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Development and improvement of a nonhydrostatic spectral model using non-constant coefficient semi-implicit and vertically conservative semi-Lagrangian schemes

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Abstract

A two-dimensional x–z nonhydrostatic spectral model using non-constant coefficient semi-implicit and vertically conservative semi-Lagrangian schemes has been developed, which is computationally efficient by allowing for long timesteps in simulations. The model incorporates several improvements related to free-slip surface boundary conditions, prognostic variables in spectral space, and the semi-Lagrangian and semi-implicit schemes. These improvements enhance the numerical stability, especially in cases with steep orography, and improve model results. The model was tested with various test cases (e.g., mountain wave test cases), confirming that a single recalculation of the non-constant coefficient linear terms is usually sufficient to stably solve the equations associated with the non-constant coefficient semi-implicit scheme, even in the cases with a steep mountain with an average slope of 45°. The model also ran stably even in the case of an extremely steep mountain with an average slope of 63.4° by performing several iterations using the preconditioned general conjugate residual method. In all these cases, good results were obtained with long timesteps.

1. Introduction

The global spectral atmospheric models used by the Japan Meteorological Agency (JMA, 2019) and Meteorological Research Institute (MRI; Yukimoto et al., 2011, 2019) are hydrostatic. In these models, a two-time-level, semi-implicit semi-Lagrangian scheme is used to allow for longer timesteps, and a vertically conservative semi-Lagrangian scheme (Yoshimura and Matsumura, 2003; Katayama et al., 2005; Yoshimura and Matsumura, 2005; Yukimoto et al., 2011) is used to improve conservation, which reduces the temperature biases around the tropical tropopause (Yoshimura and Matsumura, 2003). The reduced grid (Juang, 2004; Miyamoto, 2006, 2009) is used to save computational cost in the JMA atmospheric model and has also been introduced into the latest development version of the MRI atmospheric model. The advantage of the spectral method

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used in the hydrostatic models is that, in addition to its good accuracy, the linear simultaneous equations associated with the semi-implicit scheme can be easily solved in spectral space without iterations.

We have been developing a nonhydrostatic option for high-resolution simulations with the MRI global atmospheric spectral model. The prognostic equations used in the nonhydrostatic option are essentially the same as those in the ALADIN-NH (Non-Hydrostatic version of Aire Limitée Adaptation Dynamique Développement InterNational) nonhydrostatic limited-area spectral model (Bubnová, 1995; Bénard et al., 2010) and those in the nonhydrostatic version of the IFS (Integrated Forecasting System) global spectral model (Wedi and Smolarkiewicz, 2009; Dueben et al., 2020; Wedi et al., 2020). However, there are some differences in time discretization. In the ALADIN-NH and nonhydrostatic IFS, an iterative-centered implicit (ICI) scheme (Bénard et al., 2005) is used to improve model stability, where whole terms are recalculated at each iteration. In the nonhydrostatic option in the MRI model, a non-constant coefficient semi-implicit scheme is used, where only non-constant coefficient linear terms are recalculated per iteration and, therefore, the computational cost per iteration is lower than that for the ICI scheme. A fast-convergent iterative method, such as the general conjugate residual (GCR) method (Eisenstat et al., 1983) with preconditioning, can be used to solve the linear equations associated with the non-constant coefficient semi-implicit scheme in grid models (e.g., Skamarock et al., 1997; Thomas et al., 2003), and also can be used in the spectral model (Yoshimura 2012, Appendix E3).

The early version of the nonhydrostatic option in the previous MRI global spectral model was briefly described by Yoshimura (2012). An option using double Fourier series instead of spherical harmonics as spectral basis functions was also developed to improve the computational efficiency at high resolution (Yoshimura and Matsumura, 2005; Yoshimura, 2012; Yoshimura, 2022). Nakano et al. (2017) presented the results of typhoon simulations in the previous MRI model using the early versions of the nonhydrostatic and double Fourier series options (DFSM), along with the results of the Nonhydrostatic ICosahedral Atmospheric Model (NICAM) and the Multi-Scale Simulator for the Geoenvironment (MSSG).

To test and improve the nonhydrostatic spectral dynamical core, we have developed a nonhydrostatic spectral two-dimensional x–z model. In the model, several modifications have been introduced, including improved treatment of the free-slip surface boundary conditions, modification of the prognostic variables in wave space, and improvements related to the semi-Lagrangian and the semi-implicit schemes. The model was tested with various test cases, which confirmed that these modifications enhance the numerical stability in cases with steep orography, reduce the number of iterations needed to solve the non-constant coefficient semi-implicit equations, and improve the results.

The remainder of the paper is organized as follows. Section 2 describes the prognostic equations for the global and the two-dimensional x–z nonhydrostatic spectral models and how to discretize these, including the free-slip surface boundary conditions. Section 3 shows the results of test cases of the two-dimensional x–z nonhydrostatic model. Section 4 presents the conclusions of this study and considers future works.

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2. Nonhydrostatic model

Section 2.1 and Appendix A describe the mass-based hybrid vertical coordinate system. Section 2.2 describes the prognostic equations for the global and two-dimensional x–z nonhydrostatic spectral models, and Section 2.3 describes an improvement related to the free-slip surface boundary conditions. Section 2.4 and Appendix C describe the discretization of the prognostic equations. Appendix B describes the vertically conservative semi-Lagrangian scheme including several improvements such as a spatially averaged Eulerian treatment of mountains and a semi-Lagrangian treatment of pressure change due to horizontal advection. Appendix D describes the spectral transform of the prognostic variables, and Appendix E shows how to solve the equations for the non-constant coefficient semi-implicit scheme. Appendix F describes the Rayleigh damping.

2.1. Vertical coordinate

A mass-based vertical coordinate (Simmons and Burridge, 1981; Laprise, 1992; Bubnová, 1995) is used. The hydrostatic pressure π is given by

$$\pi = A(\eta) + B(\eta)\pi_s, \quad 0 \leq \eta \leq 1, \tag{1}$$

where η is a hybrid vertical coordinate and π_s is the surface hydrostatic pressure. By discretizing Eq. (1) vertically, the hydrostatic pressure at the half level $k - 1/2$ is given by

$$\pi_{k-1/2} = A_{k-1/2} + B_{k-1/2}\pi_s, \ \ 1 \leq k \leq K+1, \tag{2}$$

where $\pi_{1/2} = \pi(\eta = 1) = \pi_s$, $\pi_{K+1/2} = \pi(\eta = 0) = 0.0$, and K is the number of vertical full levels. The full level k represents the vertical layer between $\pi_{k-1/2}$ and $\pi_{k+1/2}$. The hydrostatic pressure depth $\Delta\pi_k$ at the full level k is given by

$$\Delta\pi_k \equiv \pi_{k-1/2} - \pi_{k+1/2} = \Delta A_k + \Delta B_k \pi_s, \tag{3a}$$

$$\Delta A_k \equiv A_{k-1/2} - A_{k+1/2}, \tag{3b}$$

$$\Delta B_k \equiv B_{k-1/2} - B_{k+1/2}. \tag{3c}$$

An example of how to obtain $A_{k-1/2}$ and $B_{k-1/2}$ is shown in Appendix A, which is used in the two-dimensional x–z nonhydrostatic model in Sect. 3.

2.2. Nonhydrostatic equations

The following prognostic equations are used, which are essentially the same as in Bubnová (1995):

$$\mathcal{P} \equiv (p - \pi)/\pi, \tag{4}$$

$$\mathcal{Q} \equiv \ln(p/\pi) = \ln(1 + \mathcal{P}) \tag{5}$$

$$m \equiv \partial\pi/\partial\eta, \tag{6}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{F}_v, \tag{7a}$$

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$$\boldsymbol{F}_{\boldsymbol{v}} \equiv -RT[\nabla \ln \pi + \nabla Q] - \left[1 + \frac{1}{m}\frac{\partial(\pi\mathcal{P})}{\partial\eta}\right]\nabla\Phi, \tag{7b}$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = F_w, \tag{8a}$$

$$F_w \equiv \frac{g}{m}\frac{\partial(\pi\mathcal{P})}{\partial\eta}, \tag{8b}$$

$$\frac{\mathrm{d}\ln T}{\mathrm{d}t} = F_{\ln T}, \tag{9a}$$

$$F_{\ln T} \equiv -\frac{R}{c_\mathrm{v}}D_3, \tag{9b}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t}\left(= \frac{\mathrm{d}\ln(1+\mathcal{P})}{\mathrm{d}t}\right) = F_Q, \tag{10a}$$

$$F_Q \equiv -\frac{1}{\pi}\frac{\mathrm{d}\pi}{\mathrm{d}t} - \frac{c_\mathrm{p}}{c_\mathrm{v}}D_3, \tag{10b}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial\pi}{\partial\eta}\right) + \nabla\cdot\left(\boldsymbol{v}\frac{\partial\pi}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(\dot{\eta}\frac{\partial\pi}{\partial\eta}\right) = 0, \tag{11}$$

$$\Phi = \Phi_\mathrm{s} + \int_\eta^1 \left[\frac{RT}{(1+\mathcal{P})\pi}m\right]\mathrm{d}\eta', \tag{12}$$

$$D_3 \equiv \nabla\cdot\boldsymbol{v} + \frac{(1+\mathcal{P})\pi}{RT}\frac{1}{m}\nabla\Phi\cdot\frac{\partial\boldsymbol{v}}{\partial\eta} - \frac{(1+\mathcal{P})\pi}{RT}\frac{g}{m}\frac{\partial w}{\partial\eta}, \tag{13}$$

where t is time, p is pressure, T is temperature, $\boldsymbol{v}$ is horizontal wind, w is vertical wind, Φ is geopotential, Φ_s is surface geopotential, D_3 is three-dimensional divergence, R is the gas constant, c_p is the specific heat capacity at a constant pressure, c_v is the specific heat capacity at a constant volume, and ∇ is the horizontal gradient on constant η surfaces. The Coriolis term is omitted in Eq. (7a).

2.3. Surface boundary conditions

Free-slip rigid surface boundary conditions can be written as follows (Bubnová et al., 1995; Bénard et al., 2010):

$$w_\mathrm{s} = \frac{1}{g}\boldsymbol{v}_\mathrm{s}\cdot\nabla\Phi_\mathrm{s} = \frac{1}{g}\boldsymbol{v}_1\cdot\nabla\Phi_\mathrm{s}, \tag{14}$$

$$\left[\frac{1}{m}\frac{\partial(\pi\mathcal{P})}{\partial\eta}\right]_\mathrm{s} = \frac{1}{g}\frac{\mathrm{d}w_s}{\mathrm{d}t}, \tag{15}$$

where the subscript s denotes the value at the surface, the subscript 1 denotes the value at the lowest full level $k = 1$, and $\boldsymbol{v}_\mathrm{s} = \boldsymbol{v}_1$ is assumed. Substituting Eq. (14) into Eq. (15) gives the following equation (Bubnová et al., 1995):

$$\left[\frac{1}{m}\frac{\partial(\pi\mathcal{P})}{\partial\eta}\right]_\mathrm{s} = \frac{1}{g^2}\left[\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t}\cdot\nabla\Phi_\mathrm{s} + {u_1}^2\frac{\partial}{\partial x}\frac{\partial\Phi_s}{\partial x} + u_1 v_1\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} + \frac{\partial}{\partial y}\frac{\partial}{\partial x}\right)\Phi_s + {v_1}^2\frac{\partial}{\partial y}\frac{\partial\Phi_s}{\partial y}\right], \tag{16}$$

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where x and y are the horizontal axes, and u_1 and v_1 are the x and y components of $\boldsymbol{v}_1$, respectively. In spherical coordinates, $\partial/\partial x = 1/(a\cos\phi)\cdot\partial/\partial\lambda$ and $\partial/\partial y = 1/a\cdot\partial/\partial\phi$, where λ is the longitude, ϕ is the latitude, and a is the radius of the sphere. Equation (7) can be discretized at the lowest full level as follows:

$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = \{-RT[\nabla\ln\pi + \nabla Q] - \nabla\Phi\}_1 - \frac{1}{2}\left\{\left[\frac{1}{m}\frac{\partial(\pi\mathcal{P})}{\partial\eta}\right]_\mathrm{s} + \left[\frac{1}{m}\frac{\partial(\pi\mathcal{P})}{\partial\eta}\right]_{1+1/2}\right\}\nabla\Phi_1. \tag{17}$$

It can be considered that Eqs. (16) and (17) are simultaneous equations for variables $\mathrm{d}\boldsymbol{v}_1/\mathrm{d}t$ and $[1/m\cdot\partial(\pi\mathcal{P})/\partial\eta]_\mathrm{s}$. Substituting Eq. (17) into Eq. (16) gives

$$\left[\frac{1}{m}\frac{\partial(\pi\mathcal{P})}{\partial\eta}\right]_\mathrm{s} = \left\{\left[[-RT(\nabla\ln\pi + \nabla Q) - \nabla\Phi]_1 - \frac{1}{2}\left[\frac{1}{m}\frac{\partial(\pi\mathcal{P})}{\partial\eta}\right]_{1+1/2}\nabla\Phi_1\right]\cdot\nabla\Phi_\mathrm{s}\right.$$
$$\left. + {u_1}^2\frac{\partial}{\partial x}\frac{\partial\Phi_s}{\partial x} + u_1 v_1\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} + \frac{\partial}{\partial y}\frac{\partial}{\partial x}\right)\Phi_s + {v_1}^2\frac{\partial}{\partial y}\frac{\partial\Phi_s}{\partial y}\right\}\Big/\left\{g^2 + \frac{1}{2}\left[\frac{\partial\Phi_s}{\partial x}\frac{\partial\Phi_1}{\partial x} + \frac{\partial\Phi_s}{\partial y}\frac{\partial\Phi_1}{\partial y}\right]\right\}. \tag{18}$$

Substituting Eq. (18) into Eq. (17) gives the equation for $\mathrm{d}\boldsymbol{v}_1/\mathrm{d}t$.

