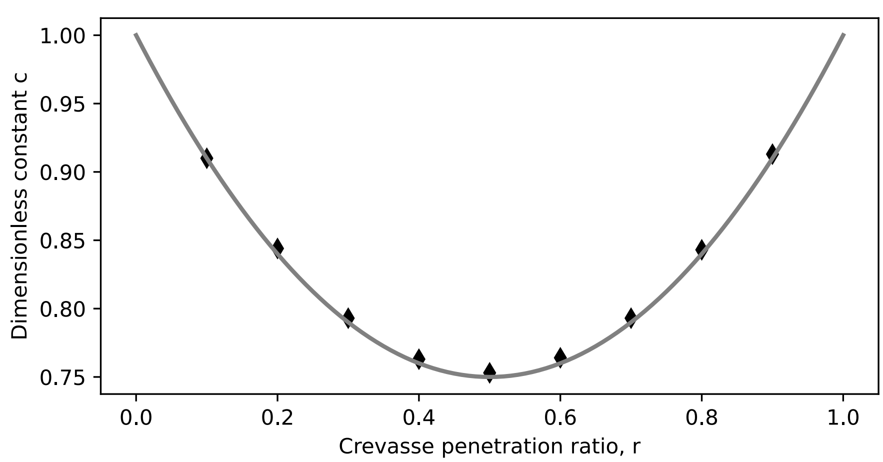

Figure R5. Comparison between the theoretical relationship defined by Equation (7) and results inferred from simulations. Black triangles represent simulation results. The solid gray line denotes the theoretical result from Equation (7). Simulations were performed using traction boundary conditions and a crevasse spacing of w=H/16 with initially narrow crevasses. Results deviate from theory for spacing larger than this.

Figure R5 shows a comparison between the theoretical value of c inferred from our numerical model versus the theoretical estimate provided in Equation (7). We clearly see that our numerical simulations provides an excellent approximation of the analytic expression we just derived, providing further confidence that both our numerical and theoretical approximations are appropriate.

Two things are clear

at this point. First, the Nye model *does* satisfy a horizontal force balance. Second, the horizontal force balance is just the Nye model in disguise.

There is, however, one important question remaining. The horizontal force balance only applies in a depth-integrated sense. As such, there are an infinite number of resistive stress distributions $R_{xx}^{intact}(z)$ that satisfy the depth-integrated force balance. By contrast, with stress/displacement applied at the boundaries, we know that the solution for stress in the interior region must be unique. This suggests that even if a resistive stress distribution satisfies the force balance, the distribution may not be consistent with closely spaced crevasses; there is a consistency issue that needs to be addressed.

The return of the matching condition.
To find a unique solution, we need to match the solution for σ_{xx}^{intact} with the solution for σ_{xx}^{c}. For the incompressible linear elastic case, we can introduce a stream function, solve the biharmonic equation and apply the matching conditions. This involves a somewhat tedious treatment of the boundary layer near the crack edges. However, we can quickly sketch out a more heuristic matching procedure, postponing a more in depth discussion to a future paper.

To do this, we assume that σ_{xx} is continuous everywhere, which is just a restatement of the Nye criterion that horizontal stress vanishes at crevasse tips. Since crevasse opening is of the order of crack spacing, w, it follows that $\sigma_{xx} \approx 0$ at $z = s$ (crevasse opening in closely spaced crevasses completely relieves the deviatoric stress at the surface). A similar argument for the bottom crevasse suggests that $\sigma_{xx} \approx -\rho_i g H$ at $z = b$. Finally, insisting that σ_{xx} is continuous in the middle portion of the ice that is untouched by crevasses hints that the solution for horizontal stress takes on an approximate form:

(8a) $$\sigma_{xx}^{intact} \approx \left[R_{xx}^{c} \left(\frac{s-z}{d_s} \right) - \rho_i g (s-z) \right], \quad s - d_s < z \leq s$$

(8b) $$\sigma_{xx}^{intact} \approx \left[R_{xx}^{c} - \rho_i g (s-z) \right], \quad b + d_b \leq z \leq s - d_s,$$

(8c) $$\sigma_{xx}^{intact} \approx \left[R_{xx}^{c} \left(\frac{z-b}{d_b} \right) - \rho_i g (s-z) \right], \quad b \leq z < b + d_b.$$

Note that this solution implies that σ_{xz} is of the order of ϵ_s and ϵ_b or smaller. However, this solution, like the vertical stress balance, does not satisfy the requirement that $\sigma_{xx} = 0$ along the crevasse walls (except at z=s and z=b). Like the vertical stress, we anticipate the need for a boundary layer near the crack walls to satisfy the “inner” traction conditions at the crack walls.

Depth-averaging equations (8a)-(8c), we find:

(9) $$\frac{1}{H}\int_b^s R_{xx}^{intact}\mathrm{d}z = \bar{R}_{xx}^{intact} = \left(1 - \frac{d}{2}\right) R_{xx}^c.$$

Finally, making use of the definition that $\bar{R}_{xx}^{intact} = cR_{xx}^c$, and then using Equation (7) and (9) we find that:

(10) $$R_{xx} r\left(S_0 r - 1\right) = 0.$$

We immediately see that for a given S_0 there are two solutions for crevasse penetration ratio: $r = 0$ and $r = 1/S_0$. The result $r = 0$ is a statement of the fact that, by definition, the forces balance when crevasse depths vanish and is analogous to the LEFM solution that small cracks will not grow. The second result $r = 1/S_0$ is just Nye's crevasse depth. We have now derived Nye's model using our numerical simulations and a version of the horizontal force balance.

We can further test the consistency of our numerical model using offset crevasses (see, e.g., our response to Roger Buck), but applying Equation 8 as a traction boundary condition for intact ice. Using this set of traction boundary conditions, we offset crevasses and once again find Nye's model is correct for closely spaced crevasses. We have now verified the Nye model with displacement boundary conditions and two sets of traction boundary conditions.

Although the matching procedure we used was slightly heuristic, we have now derived the Nye result directly from a numerical model and force balance and shown that the stress fields are consistent between numerical models and theory. This should dispel any lingering doubt that the Nye model fails to satisfy a balance of horizontal forces.

What the Nye model does not satisfy, however, is a condition where the resistive stress is constant with depth in the region between crevasses, as assumed in the horizontal force balance. This implies that there appears to be an inconsistency in the assumptions of the horizontal force balance that we alluded to earlier in our response to Roger Buck. The assumption of closely spaced crevasses combined with Equation (5) implies Equation (8) and vice versa. The horizontal force balance may thus satisfy a force balance, but perhaps not the requirements for closely spaced crevasses. Although the conditions for the horizontal force balance remain murky, at this point it should be clear that Nye was right. However, given the strong limitation on when Nye's criteria can be applied, outside of a handful of special circumstances, conditions for the Nye model seem unlikely to be readily realized in nature.