In the two-dimensional x–z model, $v = 0$ and $\partial/\partial y = 0$. Substituting Eq. (16) into Eq. (17) leads to

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = -C\frac{\mathrm{d}u_1}{\mathrm{d}t} + U, \tag{19a}$$

$$C \equiv \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\Phi_\mathrm{s}}{g}\right)\frac{\partial}{\partial x}\left(\frac{\Phi_1}{g}\right), \tag{19b}$$

where U is the remaining terms. The coefficient C is about 0.5 when the slope of a mountain is 45° and about 2.0 when the slope is 63.4°. Therefore, $C\,\mathrm{d}u_1/\mathrm{d}t$ on the right side of Eq. (19a) should be treated with caution. It is considered best to convert Eq. (19a) into the following equation for use:

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = \frac{U}{1+C}, \tag{20}$$

which is the same as the equation obtained by substituting Eq. (18) into Eq. (17).

In the previous MRI model, the following equation was used instead of Eq. (18):

$$\left[\frac{1}{m}\frac{\partial(\pi\mathcal{P})}{\partial\eta}\right]_\mathrm{s} = \left[\frac{1}{m}\frac{\partial(\pi\mathcal{P})}{\partial\eta}\right]_{3/2}. \tag{21}$$

However, it was found that the new method, using Eqs. (18) and (17), is better than the previous method that uses Eqs. (21) and (17) (Sect. 3.2).

2.4. Discretization with the semi-implicit semi-Lagrangian scheme

The prognostic equations (7)–(11) can be represented as follows:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X), \tag{22}$$

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where X represents the prognostic variables. By using the vertically conservative semi-Lagrangian scheme (Appendix B), the non-constant coefficient semi-implicit scheme, the Stable Extrapolation Two-Time-Level Scheme (SETTLS; Hortal, 2002), and second-order decentering (Simmons and Temperton, 1997; Yukimoto et al., 2011; Yoshimura, 2022), Eq. (22) can be discretized in time as follows:

$$\frac{X^{+} - X_{\mathrm{D}}^{0}}{\Delta t} = \left[\frac{F^{(+)} + \beta\left(-L_{\mathrm{C}}^{(+)} + L_{\mathrm{C}}^{0}\right) + \left(-L_{\mathrm{N}}^{(+)} + L_{\mathrm{N}}^{0}\right)}{2}\right]_{\mathrm{D}} + \frac{F^{0} + \beta\left(L_{\mathrm{C}}^{+} - L_{\mathrm{C}}^{0}\right) + \left(L_{\mathrm{N}}^{+} - L_{\mathrm{N}}^{0}\right)}{2}, \tag{23a}$$

$$F^{(+)} \equiv 2F^{0} - F^{-},\;\; L_{\mathrm{C}}^{(+)} \equiv 2L_{\mathrm{C}}^{0} - L_{\mathrm{C}}^{-},\;\; L_{\mathrm{N}}^{(+)} \equiv 2L_{\mathrm{N}}^{0} - L_{\mathrm{N}}^{-}, \tag{23b}$$

where L_{C} represents the constant coefficient linear terms; L_{N} represents the non-constant coefficient linear terms; Δt is the timestep; the superscripts $-$, 0, and $+$ are the past $(t - \Delta t)$, present (t), and future $(t + \Delta t)$ values, respectively; and the superscript $(+)$ is the future value $(t + \Delta t)$ extrapolated in time. The non-constant coefficient linear terms are used because the second term on the right side of Eq. (7b) and the second term on the right side of Eq. (13) contain $\nabla\Phi$ and cannot be linearized as constant coefficient linear terms. Since these terms are related to sound waves, the semi-implicit scheme with non-constant coefficient linear terms is used to allow long timesteps. The parameter β for second-order decentering is set to $\beta = 1.2$ instead of 1.0, because setting β to a value a little larger than 1.0 increases the effect of the semi-implicit scheme in terms of improving the numerical stability. The subscript D represents the departure point in calculating advection (Appendix B1). In the vertically conservative semi-Lagrangian scheme, computation of the advection terms is divided into the horizontal and vertical directions, and the vertical flux (vertical remapping) is evaluated with a one-dimensional conservative semi-Lagrangian scheme (Appendices B2 and B3). The horizontal advection is calculated in the usual non-conservative way (Appendices B4 and B5). A correction method (Priestley, 1993; Gravel and Staniforth, 1994) can be used for global conservation of mass and the tracers, but is not used here.

Equation (23a) can be converted as follows:

$$X^{+} - \frac{\beta\Delta t}{2} L_{\mathrm{C}}^{+} - \frac{\Delta t}{2} L_{\mathrm{N}}^{+} = R_X \tag{24a}$$

$$R_X \equiv \left[X^{0} + \frac{F^{(+)} + \beta\left(-L_{\mathrm{C}}^{0} + L_{\mathrm{C}}^{-}\right) + \left(-L_{\mathrm{N}}^{0} + L_{\mathrm{N}}^{-}\right)}{2}\Delta t\right]_{\mathrm{D}} + \frac{F^{0} + \beta\left(-L_{\mathrm{C}}^{0}\right) + \left(-L_{\mathrm{N}}^{0}\right)}{2}\Delta t, \tag{24b}$$

where R_X can be calculated explicitly. Using the discretization of Eq. (24), Eqs. (7)–(11) are discretized in time as follows:

$$\pi^{*} \equiv A(\eta) + B(\eta)\bar{\pi}_s, \tag{25}$$

$$m^{*} \equiv \partial\pi^{*}/\partial\eta, \tag{26}$$

$$\boldsymbol{v}^{+} - \frac{\beta\Delta t}{2}\boldsymbol{L}_{\mathrm{C},\boldsymbol{v}}^{+} - \frac{\Delta t}{2}\boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}^{+} = \boldsymbol{R}_{\boldsymbol{v}}, \tag{27a}$$

$$\boldsymbol{R}_{\boldsymbol{v}} \equiv \left[\boldsymbol{v}^{0} + \frac{\boldsymbol{F}_{\boldsymbol{v}}^{(+)} + \beta\left(-\boldsymbol{L}_{\mathrm{C},\boldsymbol{v}}^{0} + \boldsymbol{L}_{\mathrm{C},\boldsymbol{v}}^{-}\right) + \left(-\boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}^{0} + \boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}^{-}\right)}{2}\Delta t\right]_{\mathrm{D}} + \frac{\boldsymbol{F}_{\boldsymbol{v}}^{0} + \beta\left(-\boldsymbol{L}_{\mathrm{C},\boldsymbol{v}}^{0}\right) + \left(-\boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}^{0}\right)}{2}\Delta t, \tag{27b}$$

$$\boldsymbol{L}_{\mathrm{C},\boldsymbol{v}} \equiv R\bar{T}[\mathcal{G}^{*}(-\nabla\ln T + \nabla\mathcal{Q}) - \nabla\mathcal{Q} - \nabla\ln\pi_s], \tag{27c}$$

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$$\mathcal{G}^* X \equiv \int_\eta^1 \frac{m^*}{\pi^*} X \mathrm{d}\eta', \tag{27d}$$

$$\boldsymbol{L}_{\mathrm{N},\boldsymbol{v}} \equiv -\frac{1}{m^\#} \frac{\partial(\pi^\# Q)}{\partial \eta} \nabla \Phi^\#, \tag{27e}$$

$$w^+ = R_w, \tag{28a}$$

$$R_w \equiv \left[w^0 + \frac{F_w^{(+)}}{2} \Delta t \right]_{\mathrm{D}} + \frac{F_w^0}{2} \Delta t, \tag{28b}$$

$$w_s^+ = \frac{1}{g} \boldsymbol{v}_1^+ \cdot \nabla \Phi_s, \tag{28c}$$

$$\ln T^+ - \frac{\beta \Delta t}{2} L_{\mathrm{C},\ln T}^+ = R_{\ln T}, \tag{29a}$$

$$R_{\ln T} \equiv \ln T_D^0 + \left[\frac{F_{\ln T}^{(+)} + \beta\left(-L_{\mathrm{C},\ln T}^0 + L_{\mathrm{C},\ln T}^-\right)}{2} \Delta t \right]_{\mathrm{D}} + \frac{F_{\ln T}^0 + \beta\left(-L_{\mathrm{C},\ln T}^0\right)}{2} \Delta t, \tag{29b}$$

$$L_{\mathrm{C},\ln T} \equiv -\frac{R}{c_{\mathrm{v}}} \left(D + \hat{d} \right), \tag{29c}$$

$$Q^+ - \frac{\beta \Delta t}{2} L_{\mathrm{C},Q,}^+ = R_Q, \tag{30a}$$

$$R_Q \equiv \left[\ln(1 + \mathcal{P}^0) + \frac{F_{Q(\mathrm{div})}^{(+)} + \beta\left(-L_{\mathrm{C},Q}^0 + L_{\mathrm{C},Q}^-\right)}{2} \Delta t \right]_{\mathrm{D}} + \frac{F_{Q(\mathrm{div})}^0 + \beta\left(-L_{\mathrm{C},Q}^0\right)}{2} \Delta t + F_{Q(\mathrm{adv})} \Delta t, \tag{30b}$$

$$F_{Q(\mathrm{div})} + F_{Q(\mathrm{adv})} = F_Q, \tag{30c}$$

$$L_{\mathrm{C},Q} \equiv \mathcal{S}^* D - \frac{c_{\mathrm{p}}}{c_{\mathrm{v}}} \left(D + \hat{\hat{d}} \right), \tag{30d}$$

$$\mathcal{S}^* X \equiv \frac{1}{\pi^*} \int_0^\eta m^* X \mathrm{d}\eta', \tag{30e}$$

$$\ln \pi_s^+ - \frac{\beta \Delta t}{2} L_{\mathrm{C},\ln \pi_s}^+ = R_{\ln \pi_s}, \tag{31a}$$

$$R_{\ln \pi_s} \equiv \ln \pi_s^{+,\mathrm{explicit}} + \sum_{j=1}^{K} \left[\Delta B_j \frac{\beta\left(-L_{\mathrm{C},\ln \pi_s}^0 + L_{\mathrm{C},\ln \pi_s}^-\right)}{2} \Delta t \right]_{D_2(k)} + \frac{\beta\left(-L_{\mathrm{C},\ln \ \ s}^0\right)}{2} \Delta t, \tag{31b}$$

$$L_{\mathrm{C},\ln \pi_s} \equiv -\mathcal{N}^* D, \tag{31c}$$

$$\mathcal{N}^* X \equiv \frac{1}{\overline{\pi}_s} \int_0^1 m^* X \mathrm{d}\eta', \tag{31d}$$

where $\overline{T} = 300$ K, $\overline{\overline{T}} = 160$ K, $\overline{\pi}_s = 10^5$ Pa, $D \equiv \nabla \cdot \boldsymbol{v}$ is the horizontal divergence, $\hat{\hat{d}}$ is defined in Eq. (36b) below, and the subscript $\mathrm{D}_2(k)$ denotes the two-dimensional horizontal departure point in calculating the horizontal advection (Appendices B2 and B3). The superscript # denotes the variables used for the non-constant coefficients in the semi-implicit scheme, and

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the present values (t) of the variables are used. The explicitly calculated future value of π_s, $\pi_s^{+,\text{explicit}}$, is obtained from Eq. (B27) in Appendix B2. The variables $F_{\mathcal{Q}(\text{div})}$ and $F_{\mathcal{Q}(\text{adv})}$ are defined by Eqs. (C17b) and (C17c) in Appendix C, and $F_{\mathcal{Q}(\text{adv})}$ is calculated using the method provided in Appendix B7. Equation (28) does not include linear terms for the semi-implicit scheme, which are included after Eq. (28) is converted to Eq. (35) below. The values of $\overline{T}$ and $\overline{\overline{T}}$ affect the numerical stability (e.g., Bénard et al., 2010). Using large $\overline{T}$ and small $\overline{\overline{T}}$ tends to improve the numerical stability of the model.

In Eqs. (10) and (30), $\mathcal{Q}$ is used as a prognostic variable instead of $\mathcal{P}$, which improves the numerical stability (Bénard et al., 2010). In Eqs. (9) and (29), $\ln T$ is used as a prognostic variable instead of T, which also improves the numerical stability. When $\ln T$ is used as a prognostic variable, $\overline{T}$ is not included in the coefficient of the linear term in Eq. (29c). When T is a prognostic variable, $\overline{T}$ is included in Eq. (29c), which can reduce the accuracy of the time integration of T. Based on the results of various model tests, it appears that T should be carefully and accurately integrated in time as compared with $\boldsymbol{v}$ and w to improve numerical stability.

Equation (27a) can be converted into the following:

$$D^+ - \frac{\beta\Delta t}{2}\nabla\cdot\boldsymbol{L}_{\text{C},\boldsymbol{v}}^+ - \frac{\Delta t}{2}\nabla\cdot\boldsymbol{L}_{\text{N},\boldsymbol{v}}^+ = \nabla\cdot\boldsymbol{R}_{\boldsymbol{v}}, \tag{32}$$

$$\zeta^+ - \frac{\Delta t}{2}\left[\boldsymbol{k}\cdot\nabla\times\boldsymbol{L}_{\text{N},\boldsymbol{v}}^+\right] = \boldsymbol{k}\cdot\nabla\times\boldsymbol{R}_{\boldsymbol{v}}, \tag{33}$$

where $\boldsymbol{k}$ is the vertical unit vector and $\zeta \equiv \boldsymbol{k}\cdot\nabla\times\boldsymbol{v}$ is the horizontal rotation. In the x–z model, ζ is not used, and instead of Eq. (33), the following equation is used:

$$u_{(m=0)}^+ - \frac{\Delta t}{2}L_{\text{N},u(m=0)}^+ = R_{u(m=0)}, \tag{34}$$

where $m = 0$ is the wavenumber 0 component (i.e., the zonal mean; Appendix D).

Equation (28) is converted into the following:

$$\hat{d}^+ - \frac{\beta\Delta t}{2}L_{\text{C},\hat{d}} = R_{\hat{d}}, \tag{35a}$$

$$\hat{d} \equiv -\left[\frac{g(1+\mathcal{P})}{RT}\frac{\pi}{m}\right]^{\#}\frac{\partial w}{\partial\eta}, \tag{35b}$$

$$R_{\hat{d}} \equiv -\left[\frac{g(1+\mathcal{P})}{RT}\frac{\pi}{m}\right]^{\#}\frac{\partial}{\partial\eta}\left[\left(w^0 + \frac{F_w^{(+)}}{2}\Delta t\right)_{\text{D}} + \frac{F_w^0}{2}\Delta t\right]$$
$$- \frac{g}{R\overline{\overline{T}}}\frac{\pi^*}{m^*}\frac{\partial}{\partial\eta}\left\{\left[\frac{\beta\left(-L_{\text{C},w}^0 + L_{\text{C},w}^-\right)}{2}\Delta t\right]_{\text{D}} + \frac{\beta\left(-L_{\text{C},w}^0\right)}{2}\Delta t\right\}, \tag{35c}$$

$$F_{\hat{d}} \equiv -\left[\frac{g(1+\mathcal{P})}{RT}\frac{\pi}{m}\right]^{\#}\frac{\partial F_w}{\partial\eta}, \tag{35e}$$

$$L_{\text{C},w} \equiv \frac{g}{m^*}\frac{\partial(\pi^*\mathcal{Q})}{\partial\eta}, \tag{35d}$$

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$$L_{\mathrm{C},\hat{d}} \equiv -\frac{g}{R\bar{\bar{T}}}\frac{\pi^*}{m^*}\frac{\partial L_{\mathrm{C},w}}{\partial \eta} = -\frac{g^2}{R\bar{\bar{T}}}\mathcal{L}^* Q, \tag{35f}$$

$$\mathcal{L}^* X \equiv \frac{\pi^*}{m^*}\frac{\partial}{\partial \eta}\left(\frac{1}{m^*}\frac{\partial \pi^* X}{\partial \eta}\right) = \partial^*(\partial^* + 1)X, \tag{35g}$$

$$\partial^* X \equiv \frac{\pi^*}{m^*}\frac{\partial X}{\partial \eta}, \tag{35h}$$

where the linear terms are added using the discretization in Eq. (24). $L_{\mathrm{C},w}$ and $L_{\mathrm{C},\hat{d}}$ are the linearized terms of F_w and $F_{\hat{d}}$, respectively, with constant coefficients. Furthermore, Eq. (35a) can be converted into the following:

$$\hat{\hat{d}}^+ - L^+_{\mathrm{N},\hat{\hat{d}}} - \frac{\beta \Delta t}{2} L^+_{\mathrm{C},\hat{d}} = R_{\hat{d}}, \tag{36a}$$

$$\hat{\hat{d}} \equiv \hat{d} + \left[\frac{(1+\mathcal{P})}{RT}\frac{\pi}{m}\nabla \Phi\right]^{\#} \cdot \frac{\partial \boldsymbol{v}}{\partial \eta}, \tag{36b}$$

$$L^+_{\mathrm{N},\hat{\hat{d}}} \equiv \left[\frac{(1+\mathcal{P})}{RT}\frac{\pi}{m}\nabla \Phi\right]^{\#} \cdot \frac{\partial \boldsymbol{v}^+}{\partial \eta}, \tag{36c}$$

where $\hat{\hat{d}}$ is used instead of $\hat{d}$ as a prognostic variable in spectral space (Bénard et al., 2010), which leads to more stable calculations and enables fewer iterations (Sect. 3.3). By using $\hat{\hat{d}}$ as a prognostic variable instead of $\hat{d}$, non-constant coefficient linear terms disappear in the equations for $\ln T^+$ and Q^+ in Eqs. (29) and (30), and appear in Eq. (36). As described above, T should be carefully and accurately integrated in time for numerical stability. By using $\hat{\hat{d}}$ instead of $\hat{d}$, the errors due to the non-constant coefficient linear terms in the equations for $\ln T^+$ and Q^+ (i.e., residual $\mathbf{r}^{[n]}$ in Appendix E3) disappear, which probably improves numerical stability. In spectral space, D, ζ, $\hat{\hat{d}}$, $\ln T$, Q, and $\ln \pi_s$ are used as prognostic variables instead of $\boldsymbol{v}$, w, T, $\mathcal{P}$, and π_s.

Time integration is performed as follows.

(1) Equations (27)–(36) are discretized vertically as shown in Appendix C. The values of the variables at the departure point (with subscript D) are calculated using the vertically conservative semi-Lagrangian scheme (Appendix B). The variables $R_{\ln}$, R_Q, $R_{\ln \pi_s}$, $\boldsymbol{R}'_{\boldsymbol{v}}$, and $R'_{\hat{d}}$ in Eqs. (29), (30), (31), (E1), and (E3) are calculated in grid space and transformed into spectral space (Appendix D).

(2) The simultaneous equations (29a), (30a), (31a), (E1a), and (E3a) are solved in wave space to obtain the future values D^+, ζ^+, $\hat{\hat{d}}^+$, $\ln T^+$, Q^+, and π_s^+ (Appendix E). The vector variable $\boldsymbol{v}^+$ is calculated from D^+ and ζ^+ in spectral space (e.g., see Eqs. [I15], [I16], and [44] of Yoshimura, 2022).

(3) The future values $\boldsymbol{v}^+$, D^+, $\hat{\hat{d}}^+$, $\ln T^+$, Q^+, and $\ln \pi_s^+$ in spectral space are transformed to grid space (Appendix D). The future value $\hat{d}^+$ is calculated from $\hat{\hat{d}}^+$ using Eq. (36b). To avoid aliasing due to the calculation of $\hat{d}^+$ from $\hat{\hat{d}}^+$, especially when a mountain is steep, high wavenumber components of $\hat{d}^+$ are removed by transforming from grid to spectral space, truncating with a truncation wavenumber M (Appendix D), and transforming back from spectral to grid space. The future values w^+, T^+,

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$\mathcal{P}^+$, and π_s^+ are calculated from $\hat{d}^+$, $\ln T^+$, Q^+, and $\ln \pi_s^+$ in grid space, respectively. The horizontal gradients $\nabla \ln T$, ∇Q, and $\nabla \ln \pi_s$ are also calculated in wave space and transformed to grid space. The value w^+ in the upper levels is modified by Rayleigh damping (Appendix F) when necessary. At the next timestep, these future values are used as the present values.

3. Two-dimensional x–z nonhydrostatic model test cases

Various test cases were performed to evaluate the two-dimensional x–z nonhydrostatic model. Section 3.1 describes the two-dimensional x–z nonhydrostatic model. Sections 3.2, 3.3, and 3.4 show the results of various mountain wave test cases. Sections 3.5 and 3.6 show the results of the density current (cold bubble) and warm bubble test cases, respectively. Section 3.7 shows the results of the gravity wave test case.

3.1. Two-dimensional x–z nonhydrostatic spectral model

In the two-dimensional x–z nonhydrostatic spectral model, $v = 0$ and $\partial/\partial y = 0$ in the prognostic equations in Sect. 2. The variables D and $u_{(m=0)}$ in Eqs. (E1) and (E2) are used, and ζ is not used in spectral space. The parameters $A_{k-1/2}$ and $B_{k-1/2}$ for the mass-based hybrid vertical coordinate in Eq. (2) are given by the method described in Appendix A. Horizontally, periodic boundary conditions are used. The quadratic truncation ($M \cong I/3$) is used, where I is the number of horizontal grids and M is the truncation wave number (Appendix D). Rayleigh damping on w (Appendix F) is used to suppress the reflection of upward propagating gravity waves at the top of the model in mountain wave test cases. In this model, diffusion is not required for numerical stability, and is not used, except that specified diffusion is used in the density current test case in Sect. 3.5. To solve the linear equations for the non-constant coefficient semi-implicit scheme, the preconditioned GCR method (Appendix E3) is used for the test case involving an extremely steep mountain with an average slope of 63.4° in Sect. 3.3, the single recalculation method with $\alpha^{\mathrm{r}} = 0.8$ (Appendix E2) is used for other mountain wave test cases, and the no recalculation method (Appendix E1) is used for test cases with no mountains in Sects. 3.5–3.7. The top half-level of the model is 0 Pa, except in the gravity wave test case described in Sect. 3.7.

3.2. Mountain wave test cases of the Steep Mountain Model Intercomparison Project

The test cases of the Steep Mountain Model Intercomparison Project (St-MIP; Satomura et al., 2003) were performed. Table 1 lists the simulation conditions for seven St-MIP cases. All cases include one bell-shaped mountain:

$$z_{\mathrm{s}} = \frac{h}{1 + (x - x_0)^2/a^2}, \tag{37}$$

where x_0 is the center of the domain, h is the height of the mountain, and a is the half width of the mountain. The initial horizontal wind velocity is $U = 10\ \mathrm{m\ s^{-1}}$ for all cases. The Scorer parameter $l = N/U$ is specified for each case (Table 1),

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where N is the Brunt–Väisälä frequency. θ is the average slope angle of the mountain calculated from a and h. Δx and Δz are the horizontal and vertical grid intervals. The number of horizontal grids is 2000 in all cases. The number of vertical grids is $K = 30{,}000/\Delta z$ and the sponge layer using Rayleigh damping is placed above 15,000 m height, except for cases A3 and A4 where $K = 300$ and the sponge layer is above a height of $250\Delta z$. The integration times for cases A1, A2, A3, A4, D1 and D2 are 300, 100, 20, 10, 100 and 100 min, respectively. The timesteps for cases A1, A2, A3, A4, D1 and D2 are 60, 6, 1.2, 0.3, 3.0 and 3.0 s, respectively. In these cases, to solve the linear simultaneous equations related to the semi-implicit scheme, the single recalculation method with $\alpha^{\mathrm{r}} = 0.8$ is used, which is computationally more efficient and numerically less stable than the preconditioned GCR method (Appendix E). Using $\alpha^{\mathrm{r}} = 0.8$ instead of $\alpha^{\mathrm{r}} = 1.0$ in this method allows stable calculations at long timesteps even in cases A4 and D2, which have a steep mountain with an average slope of 45°.

Figure 1 shows the simulated vertical velocity near the mountain. The results are similar to those of the three models in Satomura et al. (2003). In case A1 where $al \gg 1$, nearly hydrostatically balanced mountain waves appear, which propagate upward above the mountain. In case A2 where $al \approx 1$, nonhydrostatic mountain waves appear, which propagate upward and downstream relative to the mountain. In cases A3 and A4 where $al \ll 1$, wave structures diminish, and simple flow that climbs over and goes down the mountain is simulated (Satomura et al., 2003). In cases D1 and D2 where $hl \ll 1$, the mountain is high enough that nonlinear effects are taken into account (e.g., Smith, 1979; Satomura et al., 2003). In these cases, the amplitude of the vertical velocity is slightly larger than that derived from linear theory, which is probably because of nonlinear effects (Satomura et al., 2003; Li et al., 2016).

Figure 2 shows the result for case A4 when Eq. (21) is used instead of Eq. (18) for the surface boundary conditions. The result using Eq. (18) (Fig. 1d) is more similar to the model results and the linear theory in Satomura et al. (2003) than that using Eq. (21) (Fig. 2). Moreover, when Eq. (21) is used, the numerical stability is reduced, and the use of the GCR method with at least two iterations becomes necessary in case A4 for stable calculations. Therefore, the method using Eq. (18) is better than that using Eq. (21).

When $\hat{d}$ is used instead of $\hat{\hat{d}}$, the numerical stability is decreased, and the use of the GCR method with at least two iterations becomes necessary in case A4.

Table 1. Conditions of the St-MIP simulation.

	a (m)	h (m)	l (m^{-1})	$\Delta x, \Delta z$ (m)	θ (deg)
A1	5000	100	2×10^{-3}	1000, 250	0.57
A2	500	100	2×10^{-3}	100, 100	5.7
A3	100	100	2×10^{-3}	20, 20	26.5
A4	50	100	2×10^{-3}	5, 5	45
D1	500	500	1×10^{-3}	50, 50	26.5
D2	250	500	1×10^{-3}	50, 50	45

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3.3. Extremely steep mountain test case

A case with an extremely steep mountain with an average slope of 63.4° (herein, case S1) was also simulated. Table 2 provides the simulation conditions for case S1, where h and l are the same as in case D2, and a is halved as compared with case D2. The number of vertical grids is $K = 10{,}000/\Delta z$ and the sponge layer using Rayleigh damping is placed above 7,500 m height. The integration time is set to 50 min and the timestep is set to 0.3 s. In case S1, the preconditioned GCR method with a maximum number of iterations $N_{\mathrm{max\_iter}} = 4$ (Appendix E3) is used here for the semi-implicit calculation, because the single recalculation method or the GCR method with $N_{\mathrm{max\_iter}} < 4$ leads to numerical instability.

Figure 3a shows the vertical velocity in case S1. After the wind passes over the mountain, a very strong downslope wind appears near the surface. Figure 3b shows the wind vectors (u, w) in case S1. The wind over the mountain splits into two parts: one that moves eastward and the other that descends near the surface. Figures 3c–d are the same as Figs. 3a–b, except that the initial condition of w is given by $w = \boldsymbol{v} \cdot \nabla\Phi/g$ instead of $w = 0$. A large rotor forms downwind of the mountain. Notably, two completely different states appear under identical conditions, except for the initial conditions of w. When the previous vertical interpolation method is applied to compute vertical fluxes at the lowest and highest levels (Appendix B3), results similar to those in Figures 9a–b are obtained for both initial conditions. Thus, the results of case S1 appear to be highly sensitive to both the initial conditions of w and the computing method applied near the surface.

Table 2. Conditions for the extremely steep mountain test case.

	a (m)	h (m)	l (m^{-1})	$\Delta x, \Delta z$ (m)	θ (deg)
S1	125	500	1×10^{-3}	10, 10	63.4

3.4. Mountain wave test cases of Schär et al. (2002)

Two mountain wave test cases of Schär et al. (2002) were performed, which are referred to as case Schär1 and case Schär2 in this paper. The surface height is given as

$$z_s = h_0 \exp\left[-\left(\frac{x}{a}\right)^2\right] \cos^2\left(\frac{\pi x}{\lambda}\right), \tag{38}$$

where $h_0 = 250$ m, $a = 5$ km, and $\lambda = 4$ km. Table 3 provides the simulation conditions of cases Schär1 and Schär2, where the Brunt–Väisälä frequency N, initial horizontal wind velocity U, Scorer parameter $l = N/U$, grid intervals Δx and Δz, upstream surface temperature T_0, and surface pressure p_0 are specified. The horizontal extent is $L = 200$ km. The number of vertical levels is $K = 65$, and Rayleigh damping is used above a height of $35\Delta z$ to minimize reflections. The integration time is 600 min.

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Figure 4 shows the results for case Schär1. Figures 4a–c show the vertical velocity when using timesteps of 5, 30, and 60 s, respectively. These results are similar to the analytical solution of Schär et al. (2002), even with long timesteps. In Schär et al. (2002), the results were good when using a smooth level vertical (SLEVE) coordinate, but not good when using the usual hybrid height coordinate. In our model, good results are obtained with the usual hybrid mass-based coordinate, possibly because of the use of the vertically conservative semi-Lagrangian scheme. Figures 4d–f are the same as Figs. 4a–c, except for not using the semi-Lagrangian treatment of the pressure change (SLTPC) that is explained in Appendix B7. The result with the long timestep of 60 s is not good when not using the SLTPC, which indicates the SLTPC is effective when using a long timestep.

Figure 5 shows the results for case Schär2. Figures 5a–c show the vertical velocity with timesteps of 5, 30, and 60 s, respectively, which are also similar to the analytical solution in Schär et al. (2002) even with long timesteps. Figures 5d–f are the same as Figs. 5a–c, except for not using the spatially averaged Eulerian treatment of mountains (SAETM; Appendix B6) for the horizontal mass advection, which yields poor results. Figures 5g–h are the same as Fig. 5a–c, except for not using the cubic Hermite interpolation (Appendix B8) and using the Lagrange interpolation for SLTPC. The result around $x = -5$ km with a short timestep of $\Delta t = 5$ s when not using the cubic Hermite interpolation (Fig. 5g) is not as good as that when using the interpolation (Fig. 5a).

Table 3. Conditions for the mountain wave test cases of Schär et al. (2002).

	N (s^{-1})	U ($\mathrm{m\ s}^{-1}$)	l (m^{-1})	$\Delta x, \Delta z$ (m)	T_0 (K)	p_0 (Pa)
Schär1	0.01	10	1×10^{-3}	500, 300	288	10^5
Schär2	0.01871	18.71	1×10^{-3}	1000, 300	273.16	10^5

3.5. Density current test case

The density current test case of Straka et al. (1993) was performed. The basic state values are given as follows:

$$\bar{T} = T_s - zg/c_p, \tag{39}$$

$$\bar{\theta} = T_s = 300\ \mathrm{K}. \tag{39′}$$

To initiate a density current, the following function is used to specify an initial temperature perturbation:

$$\Delta T = \begin{cases} 0.0\ \mathrm{°C} & \text{if } L > 1.0, \\ -15.0\,[\cos(\pi L) + 1.0]/2\ \mathrm{°C} & \text{if } L \le 1.0, \end{cases} \tag{40}$$

where $L = \{[(x - x_c)x_r^{-1}]^2 + [(z - z_c)z_r^{-1}]^2\}^{0.5}$, $x_\mathrm{c} = 0.0$ km, $x_\mathrm{r} = 4.0$ km, $z_\mathrm{c} = 3.0$ km and $z_\mathrm{r} = 2.0$ km. The horizontal extent is from $x = -25.6$ km to $x = 25.6$ km, and the number of vertical grids is $K = 6400/\Delta z$. Second-order x–z diffusion with a diffusion coefficient of $75.0\ \mathrm{m^2\ s^{-1}}$, as specified by Straka et al. (1993), is used for T, Q, D, and $\hat{\hat{d}}$ in wave space. In test cases without mountains, no recalculation of the non-constant coefficient linear terms $\boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}$ and $L_{\mathrm{N},\hat{\hat{d}}}$ is necessary to solve

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the semi-implicit equations without stability problems, because the values of $\boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}$ and $L_{\mathrm{N},\hat{\hat{d}}}$, which contain $\nabla\Phi^{\#}$, are small (Appendix E1).

Figure 6 shows the simulated potential temperature perturbations at 900 s. The result with $\Delta x = \Delta z = 50$ m and $\Delta t = 3$ s (Fig. 6a) is similar to the result of the reference model (REFC) with $\Delta x = \Delta z = 50$ m and $\Delta t = 1.5625 \times 10^{-2}$ s in Straka et al. (1993) despite the long timestep of 3 s. The result at low resolution with $\Delta x = \Delta z = 400$ m and $\Delta t = 25$ s (Fig. 6d) is more similar to the result with $\Delta x = \Delta z = 50$ m, compared with the results of Straka et al. (1993) at the same resolution. This is probably because the model has good computational accuracy by using the spectral method and good numerical stability by using the semi-implicit semi-Lagrangian scheme.

3.6. Warm bubble convection test case

The warm bubble convection test case of Robert (1993) was simulated. The basic state of the atmosphere is isentropic with a potential temperature of 30 °C. The initial perturbation of the potential temperature within the bubble is given by

$$\theta' = \begin{cases} A, & r \le a \\ A\exp[-(r-a)^2/s^2], & r > a, \end{cases} \tag{41}$$

with

$$r^2 = (x - x_0)^2 + (z - z_0)^2, \tag{42}$$

where $A = 0.5$ °C, $a = 50$ m, $s = 100$ m, $x_0 = 500$ m and $z_0 = 260$ m. This produces a bubble with a Gaussian profile with a small flat region at its center. The grid intervals are $\Delta x = 10$ m and $\Delta z = 10$ m, the horizontal length of the domain is $L =$ 1 km, and the number of vertical grids is $K = 1{,}500/\Delta z$.

Figures 7a–d show the potential temperature perturbations at $t = 0$, 6, 12, and 18 min, respectively. The results at $t = 0$, 6, and 12 min are very similar to those of Robert (1993). The result at $t =$18 min is similar to that of Robert (1993), but there are some differences. Figures 8a–d show the results at $t =$18 min with different settings. Figures 8a and b are the same as Fig. 7d, except for using the linear truncation ($M \cong I/2 - 1$) and the cubic truncation ($M \cong I/4$), respectively, instead of the quadratic truncation ($M \cong I/3$); There are also some differences among their results. Figure 8c is the same as Fig. 7d, except for using $\Delta t = 2.5$ s instead of $\Delta t = 5$ s; these two results are similar. Figure 8d is the same as Fig. 8c, except for using $\Delta x =$ 5 m and $\Delta z = 5$ m; small-scale sharp structures appear due to the higher resolution.

3.7. Gravity wave test case

The gravity wave test case of Skamarock and Klemp (1994) was simulated. The initial perturbation of potential temperature θ' is given as follows:

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$$\theta' = \Delta\theta_0 \frac{\sin\left(\frac{\pi z}{H}\right)}{1 + \frac{(x - x_c)^2}{a^2}}, \tag{43}$$

where $\Delta\theta_0 = 10^{-2}$ K, $a = 5$ km, the height of the domain $H = 10$ km, the horizontal length of the domain $L = 300$ km, and the central position of the initial perturbation $x_c = L/3$. A constant Brunt–Väisälä frequency $N = 10^{-2}\ \mathrm{s}^{-1}$ and a surface temperature of 300 K give the basic state of the vertical temperature profile. The initial horizontal wind velocity is $U = 20\ \mathrm{m/s}$. The grid intervals are $\Delta x = \Delta z = 500$ m.

Figures 9a–b show the potential temperature perturbations at $t = 3000$ s using the timesteps $\Delta t = 6\ s$ and $\Delta t = 30\ s$, respectively. The results are similar to those of Wong et al. (2014). Figures 9c–d show the results using the previous method of vertical interpolation for the vertical flux calculation in the lowest and highest vertical levels (Appendix B3). The present method (Figs. 9a–b) improves the results as compared with the previous method.

4. Conclusions

We have developed a two-dimensional x–z nonhydrostatic spectral model using non-constant coefficient semi-implicit and vertically conservative semi-Lagrangian schemes to allow long timesteps. The model was examined using various test cases and good results were obtained with long timesteps and even with steep orography, because of the following features and improvements of the model.

(1) The non-constant coefficient semi-implicit scheme is computationally efficient. A single recalculation of non-constant coefficient linear terms (i.e., the single recalculation method; Appendix E2) is usually sufficient to solve the equations associated with the non-constant coefficient semi-implicit scheme with the help of (2), (3), and (4), even for cases with a steep mountain with an average slope of 45° (Sect. 3.2). In the case of an extremely steep mountain with an average slope of 63.4° (Sect. 3.3), several iterations using the preconditioned GCR method (Appendix E3) are necessary to solve the semi-implicit equations, where only non-constant coefficient linear terms are recalculated in each iteration. In the test cases without mountains (Sects. 3.5–3.7), no recalculation is necessary to solve the semi-implicit equations (Appendix E1).

(2) In the single recalculation method, setting $\alpha^{\mathrm{r}} = 0.8$ instead of $\alpha^{\mathrm{r}} = 1.0$ improves the numerical stability (Appendix E2; Sect. 3.2).

(3) By improving the treatment of the free-slip boundary conditions, the number of iterations for solving the semi-implicit equations are reduced (Sect. 2.3), and the results for the test cases with a steep mountain are improved (Sect. 3.2).

(4) By using $\hat{\hat{d}}$, $\mathcal{Q}$ (Bénard et al., 2010), and $\ln T$ instead of $\hat{d}$, $\mathcal{P}$, and T as prognostic variables in spectral space, the numerical stability is improved, and the number of iterations needed to solve the semi-implicit equations is reduced (Sects. 2.4 and 3.2).

(5) A vertically conservative semi-Lagrangian scheme is used, where a conservative semi-Lagrangian method is used in the vertical direction (Appendix B). Mass and other variables are advected in a consistent manner.

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(6) By using a spatially averaged Eulerian treatment of mountains with horizontal mass advection similar to Ritchie and Tanguay (1996) (Appendix B6), the results for the test cases with mountains are improved (Sect. 3.4).

(7) By treating the hydrostatic pressure change with horizontal advection in a semi-Lagrangian fashion (Appendices B7 and B8), the results for the test cases with mountains are improved (Sect. 3.4).

(8) By modifying the vertical interpolation method for the vertical flux calculation in the lowest and highest vertical levels (Appendix B3), the results for the gravity wave test case are improved (Sect. 3.7).

We have been developing a global nonhydrostatic spectral model using non-constant coefficient semi-implicit and vertically conservative semi-Lagrangian schemes, as briefly described by Yoshimura (2012). However, improvements (2)–(4) and (6)–(8) in the above list have not been introduced into the global model. These improvements will be introduced into the global model and tested in future work. Similar to the two-dimensional x–z nonhydrostatic model, the global nonhydrostatic model is expected to be very efficient: its high numerical stability allows long timesteps, and the single recalculation method is sufficient for stable time integration since there are no extremely steep mountains (such as with a 63.4° slope) in the model. Furthermore, not only in this model but also in other nonhydrostatic models, introducing some improvements described in this paper is expected to enhance computational stability, efficiency, and accuracy, particularly in cases with steep mountains.

Code availability. The source code of the two-dimensional x–z nonhydrostatic model is available on Zenodo at https://doi.org/10.5281/zenodo.17935265 (Yoshimura, 2025) and is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International license (CC BY-NC-SA 4.0).

Data availability. The output data from the model experiments are available at https://climate.mri-jma.go.jp/pub/archives/Yoshimura_Nonhydro_2D_XZ_Testcase.

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Appendix A: How to obtain the coefficients $A_{k-1/2}$ and $B_{k-1/2}$ in the vertical coordinate of the two-dimensional x–z model

In the two-dimensional x–z model, $A_{k-1/2}$ and $B_{k-1/2}$ in Eq. (2) are defined as follows:

$$A_{k-1/2} \equiv \mu_{k-1/2} \cdot 10^5 \cdot \eta_{k-1/2}, \tag{A1a}$$

$$B_{k-1/2} \equiv \left(1 - \mu_{k-1/2}\right) \cdot \eta_{k-1/2}, \tag{A1b}$$

$$\mu_{k-1/2} \equiv \begin{cases} 0 & \text{for } \gamma_{k-1/2} \le 0, \\ \gamma_{k-1/2} + 0.125\left[-16\left(\gamma_{k-1/2} - 0.5\right)^5 + \gamma_{k-1/2} - 0.5\right] & \text{for } 0 \le \gamma_{k-1/2} \le 1, \\ 1 & \text{for } \gamma_{k-1/2} \ge 1, \end{cases} \tag{A1c}$$

$$\gamma_{k-1/2} \equiv \frac{\ln \eta_{[2]} - \ln \eta_{k-1/2}}{\ln \eta_{[2]} - \ln \eta_{[1]}}, \quad \eta_{[1]} = 0.1, \quad \eta_{[2]} = 1.0. \tag{A1d}$$

When $\pi_s = 10^5\ \mathrm{Pa}$, Eq. (2) becomes $\pi_{k-1/2} = 10^5 \cdot \eta_{k-1/2}$. The vertical profile of $\eta_{k-1/2}$ is given by the following equations:

$$\eta_{1/2} = 1 \quad \left(\log \eta_{1/2} = 0\right), \tag{A2a}$$

$$\log \eta_{k+1/2} = \log \eta_{k-1/2} - \frac{g\Delta z}{RT_k}, \quad (1 \le k \le K-1), \tag{A2b}$$

$$\eta_{K+1/2} = 0, \tag{A2c}$$

where Δz is the vertical resolution. Exceptionally, in the gravity wave test case in Sect. 3.7, $\eta_{[1]} = 0.5$ is used in Eq. (A1d), and $\eta_{K+1/2}$ is given using Eq. (A2b) instead of (A2c).

The vertical profiles of T_k in Eq. (A2b) are calculated as shown below. When the Brunt–Väisälä frequency N is given, the following relationship is satisfied:

$$\frac{\mathrm{d}T}{\mathrm{d}z} = \frac{N^2}{g} T - \frac{g}{c_p}. \tag{A3}$$

Equation (A3) is discretized vertically as follows:

$$\frac{T_{k+1/2} - T_{k-1/2}}{\Delta z} = \frac{N^2}{g} \frac{T_{k+1/2} + T_{k-1/2}}{2} - \frac{g}{c_p}, \tag{A4}$$

which leads to:

$$T_{k+1/2} = \frac{\left(1 + \dfrac{N^2 \Delta z}{2g}\right) T_{k-1/2} - \dfrac{g\Delta z}{c_p}}{1 - \dfrac{N^2 \Delta z}{2g}}. \tag{A5}$$

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When the surface temperature $T_{1/2}$ is given, $T_{k+1/2}$ $(k = 1, \ldots, K-1)$ is calculated using Eq. (A5), and T_k in Eq. (A2b) is given by:

$$T_k = \frac{T_{k-1/2} + T_{k+1/2}}{2} \quad (1 \leq k \leq K-1). \tag{A6}$$

Appendix B: Vertically conservative semi-Lagrangian scheme

A simple explanation of the vertically conservative semi-Lagrangian scheme was given by Yoshimura and Matsumura (2003) and Yukimoto et al. (2011). The scheme has some similarities to the vertically Lagrangian scheme of Lin (2004). Here, the scheme is described in detail, including some improvements.

B1. Advection with the usual semi-Lagrangian scheme

The advection equation with variable T can be expressed as follows:

$$\frac{\mathrm{d}T}{\mathrm{d}t}\left(= \frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T + \dot{\eta}\frac{\partial T}{\partial \eta}\right) = 0. \tag{B1}$$

In the usual non-conservative semi-Lagrangian scheme (e.g., Robert, 1981; Temperton, 2001; Hortal, 2002), Eq. (B1) is discretized in time as

$$\frac{T^{+} - T_{\mathrm{D}}^{0}}{\Delta t} = 0, \tag{B2}$$

where the subscript D represents the value at the three-dimensional departure point $\boldsymbol{x}_{\mathrm{D}}$ and the absence of the subscript D means the value at the arrival point $\boldsymbol{x}$, which is on the grid point. The departure point $\boldsymbol{x}_{\mathrm{D}}$ is located upstream from the arrival point $\boldsymbol{x}$ along the wind vector. Equation (B2) is the discretization when the right side of Eq. (B1) is zero. Section 2.4 shows the discretization when the right side of Eq. (B1) is not zero.

B2. Advection of mass and other variables with the vertically conservative semi-Lagrangian scheme

In the vertically conservative semi-Lagrangian scheme, the value $T^{+}(= T_{\mathrm{D}}^{0})$ in Eq. (B2) is obtained by calculating horizontal and vertical advection separately. A conservative semi-Lagrangian scheme is used in the vertical direction. The continuous equation in Eq. (11) is converted to

$$\frac{\mathrm{d_H}}{\mathrm{d}t}\left(\frac{\partial \pi}{\partial \eta}\right) = -D\frac{\partial \pi}{\partial \eta} - \frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial \pi}{\partial \eta}\right), \tag{B3}$$

where D is the horizontal divergence and $\mathrm{d_H}/dt \equiv \partial/\partial t + \boldsymbol{v} \cdot \nabla$ is the horizontal advection. Using Eq. (B3), Eq. (B1) is converted into:

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$$\frac{\mathrm{d_H}}{\mathrm{d}t}\left(\frac{\partial \pi}{\partial \eta}T\right) = -D\frac{\partial \pi}{\partial \eta}T - \frac{\partial}{\partial \eta}\left(\dot{\eta}\frac{\partial \pi}{\partial \eta}T\right). \tag{B4}$$

Using Eqs. (B3) and (B4), the mass and other variables are advected in a consistent manner. Using Eq. (3), Eqs. (B3) and (B4) are discretized in the vertical direction as follows:

$$\frac{\mathrm{d_H}}{\mathrm{d}t}(\Delta\pi_k) = -D_k\Delta\pi_k - \left(\dot{\eta}\frac{\partial \pi}{\partial \eta}\right)_{k-1/2} + \left(\dot{\eta}\frac{\partial \pi}{\partial \eta}\right)_{k+1/2}, \tag{B5}$$

$$\frac{\mathrm{d_H}}{\mathrm{d}t}(\Delta\pi_k T_k) = -D_k\Delta\pi_k T_k - \left(\dot{\eta}\frac{\partial \pi}{\partial \eta}T\right)_{k-1/2} + \left(\dot{\eta}\frac{\partial \pi}{\partial \eta}T\right)_{k+1/2}. \tag{B6}$$

By using the SETTLS discretization, Eqs. (B5) and (B6) are discretized in time at each layer k as follows:

$$\begin{aligned}\Delta\pi_k^+ = &\left\{\Delta\pi_k^0 + \frac{\Delta t}{2}(-D_k\Delta\pi_k)^{(+)} + \frac{\Delta \mathrm{t}}{2}\left[-\left(\dot{\eta}\frac{\partial \pi}{\partial \eta}\right)_{k-1/2}^{(+)} + \left(\dot{\eta}\frac{\partial \pi}{\partial \eta}\right)_{k+1/2}^{(+)}\right]\right\}_{\mathrm{D}_2(k)} \\ &+ \frac{\Delta \mathrm{t}}{2}(-D_k\Delta\pi_k)^0 + \frac{\Delta \mathrm{t}}{2}\left[-\left(\dot{\eta}\frac{\partial \pi}{\partial \eta}\right)_{k-1/2}^0 + \left(\dot{\eta}\frac{\partial \pi}{\partial \eta}\right)_{k+1/2}^0\right],\end{aligned} \tag{B7}$$

$$\begin{aligned}\Delta\pi_k^+ T_k^+ = &\left\{\Delta\pi_k^0 T_k^0 + \frac{\Delta t}{2}(-D_k\Delta\pi_k)^{(+)}T_k^0 + \frac{\Delta \mathrm{t}}{2}\left[-\left(\dot{\eta}\frac{\partial \pi}{\partial \eta}T\right)_{k-1/2}^{(+)} + \left(\dot{\eta}\frac{\partial \pi}{\partial \eta}T\right)_{k+1/2}^{(+)}\right]\right\}_{\mathrm{D}_2(k)} \\ &+ \frac{\Delta \mathrm{t}}{2}(-D_k\Delta\pi_k)^0 T_k^+ + \frac{\Delta \mathrm{t}}{2}\left[-\left(\dot{\eta}\frac{\partial \pi}{\partial \eta}T\right)_{k-1/2}^0 + \left(\dot{\eta}\frac{\partial \pi}{\partial \eta}T\right)_{k+1/2}^0\right],\end{aligned} \tag{B8}$$

where the subscript $\mathrm{D}_2(k)$ indicates the value at the two-dimensional horizontal departure point in the vertical level k, and the absence of the subscript $\mathrm{D}_2(k)$ indicates the value at the arrival point.

Time integration of Eqs. (B7) and (B8) is performed in the following order:

(1) Initial values to be advected are

$$\Delta\pi_k[1] = \Delta\pi_k^0, \tag{B9}$$

$$T_k[1] = T_k^0. \tag{B10}$$

(2) Due to horizontal divergence, the values of $\Delta\pi_k$ and T_k change as follows:

$$\Delta\pi_k[2] = \Delta\pi_k[1] + \frac{\Delta t}{2}(-D_k\Delta\pi_k)^{(+)}, \tag{B11}$$

$$\Delta\pi_k[2]T_k[2] = \Delta\pi_k[1]T_k[1] + \frac{\Delta t}{2}(-D_k\Delta\pi_k)^{(+)}T_k[1]. \tag{B12}$$

From Eqs. (B11) and (B12),

$$T_k[2] = T_k[1]. \tag{B13}$$

(3) Due to the vertical flux, the values of $\Delta\pi_k$ and T_k change as follows:

$$\Delta\pi_k[3] = \Delta\pi_k[2] + \frac{\Delta t}{2}\left[-\left(\dot{\eta}\frac{\partial \pi}{\partial \eta}\right)_{k-1/2}^{(+)} + \left(\dot{\eta}\frac{\partial \pi}{\partial \eta}\right)_{k+1/2}^{(+)}\right], \tag{B14}$$

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$$\Delta\pi_k[3]T_k[3] = \Delta\pi_k[2]T_k[2] + \frac{\Delta t}{2}\left[-\left(\dot{\eta}\frac{\partial\pi}{\partial\eta}T\right)_{k-1/2}^{(+)} + \left(\dot{\eta}\frac{\partial\pi}{\partial\eta}T\right)_{k+1/2}^{(+)}\right]. \tag{B15}$$

Here, $(\dot{\eta}\,\partial\pi/\partial\eta)_{k-1/2}$ is calculated from the following equation (e.g., Ritchie et al., 1995):

$$\left(\dot{\eta}\frac{\partial\pi}{\partial\eta}\right)_{k-1/2} = B_{k-1/2}\sum_{j=1}^{K}\left[D_j\Delta\pi_j + \pi_s(\boldsymbol{v_j}\cdot\nabla\ln\pi_s)\Delta B_j\right] - \sum_{j=k}^{K}\left[D_j\Delta\pi_j + \pi_s(\boldsymbol{v_j}\cdot\nabla\ln\pi_s)\Delta B_j\right], \tag{B16}$$

where $(\dot{\eta}\,\partial\pi/\partial\eta)_{1/2} = (\dot{\eta}\,\partial\pi/\partial\eta)_{K+1/2} = 0$. The value $T_k[3]$ is calculated from $T_k[2]$ using Eq. (B15) via the remapping method described in Appendix B3.

(4) Due to horizontal advection, the values of $\Delta\pi_k$ and T_k change as follows:

$$\Delta\pi_k[4] = (\Delta\pi_k[3])_{\mathrm{D}_2(k)}, \tag{B17}$$

$$\Delta\pi_k[4]T_k[4] = (\Delta\pi_k[3])_{\mathrm{D}_2}(T_k[3])_{\mathrm{D}_2(k)}. \tag{B18}$$

When adopting the spatially averaged Eulerian treatment of the mountains described in Appendix B6, Eq. (B44) below is used instead of Eq. (B17). From Eqs. (B17) and (B18),

$$T_k[4] = (T_k[3])_{\mathrm{D}_2(k)}. \tag{B19}$$

How to calculate the horizontal departure point and the value at the horizontal departure point with the subscript $\mathrm{D}_2(k)$ is explained in Appendix B4.

(5) Due to the vertical flux, the values of $\Delta\pi_k$ and T_k change as follows:

$$\Delta\pi_k[5] = \Delta\pi_k[4] + \frac{\Delta t}{2}\left[-\left(\dot{\eta}\frac{\partial\pi}{\partial\eta}\right)_{k-1/2}^{0} + \left(\dot{\eta}\frac{\partial\pi}{\partial\eta}\right)_{k+1/2}^{0}\right], \tag{B20}$$

$$\Delta\pi_k[5]T_k[5] = \Delta\pi_k[4]T_k[4] + \frac{\Delta t}{2}\left[-\left(\dot{\eta}\frac{\partial\pi}{\partial\eta}T\right)_{k-1/2}^{0} + \left(\dot{\eta}\frac{\partial\pi}{\partial\eta}T\right)_{k+1/2}^{0}\right]. \tag{B21}$$

Here, $(\dot{\eta}\,\partial\pi/\partial\eta)_{k-1/2}$ calculated from Eq. (B16) is not used, and $\Delta\pi_k[5]$ is calculated from Eq. (B29) below so that Eq. (B28) below is satisfied. After calculating $\Delta\pi_k[5]$, $T_k[5]$ is calculated from $T_k[4]$ using Eq. (B21) by the remapping method described in Appendix B3.

(6) Due to horizontal divergence, the values of $\Delta\pi_k$ and T_k change as follows:

$$\Delta\pi_k[6] = \Delta\pi_k[5] + \frac{\Delta t}{2}(-D_k\Delta\pi_k)^0, \tag{B22}$$

$$\Delta\pi_k[6]T_k[6] = \Delta\pi_k[5]T_k[5] + \frac{\Delta t}{2}(-D_k\Delta\pi_k)^0 T_k[5]. \tag{B23}$$

From Eqs. (B22) and (B23),

$$T_k[6] = T_k[5]. \tag{B24}$$

(7) The values after advection are obtained as follows:

$$\Delta\pi_k^+ = \Delta\pi_k[6], \tag{B25}$$

$$T_k^+(=T_\mathrm{D}^0) = T_k[6]. \tag{B26}$$

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From Eqs. (B25), (B22), and (B20), and using the boundary condition $(\dot{\eta}\,\partial\pi/\partial\eta)_{1/2} = (\dot{\eta}\,\partial\pi/\partial\eta)_{K+1/2} = 0$, the following equation is derived:

$$\pi_s^{+,\text{explicit}} = \pi_{\text{top}} + \sum_{j=1}^{K} \Delta\pi_j^{+} = \pi_{\text{top}} + \sum_{j=1}^{K}\left[\Delta\pi_k[4] + \frac{\Delta t}{2}(-D_k\Delta\pi_k)^0\right], \tag{B27}$$

where $\pi_{\text{top}} = \pi_{K+1/2}$ is π at the top of the model and usually $\pi_{\text{top}} = 0$. The variable $\pi_s^{+,\text{explicit}}$ is the explicitly calculated $\pi_s(t+\Delta t)$, which is used in Eq. (31b). By using Eq. (3a), the following equation is obtained:

$$\Delta\pi_k^{+} = \Delta A_k + \Delta B_k \pi_s^{+,\text{explicit}}. \tag{B28}$$

From Eqs. (B28), (B25), and (B22), the following equation is obtained:

$$\Delta\pi_k[5] = \Delta\pi_k^{+} + \frac{\Delta t}{2}(D_k\Delta\pi_k)^0. \tag{B29}$$

By using $\Delta\pi_k[4]$ and $\Delta\pi_k[5]$ obtained from Eqs. (B17) and (B29), $T_k[5]$ in Eq. (B21) is calculated.

Equations (B9)–(B29) are used to calculate the value at the three-dimensional departure point. For example, in Eq. (29b), when setting $T[1] = T^0$, $T[6]$ is calculated as the value of T_D^0. When setting $T[1] = \left(F_{\ln T}^{(+)} + \beta_{\text{C}}\left(-L_{\text{C},\ln T}^{0} + L_{\text{C},\ln T}^{-}\right)\right)\Delta t/2$, $T[6]$ is calculated as the value of $\left[\left(F_{\ln T}^{(+)} + \beta_{\text{C}}\left(-L_{\text{C},\ln}^{0} \;\; + L_{\text{C},\ln T}^{-}\right)\right)\Delta t/2\right]_{\text{D}}$.

The calculation of the advection of $w_{k+1/2}$ at the half level is described in Appendix B5.

B3. Vertical remapping for the vertical flux calculation

The vertical flux is calculated with a conservative semi-Lagrangian scheme. When $\Delta\pi_k[2]$ and $\Delta\pi_k[3]$ are given, $\pi_{k-1/2}[2]$ and $\pi_{k-1/2}[3]$ can be calculated as follows:

$$\pi_{k-1/2} = \pi_{\text{top}} + \sum_{l=k}^{K} \Delta\pi_k, \tag{B30}$$

where $\pi_{\text{top}} = \pi_{K+1/2}$ is π at the top of the model, and usually $\pi_{\text{top}} = 0$. Figure B1 shows the change of T in the vertical flux calculation from [2] to [3]. The left image in Fig. B1 is before the vertical flux calculation, and the vertical profile of $T(\pi)$ satisfying the following equation is used to calculate the vertical flux:

$$\int_{\pi_{k+1/2}[2]}^{\pi_{k-1/2}[2]} T(\pi)d\pi = T_k[2]\Delta\pi_k[2]. \tag{B31}$$

The shaded areas of the middle image in Fig. B1 show the vertical flux of T moving from the upper to lower levels. The value $T_k[3]$ after the vertical flux calculation in the right image in Fig. B1 is obtained from the following equation:

$$T_k[3] = \frac{1}{\Delta\pi_k[3]}\int_{\pi_{k+1/2}[3]}^{\pi_{k-1/2}[3]} T(\pi)d\pi. \tag{B32}$$

This calculation is considered to be conservative vertical remapping.

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The integrated value $S(\pi)$ from the top in the following equation is used instead of $T(\pi)$:

$$S(\pi) = \int_{\pi_{\mathrm{top}}}^{\pi} T(\pi')d\pi'. \tag{B33}$$

The value $S(\pi_{k-1/2[2]})$ is obtained from:

$$S(\pi_{k-1/2[2]}) = \int_{\pi_{\mathrm{top}}}^{\pi_{k-1/2}[2]} T(\pi')d\pi' = \sum_{l=k}^{K} T_l[2]\Delta\pi_l[2]. \tag{B34}$$

There is arbitrariness in how to obtain the vertical profile of $S(\pi)$. The third-order Lagrange interpolation is used to calculate the value $S(\pi)$ from the values of $S(\pi_{k-1/2}[2])$ at the nearest four grid points when $\pi_{3/2}[2] \le \pi \le \pi_{K-1/2}[2]$. Previously, when π is in the lowest or highest vertical levels, which is when $\pi_{1/2}[2] \le \pi \le \pi_{3/2}[2]$ or $\pi_{K-1/2}[2] \le \pi \le \pi_{K+1/2}[2] = \pi_{\mathrm{top}}$, the third-order Lagrange interpolation was employed (i.e., the previous method) to calculate $S(\pi)$ by using the extrapolated values of $S(\pi_{-1/2}[2])$ and $S(\pi_{K+3/2}[2])$:

$$S(\pi_{-1/2}[2]) = S(\pi_{1/2}[2]) + T_1[2]\Delta\pi_1[2], \tag{B35a}$$

$$S(\pi_{K+3/2}[2]) = S(\pi_{K+1/2}[2]) - T_K[2]\Delta\pi_K[2]. \tag{B35b}$$

Here, the second-order Lagrange interpolation is used (i.e., the present method) instead of the third-order Lagrange interpolation when π is in the lowest or highest vertical levels. The values of $S(\pi_{1/2}[2])$, $S(\pi_{3/2}[2])$, and $S(\pi_{5/2}[2])$ are used for interpolation when $\pi_{1/2}[2] < \pi < \pi_{3/2}[2]$, and the values of $S(\pi_{K-3/2}[2])$, $S(\pi_{K-1/2}[2])$, and $S(\pi_{K+1/2}[2])$ are used when $\pi_{K-1/2}[2] < \pi < \pi_{K+1/2}[2]$. In the gravity wave test case (Sect. 3.7), the results of the present method are better than those of the previous method. Using the values of $S(\pi_{k-1/2}[3])$ calculated by the Lagrange interpolation, $T_k[3]$ is calculated as follows:

$$T_k[3] = \frac{1}{\Delta\pi_k[3]}\left[S(\pi_{k-1/2}[3]) - S(\pi_{k+1/2}[3])\right]. \tag{B36}$$

The value of $T_k[5]$ can be calculated from $T_k[4]$ in the same way using the values of $\Delta\pi_k[4]$ and $\Delta\pi_k[5]$.

In calculating the vertical flux of substances such as water vapor, the piecewise rational method (PRM; Xiao and Peng, 2004), which is a monotonic scheme, can be used to avoid negative values.

B4. Horizontal advection

Horizontal advection from [3] to [4] with Eqs. (B17) and (B19) is calculated using a typical two-dimensional horizontal semi-Lagrangian scheme. The horizontal departure point $(\boldsymbol{x}_k)_{\mathrm{D}_2(k)}$ is located upstream from the arrival point $\boldsymbol{x}_k$ along the wind vector between $(\boldsymbol{x}_k)_{\mathrm{D}_2(k)}$ and $\boldsymbol{x}_k$.

One way to calculate $(\boldsymbol{x}_k)_{\mathrm{D}_2(k)}$ is to use the following equation:

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$$(\boldsymbol{x}_k)_{\mathrm{D}_2(k)} = \boldsymbol{x}_k - \Delta t \left[\frac{\left(\boldsymbol{v}_k^{(+)}[3]\right)_{\mathrm{D}_2} + \boldsymbol{v}_k^0[4']}{2} \right], \tag{B37}$$

where SETTLS is adopted, which uses the time-extrapolated future value $\boldsymbol{v}_k^{(+)} = 2\boldsymbol{v}_k^0 - \boldsymbol{v}_k^-$ at the departure point $(\boldsymbol{x}_k)_{\mathrm{D}_2(k)}$. The variable $\boldsymbol{v}_k^{(+)}[3]$ is calculated by vertical remapping of $\boldsymbol{v}_k^{(+)}[2]$ using $\Delta\pi_k[2]$ and $\Delta\pi_k[3]$ (Appendix B3). The variable $\boldsymbol{v}_k^0[4']$ is calculated by vertical remapping of $\boldsymbol{v}_k^0[5'](= \boldsymbol{v}_k^0)$, where $\Delta\pi_k[4']$ and $\Delta\pi_k[5']$ calculated from the following equations are used:

$$\Delta\pi_k[6'] = \Delta A_k + \Delta B_k \pi_s^0, \tag{B38a}$$

$$\Delta\pi_k[5'] = \Delta\pi_k[6'] + \frac{\Delta t}{2}(D_k \Delta\pi_k)^0, \tag{B38b}$$

$$\Delta\pi_k[4'] = \Delta\pi_k[5'] + \frac{\Delta t}{2}\left[\left(\dot{\eta}\frac{\partial\pi}{\partial\eta}\right)^0_{k-1/2} - \left(\dot{\eta}\frac{\partial\pi}{\partial\eta}\right)^0_{k+1/2}\right]. \tag{B38c}$$

Here, Eq. (B38a) is derived from Eqs. (B28) and (B25), where $\pi_s^{+,\mathrm{explicit}}$ is approximated by π_s^0. Equations (B38b) and (B38c) are derived from Eqs. (B22) and (B20), respectively. It is better to use $\boldsymbol{v}_k^0[4]$ instead of $\boldsymbol{v}_k^0[4']$ in Eq. (B37). However, when calculating horizontal advection from [3] to [4], $\Delta\pi_k[4]$ and $\Delta\pi_k[5]$ are not yet obtained, and $\boldsymbol{v}_k^0[4]$ cannot be calculated. Therefore, in Eq. (B37), $\boldsymbol{v}_k^0[4']$ is used as a sufficiently accurate approximation of $\boldsymbol{v}_k^0[4]$.

By using Eq. (B37), the horizontal departure point $(\boldsymbol{x}_k)_{\mathrm{D}_2}$ can be calculated by iteration (e.g., Ritchie et al., 1995; Temperton et al., 2001). The value of $\left(\boldsymbol{v}_k^{(+)}[3]\right)_{\mathrm{D}_2}$ at the horizontal departure point $(\boldsymbol{x}_k)_{\mathrm{D}_2}$ is calculated by the Lagrange interpolation from the values at the grid points near $(\boldsymbol{x}_k)_{\mathrm{D}_2}$. Here, the fifth-order Lagrange interpolation is used.

The explicitly calculated provisional future value of the wind (Yoshimura, 2022) can also be used instead of the extrapolated future value $\boldsymbol{v}_k^{(+)}$. $\boldsymbol{v}_k^{(+)}$ is used here because the results of the test cases in Sect. 3 are almost identical when the explicitly calculated provisional future value is used.

B5. Horizontal advection of a variable at half-levels

When calculating the advection $w_{k+1/2}$ at half levels to obtain $\left(w_{k+1/2}\right)_{\mathrm{D}}$, $w_{k+1/2}$ is converted to w_k at full levels using the following equation:

$$w_k = \frac{w_{k-1/2} + w_{k+1/2}}{2}, \tag{B39}$$

and then the advection of w_k is calculated using the vertically conservative semi-Lagrangian scheme provided in Appendix B2 to obtain $(w_k)_{\mathrm{D}}$. The value of $\left(w_{k+1/2}\right)_{\mathrm{D}}$ is calculated from the following equation:

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$$\left(w_{k+1/2}\right)_{\mathrm{D}} = w_{k+1/2} + \frac{\ln\left(\frac{\pi_{k-1/2}}{\pi_{k+1/2}}\right)\left[(w_k)_{\mathrm{D}} - w_k\right] + \ln\left(\frac{\pi_{k+1/2}}{\pi_{k+3/2}}\right)\left[(w_{k+1})_{\mathrm{D}} - w_{k+1}\right]}{\ln\left(\frac{\pi_{k-1/2}}{\pi_{k+1/2}}\right) + \ln\left(\frac{\pi_{k+1/2}}{\pi_{k+3/2}}\right)}. \quad \text{(B40)}$$

The method of vertically interpolating $(w_k)_{\mathrm{D}} - w_k$ using Eq. (B40) to obtain $\left(w_{k+1/2}\right)_{\mathrm{D}}$ is more accurate than simply vertically interpolating $(w_k)_{\mathrm{D}}$.

B6. Spatially averaged Eulerian treatment of mountains with horizontal mass advection

The spatially averaged Eulerian treatment of mountains, similar to that of Ritchie and Tanguay (1996), is adopted when the horizontal mass advection is calculated. The approximate effect of Φ_s on the surface pressure is defined here as

$$\pi_s^{\Phi} \equiv \tilde{\pi}_s \exp\left(\frac{-\Phi_s}{R\tilde{T}}\right), \quad \text{(B41)}$$

where $\tilde{\pi}_s = 10^5$ (Pa) and $\tilde{T} = 280$ (K). Since $\partial \pi_s^{\Phi}/\partial t = 0$, the following equation is satisfied:

$$\frac{\mathrm{d}_H}{\mathrm{d}t}(\Delta \pi_k^{\Phi}) = \frac{\mathrm{d}_H}{\mathrm{d}t}(\Delta B_k \pi_s^{\Phi}) = \Delta B_k \boldsymbol{v}_k \cdot \nabla \pi_s^{\Phi}, \quad \text{(B42)}$$

where $\Delta \pi_k^{\Phi} \equiv \Delta A_k + \Delta B_k \pi_s^{\Phi}$. Using Eq. (B42), Eq. (B5) becomes

$$\frac{\mathrm{d}_H}{\mathrm{d}t}(\Delta \pi_k - \Delta B_k \pi_s^{\Phi}) = -\Delta B_k \boldsymbol{v}_k \cdot \nabla \pi_s^{\Phi} - D_k \Delta \pi_k - \left(\dot{\eta}\frac{\partial \pi}{\partial \eta}\right)_{k-1/2} + \left(\dot{\eta}\frac{\partial \pi}{\partial \eta}\right)_{k+1/2}. \quad \text{(B43)}$$

By discretizing Eq. (B43) in time in the same way as Eq. (B7), and using Eqs. (B9), (B11), (B14), (B20), (B22), and (B25), the following equation is obtained:

$$\Delta \pi_k[4] = \left[\Delta \pi_k[3] - \Delta B_k \pi_s^{\Phi} - \frac{\Delta t}{2}\Delta B_k \boldsymbol{v}_k^{(+)}[3] \cdot \nabla \pi_s^{\Phi}\right]_{\mathrm{D}_2(k)} + \Delta B_k \pi_s^{\Phi} - \frac{\Delta t}{2}\Delta B_k \boldsymbol{v}_k[4'] \cdot \nabla \pi_s^{\Phi}. \quad \text{(B44)}$$

By using Eq. (B44) instead of Eq. (B17), the results near mountains are improved (Sect. 3.4).

B7. Semi-Lagrangian treatment of hydrostatic pressure change due to horizontal advection

Equation (C17c) below includes:

$$(\delta\pi)_k \equiv B_{k+1/2}\pi_s \boldsymbol{v}_k \cdot \nabla \ln \pi_s - \sum_{l=k+1}^{K}(\Delta B_l \pi_s \boldsymbol{v}_l \cdot \nabla \ln \pi_s) = \boldsymbol{v}_k \cdot \nabla \pi_{k+1/2} - \sum_{l=k+1}^{K}\left(\boldsymbol{v}_l \cdot \nabla(\Delta \pi_l)\right), \quad \text{(B45)}$$

which represents the pressure change due to horizontal advection. The term $\boldsymbol{v}_k \cdot \nabla \pi_{k+1/2}$ represents the pressure change due to horizontal advection in the kth layer, and the term $-\sum_{l=k+1}^{K}\left(\boldsymbol{v}_l \cdot \nabla(\Delta \pi_l)\right)$ represents the pressure change due to horizontal advection above the kth layer. The value $\delta\pi$ can be calculated in a semi-Lagrangian fashion as follows:

$$(\delta\pi)_k = \frac{\pi_{k+1/2}[4] - \pi_{k+1/2}[3]_{\ \mathrm{D}_2(k)}}{\Delta t}. \quad \text{(B46)}$$

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The nonlinear terms in Eq. (B45) can be one of the sources of numerical instability. Using Eqs. (B46) and (B17), Eq. (C17c) can be discretized as follows:

$$\begin{aligned}\left(F_{Q(\mathrm{adv})}\right)_k\Delta t &= -\frac{1}{2}\left[\frac{1}{\Delta\pi_k[4]}\ln\left(\frac{\pi_{k-1/2}[4]}{\pi_{k+1/2}[4]}\right)+\left\{\frac{1}{\Delta\pi_k[3]}\ln\left(\frac{\pi_{k-1/2}[3]}{\pi_{k+1/2}[3]}\right)\right\}_{\mathrm{D}_2(k)}\right](\delta\pi)_k\Delta t\\ &= -\frac{1}{2\Delta\pi_k[4]}\ln\left(\frac{\pi_{k-1/2}[4]}{\pi_{k+1/2}[4]}\cdot\frac{\pi_{k+1/2}[3]_{\ \mathrm{D}_2(k)}+\Delta\pi_k[4]}{\pi_{k+1/2}[3]_{\ \mathrm{D}_2(k)}}\right)\left(\pi_{k+1/2}[4]-\pi_{k+1/2}[3]_{\ \mathrm{D}_2(k)}\right).\end{aligned} \tag{B47}$$

Using Eq. (B47), instead of discretizing Eq. (C17c) in the usual way, improves the results in some cases when the timestep is long (Sect. 3.4).

B8. Cubic Hermite interpolation to calculate the change in hydrostatic pressure with horizontal advection

To calculate $\pi_{k+1/2}[3]_{\mathrm{D}_2(k)}$ at the horizontal departure point in Eq. (B47), piecewise cubic Hermite interpolation is used. The cubic Hermite interpolation requires the horizontal derivative and is computationally expensive if the horizontal derivative of $\pi_{k+1/2}[3]$ is calculated every timestep by transforming $\pi_{k+1/2}[3]$ from grid to spectral space and the horizontal derivative of $\pi_{k+1/2}[3]$ from spectral to grid space. Since $\pi_{k+1/2}[3]$ has similar values to $f_{k+1/2} = A_{k+1/2} + B_{k+1/2}\pi_s^{\Phi}$ (Eq. [B41]) and the horizontal derivative $\partial f_{k+1/2}/\partial x = B_{k+1/2}\,\partial\pi_s^{\Phi}/\partial x$ only needs to be calculated once rather than at every timestep, $f_{k+1/2}$ is interpolated using the cubic Hermite interpolation and the remaining $g_{k+1/2} = \pi_{k+1/2}[3] - f_{k+1/2}$ is interpolated using the fifth-order Lagrange interpolation. This method yields a slight improvement than only using the fifth-order Lagrange interpolation in the results for case Schär2 when the timestep is short and the departure point is near the grid point (Sect. 3.4).

Appendix C: Vertical discretization

The following vertical discretization method is used, which is similar to that of Bubnová (1995), except for Eqs. (C1)–(C3) and (C19)–(C21) where the discretization is similar to that of Simmons and Burridge (1981):

$$\delta_k \equiv \begin{cases}\ln\dfrac{\pi_{k-1/2}}{\pi_{k+1/2}} & \text{for } k<K\\ 2\ln 2 & \text{for } k=K,\end{cases} \tag{C1}$$

$$\tilde{\pi}_k = \frac{\Delta\pi_k}{\delta_k}, \tag{C2}$$

$$\alpha_k \equiv \begin{cases}1-\dfrac{\pi_{k+1/2}}{\tilde{\pi}_k} & \text{for } k<K\\ \ln 2 & \text{for } k=K,\end{cases} \tag{C3}$$

$$\nabla\pi_s = \pi_s\nabla\ln\pi_s, \tag{C4}$$

$$\Phi_{k-1/2} = \Phi_s + \sum_{l=1}^{k-1}\left(\delta_l\frac{RT_l}{1+\mathcal{P}_l}\right), \tag{C5}$$

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$$\Phi_k = \Phi_{k-1/2} + \alpha_k \frac{RT_k}{1+\mathcal{P}_k}, \tag{C6}$$

$$\nabla\Phi_{k-1/2} = \nabla\Phi_s + \sum_{l=1}^{k-1}\left[\delta_l R\left(\frac{\nabla T_l - T_l \nabla Q_l}{1+\mathcal{P}_l}\right) + \frac{RT_l}{1+\mathcal{P}_l}\left(\frac{B_{l-1/2}}{\pi_{l-1/2}} - \frac{B_{l+1/2}}{\pi_{l+1/2}}\right)\nabla\pi_s\right], \tag{C7}$$

$$(\nabla \ln \pi)_k = \frac{\delta_k B_{k+1/2} + \alpha_k \Delta B_k}{\Delta\pi_k}\nabla\pi_s, \tag{C8}$$

$$\nabla\Phi_k = \nabla\Phi_{k-1/2} + \alpha_k R\left(\frac{\nabla T_k - T_k\nabla Q_k}{1+\mathcal{P}_k}\right) + \frac{RT_k}{1+\mathcal{P}_k}\frac{B_{k-1/2}}{\pi_{k-1/2}}\nabla\pi_s - \frac{RT_k}{1+\mathcal{P}_k}(\nabla\ln\pi)_k, \tag{C9}$$

$$\left(\frac{1}{m}\frac{\partial\pi\mathcal{P}}{\partial\eta}\right)_{k-1/2} = \begin{cases} \text{See Sect. 2.3} & \text{for } k=1 \\ \dfrac{\tilde{\pi}_k\mathcal{P}_k - \tilde{\pi}_{k+1}\mathcal{P}_{k+1}}{\tilde{\pi}_k - \tilde{\pi}_{k+1}} & \text{for } 2 \le k \le K \\ \mathcal{P}_k & \text{for } k = K+1, \end{cases} \tag{C10}$$

$$(\boldsymbol{F}_v)_k = -RT_k[(\nabla\ln\pi)_k + \nabla Q_k] - \left\{1 + \frac{1}{2}\left[\left(\frac{1}{m}\frac{\partial\pi\mathcal{P}}{\partial\eta}\right)_{k-1/2} + \left(\frac{1}{m}\frac{\partial\pi\mathcal{P}}{\partial\eta}\right)_{k+1/2}\right]\right\}\nabla\Phi_k, \tag{C11}$$

$$(F_w)_{k-1/2} = g\left(\frac{1}{m}\frac{\partial\pi\mathcal{P}}{\partial\eta}\right)_{k-1/2}, \tag{C12}$$

$$\boldsymbol{v}_{k-1/2} = \frac{\delta_k\boldsymbol{v}_{k-1} + \delta_{k-1}\boldsymbol{v}_k}{\delta_{k-1} + \delta_k}, \tag{C13}$$

$$\begin{aligned}(D_3)_k &= \left\{\nabla\cdot\boldsymbol{v} + \frac{(1+\mathcal{P})}{RT}\frac{\pi}{m}\left[\frac{\partial}{\partial\eta}(\nabla\Phi\cdot\boldsymbol{v}) - \frac{\partial\nabla\Phi}{\partial\eta}\cdot\boldsymbol{v}\right] - \frac{(1+\mathcal{P})\pi}{RT}\frac{g}{m}\frac{\partial w}{\partial\eta}\right\}_k \\ &= D_k + \frac{1+\mathcal{P}_k}{RT_k}\frac{\tilde{\pi}_k}{\Delta\pi_k}\big[\nabla\Phi_{k-1/2}\cdot\boldsymbol{v}_{k-1/2} - \nabla\Phi_{k+1/2}\cdot\boldsymbol{v}_{k+1/2} \\ &\qquad -(\nabla\Phi_{k-1/2} - \nabla\Phi_{k+1/2})\cdot\boldsymbol{v}_k - g(w_{k-1/2} - w_{k+1/2})\big],\end{aligned} \tag{C14}$$

$$(D_3)_{k=1} = D_k + \frac{1+\mathcal{P}_1}{RT_1}\frac{\tilde{\pi}_1}{\Delta\pi_1}\left[-\nabla\Phi_{3/2}\cdot\boldsymbol{v}_{3/2} - (\nabla\Phi_{1/2} - \nabla\Phi_{3/2})\cdot\boldsymbol{v}_1 - g(-w_{3/2})\right], \tag{C14'}$$

$$(F_{\ln T})_k = -\frac{R}{c_v}(D_3)_k, \tag{C15}$$

$$\left(\frac{1}{\pi}\frac{\mathrm{d}\pi}{\mathrm{d}t}\right)_k = \frac{1}{\Delta\pi_k}\left\{\ln\left(\frac{\pi_{k-1/2}}{\pi_{k+1/2}}\right)\left[B_{k+1/2}\pi_s\boldsymbol{v}_k\cdot\nabla\ln\pi_s - \sum_{l=k+1}^{K}(\Delta B_l\pi_s\boldsymbol{v}_l\cdot\nabla\ln\pi_s + D_l\Delta\pi_l)\right] - \alpha_k D_k\Delta\pi_k\right\}, \tag{C16}$$

$$F_Q = -\left(\frac{1}{\pi}\frac{\mathrm{d}\pi}{\mathrm{d}t}\right)_k - \frac{c_\mathrm{p}}{c_\mathrm{v}}(D_3)_k = \left(F_{Q(\mathrm{div})}\right)_k + \left(F_{Q(\mathrm{adv})}\right)_k, \tag{C17a}$$

$$\left(F_{Q(\mathrm{div})}\right)_k \equiv -\frac{c_p}{c_v}(D_3)_k + \left[\frac{1}{\Delta\pi_k}\ln\left(\frac{\pi_{k-1/2}}{\pi_{k+1/2}}\right)\sum_{l=k+1}^{K}(D_l\Delta\pi_l) + \alpha_k D_k\Delta\pi_k\right], \tag{C17b}$$

$$\left(F_{Q(\mathrm{adv})}\right)_k \equiv \frac{1}{\Delta\pi_k}\ln\left(\frac{\pi_{k-1/2}}{\pi_{k+1/2}}\right)\left[-B_{k+1/2}\pi_s\boldsymbol{v}_k\cdot\nabla\ln\pi_s + \sum_{l=k+1}^{K}(\Delta B_l\pi_s\boldsymbol{v}_l\cdot\nabla\ln\pi_s)\right], \tag{C17c}$$

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$$\Delta\pi_k^* = \Delta A_k + \Delta B_k \bar{\pi}_s, \tag{C18}$$

$$\delta_k^* = \begin{cases} \ln\left(\pi_{k-1/2}^*/\pi_{k+1/2}^*\right) & \text{for } 1 \le k \le K-1 \\ 2\ln 2 & \text{for } k = K, \end{cases} \tag{C19}$$

$$\tilde{\pi}_k^* = \Delta\pi_l^*/\delta_l^*, \tag{C20}$$

$$\alpha_k^* = \begin{cases} 1 - \dfrac{\pi_{k+1/2}^*}{\tilde{\pi}_k^*} & \text{for } 1 \le k \le K-1 \\ \ln 2 & \text{for } k = K, \end{cases} \tag{C21}$$

$$\left(\frac{\pi^*}{m^*}\frac{\partial \mathrm{X}}{\partial\eta}\right)_k = \frac{X_{k-1/2} - X_{k+1/2}}{\delta_k^*}, \tag{C22}$$

$$\left(\boldsymbol{L}_{\mathrm{C},\boldsymbol{v}}\right)_k = R\bar{T}\left[\{\mathbf{G}^*(-\nabla\ln T + \nabla\mathcal{Q})\}_k - \nabla\mathcal{Q}_k - \nabla\ln\pi_s\right], \tag{C23a}$$

$$(\mathbf{G}^* X)_k \equiv \sum_{l=1}^{k-1} \delta_l^* X_l + \alpha_k^* X_k, \tag{C23b}$$

$$\left(\boldsymbol{L}_{\mathrm{N},\boldsymbol{v},}\right)_k = -\left[\nabla\Phi^{\#}\frac{1}{m^{\#}}\frac{\partial(\pi^{\#}\mathcal{P})}{\partial\eta}\right]_k = -\nabla\Phi_k^{\#}\left[\left(\frac{1}{m^{\#}}\frac{\partial\pi^{\#}\mathcal{P}}{\partial\eta}\right)_{k-1/2} + \left(\frac{1}{m^{\#}}\frac{\partial\pi^{\#}\mathcal{P}}{\partial\eta}\right)_{k+1/2}\right], \tag{C24}$$

$$\left(\frac{1}{m^*}\frac{\partial\pi^*\mathcal{Q}}{\partial\eta}\right)_{k-1/2} = \begin{cases} 0 & \text{for } k = 1 \\ \dfrac{\tilde{\pi}_k^*\mathcal{Q}_k - \tilde{\pi}_{k+1}^*\mathcal{Q}_{k+1}}{\tilde{\pi}_k^* - \tilde{\pi}_{k+1}^*} & \text{for } 2 \le k \le K \\ \mathcal{Q}_k & \text{for } k = K+1, \end{cases} \tag{C25}$$

$$\left(L_{\mathrm{C,w,}}\right)_{k-1/2} = \left(\frac{g}{m^*}\frac{\partial\pi^*\mathcal{Q}}{\partial\eta}\right)_{k-1/2}, \tag{C26}$$

$$\hat{\hat{d}}_k = \left[\frac{1+\mathcal{P}_k}{RT_k}\frac{\tilde{\pi}_k}{\Delta\pi_k}\right]^{\#}\left[\nabla\Phi_{k-1/2}^{\#}\cdot\boldsymbol{v}_{k-1/2} - \nabla\Phi_{k+1/2}^{\#}\cdot\boldsymbol{v}_{k+1/2}\right.$$
$$\left. -\left(\nabla\Phi_{k-1/2}^{\#} - \nabla\Phi_{k+1/2}^{\#}\right)\cdot\boldsymbol{v}_k - g\left(w_{k-1/2} - w_{k+1/2}\right)\right], \tag{C27}$$

$$\hat{\hat{d}}_{k=1} = \left[\frac{1+\mathcal{P}_1}{RT_1}\frac{\pi_1}{\Delta\pi_1}\right]^{\#}\left[-\nabla\Phi_{3/2}^{\#}\cdot\boldsymbol{v}_{3/2} - \left(\nabla\Phi_{1/2}^{\#} - \nabla\Phi_{3/2}^{\#}\right)\cdot\boldsymbol{v}_1 - g\left(-w_{3/2}\right)\right], \tag{C27′}$$

$$\left(L_{\mathrm{C},\ln T}\right)_k = -\frac{R}{c_{\mathrm{v}}}\left(D_k + \hat{\hat{d}}_k\right), \tag{C28}$$

$$\left(L_{\mathrm{C},\mathcal{Q}}\right)_k = (\mathbf{S}^* D)_k - \frac{c_{\mathrm{p}}}{c_{\mathrm{v}}}\left(D_k + \hat{\hat{d}}_k\right), \tag{C29a}$$

$$(\mathbf{S}^* X)_k \equiv \frac{1}{\tilde{\pi}_k^*}\sum_{l=k+1}^{L}\Delta\pi_l^* X_l + \alpha_k^* X_k, \tag{C29b}$$

$$\left(L_{\mathrm{C},\ln\pi_s}\right)_k = -(\mathbf{N}^* D)_k, \tag{C30a}$$

$$(\mathbf{N}^* X)_k \equiv \frac{1}{\bar{\pi}_s}\sum_{l=1}^{L}\Delta\pi_l^* X_l, \tag{C30b}$$

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$$\left(L_{\mathrm{C},\hat{d}}\right)_k = -\frac{g^2}{R\bar{\bar{T}}}(\mathbf{L}^* X)_k, \tag{C31a}$$

$$(\mathbf{L}^* X)_k \equiv \begin{cases} \dfrac{1}{\delta_k^*}\left(-\dfrac{\tilde{\pi}_k^*}{\tilde{\pi}_k^* - \tilde{\pi}_{k+1}^*}X_k + \dfrac{\tilde{\pi}_{k+1}^*}{\tilde{\pi}_k^* - \tilde{\pi}_{k+1}^*}X_{k+1}\right) & \text{for } k = 1 \\ \dfrac{1}{\delta_k^*}\left[\dfrac{\tilde{\pi}_{k-1}^*}{\tilde{\pi}_{k-1}^* - \tilde{\pi}_k^*}X_{k-1} - \left(\dfrac{\tilde{\pi}_k^*}{\tilde{\pi}_{k-1}^* - \tilde{\pi}_k^*} + \dfrac{\tilde{\pi}_k^*}{\tilde{\pi}_k^* - \tilde{\pi}_{k+1}^*}\right)X_k + \dfrac{\tilde{\pi}_{k+1}^*}{\tilde{\pi}_k^* - \tilde{\pi}_{k+1}^*}X_{k+1}\right] & \text{for } 1 \le k \le K-1 \\ \dfrac{1}{\delta_k^*}\left(\dfrac{\tilde{\pi}_{k-1}^*}{\tilde{\pi}_{k-1}^* - \tilde{\pi}_k^*}X_{k-1} - \dfrac{\tilde{\pi}_{k-1}^*}{\tilde{\pi}_{k-1}^* - \tilde{\pi}_k^*}X_k\right) & \text{for } k = K, \end{cases} \tag{C31b}$$

$$\left(\frac{\pi^*}{m^*}\frac{\partial \mathrm{X}}{\partial \eta}\right)_k = \frac{X_{k-1/2} - X_{k+1/2}}{\delta_k^*}. \tag{C32}$$

In Eq. (C14), the relationship $\nabla\Phi \cdot \partial\boldsymbol{v}/\partial\eta = \partial(\nabla\Phi \cdot \boldsymbol{v})/\partial\eta - \partial\nabla\Phi/\partial\eta \cdot \boldsymbol{v}$ is used (Bubnová et al., 1995). By using Eq. (14), Eq. (C14) becomes Eq. (C14') for $k = 1$. In Eq. (C17c), $\left(F_{Q(\mathrm{adv})}\right)_k$ can be calculated using a semi-Lagrangian treatment of the pressure change due to horizontal advection (Appendix B7). In Eq. (C24), $(1/m^\# \cdot \partial\pi^\#\mathcal{P}/\partial\eta)$ is discretized as in Eq. (C10). See Eqs. (C14) and (C14') for the derivation of Eqs. (C27) and (C27'). Note that $w_{1/2}$ and $\boldsymbol{v}_{1/2}$ do not appear in Eqs. (C14') and (C27').

Appendix D: Spectral transform

In the x–z non-hydrostatic spectral model, the variables $\ln T$, $\mathcal{Q}$, u, $\hat{\hat{d}}$, and $\ln\pi_s$ are expanded in the x direction using the discrete Fourier transform. For example, $\mathcal{Q}(x,z)$ is expanded in the x direction as follows:

$$\mathcal{Q}(x,z) \cong \sum_{m=-M}^{M} \mathcal{Q}_m(z)\exp\left(i\frac{2\pi m x}{L}\right), \tag{D1a}$$

$$\mathcal{Q}_{-m}(z) \equiv \mathcal{Q}_m^*(z), \tag{D1b}$$

where m is the wavenumber, M is the truncation wavenumber, $\mathcal{Q}_m(z)$ is a complex variable, and the superscript $*$ represents the complex conjugate. Equation (D1a) can be converted as follows:

$$\mathcal{Q}(x,z) \cong \mathcal{Q}_0(z) + \sum_{m=1}^{M}\left[\mathcal{Q}_m^{\mathrm{c}}(z)\cos\left(\frac{2\pi m x}{L}\right) + \mathcal{Q}_m^{\mathrm{s}}(z)\sin\left(\frac{2\pi m x}{L}\right)\right], \tag{D2a}$$

$$\mathcal{Q}_m^{\mathrm{c}}(z) \equiv 2\,\mathrm{Re}\big(\mathcal{Q}_m(z)\big),\quad \mathcal{Q}_m^{\mathrm{s}}(z) \equiv -2\,\mathrm{Im}\big(\mathcal{Q}_m(z)\big), \tag{D2b}$$

where Re( ) represents the real part and Im( ) the imaginary part.

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Appendix E: How to solve linear equations for the non-constant coefficient semi-implicit scheme

The method to solve the simultaneous equations (29a), (30a), (31a), (32), (33), and (36a) in the spectral model is described here. There are non-constant coefficient linear terms, $\boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}^{+}$ and $L_{\mathrm{N},\hat{\hat{d}}}^{+}$, in Eqs. (32), (33), and (36a), which contain $\nabla\Phi^{\#}$. Because of these non-constant coefficient linear terms, the simultaneous equations cannot be easily solved.

710 The first approximate solution is obtained using the method described in Appendix E1. One of the following three methods is used to obtain the solution.

(1) The no recalculation method that considers the first approximate solution as the final solution (Appendix E1).

(2) The single recalculation method that only recalculates non-constant coefficient linear terms once (Appendix E2).

(3) The preconditioned GCR method (Appendix E3).

715 The computational cost and numerical stability change from low to high from (1) to (3).

E1. No recalculation method to obtain a first approximate solution

Substituting the extrapolated future value $\boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}^{(+)} = 2\boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}^{0} - \boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}^{-}$ into $\boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}^{+}$ in Eqs. (32) and (33) gives

$$D^{+} - \frac{\Delta t}{2} R T_1^{*} [\mathcal{G}^{*}(-\nabla^2 \ln T^{+} + \nabla^2 \mathcal{Q}^{+}) - \nabla^2 \mathcal{Q}^{+} - \nabla^2 \ln \pi_s^{+}] = \nabla \cdot \boldsymbol{R}_{\boldsymbol{v}}', \tag{E1a}$$

720
$$\zeta^{+} = \boldsymbol{k} \cdot \nabla \times \boldsymbol{R}_{\boldsymbol{v}}', \tag{E1b}$$

$$\boldsymbol{R}_{\boldsymbol{v}}' \equiv \boldsymbol{R}_{\boldsymbol{v}} + \frac{\Delta t}{2}\left(2\boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}^{0} - \boldsymbol{L}_{\mathrm{N},\boldsymbol{v}}^{-}\right). \tag{E1c}$$

In the x–z model, substituting the extrapolated future value $L_{\mathrm{N},u(m=0)}^{(+)} = 2L_{\mathrm{N},u(m=0)}^{0} - L_{\mathrm{N},u(m=0)}^{-}$ into $L_{\mathrm{N},u(m=0)}^{+}$ in Eq. (34) gives

$$u_{u(m=0)}^{+} = R_{u(m=0)}', \tag{E2a}$$

725
$$R_{u(m=0)}' \equiv R_{u(m=0)} + \frac{\Delta t}{2}\left(2L_{\mathrm{N},u(m=0)}^{0} - L_{\mathrm{N},u(m=0)}^{-}\right), \tag{E2b}$$

which is used instead of Eqs. (E1b) and (E1c). Substituting the extrapolated future value $\boldsymbol{v}^{(+)} = 2\boldsymbol{v}^{0} - \boldsymbol{v}^{-}$ into $\boldsymbol{v}^{+}$ in Eq. (36) gives

$$\hat{\hat{d}}^{+} - \frac{\beta\Delta t}{2}\left(-\frac{g^2}{R\overline{\overline{T}}}\mathcal{L}^{*}\mathcal{Q}^{+}\right) = R_{\hat{\hat{d}}}', \tag{E3a}$$

$$R_{\hat{\hat{d}}}' \equiv R_{\hat{\hat{d}}} + \left[\frac{(1+\mathcal{P})}{RT}\frac{\pi}{m}\nabla\Phi\right]^{\#} \cdot \frac{\partial(2\boldsymbol{v}^{0} - \boldsymbol{v}^{-})}{\partial\eta}. \tag{E3b}$$

730 The right sides of Eqs. (E1a), (E1b), (E3a), (29a), (30a), and (31a) are explicitly calculated in grid space and transformed to spectral space. Since the left sides of these equations contain only constant coefficient linear terms, these simultaneous

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equations can be easily solved in spectral space in a similar way as in Bubnová et al. (1995), as shown below. Substituting $\ln T^+$ from Eq. (29a), Q^+ from Eq. (30a), and $\ln \pi_s^+$ from Eq. (31a) into Eq. (E1a), and then using the relationship

$$R = c_\mathrm{p} - c_\mathrm{v}, \tag{E4}$$

yields the following equations:

$$\left[1 - \left(\frac{\beta\Delta t}{2}\right)^2 R\bar{T}\left(-\mathcal{A}_1 + \frac{c_\mathrm{p}}{c_\mathrm{v}}\right)\nabla^2\right] D^+ - \left(\frac{\beta\Delta t}{2}\right)^2 R\bar{T}\left(-\mathcal{G}^* + \frac{c_\mathrm{p}}{c_\mathrm{v}}\right)\nabla^2 \hat{\hat{d}}^+ = G, \tag{E5a}$$

$$\mathcal{A}_1 \equiv -\mathcal{G}^*\mathcal{S}^* + \mathcal{G}^* + \mathcal{S}^* - \mathcal{N}^*, \tag{E5b}$$

$$G \equiv \nabla \cdot \boldsymbol{R}_v' + \frac{\beta\Delta t}{2} R\bar{T}\left[\mathcal{G}^*\left(-\nabla^2 R_{\ln} + \nabla^2 R_Q\right) - \nabla^2 R_Q - \nabla^2 R_{\ln\ s}\right]. \tag{E5c}$$

Substituting Q^+ into Eq. (30a) and into Eq. (E3a) leads to

$$\left[1 - \left(\frac{\beta\Delta t}{2}\right)^2 \frac{g^2}{R\bar{\bar{T}}} \frac{c_\mathrm{p}}{c_\mathrm{v}} \mathcal{L}^*\right] \hat{\hat{d}}^+ - \left(\frac{\beta\Delta t}{2}\right)^2 \frac{g^2}{R\bar{\bar{T}}} \mathcal{L}^*\left(-\mathcal{S}^* + \frac{c_\mathrm{p}}{c_\mathrm{v}}\right) D^+ = H, \tag{E6a}$$

$$H \equiv R_{\hat{\hat{d}}}' - \frac{\beta\Delta t}{2} \frac{g^2}{R\bar{\bar{T}}} \mathcal{L}^* R_Q. \tag{E6b}$$

When the vertical discretization of Bubnová (1995) is used, $\mathcal{A}_1 = 0$ is satisfied. In the model, $\mathcal{A}_1 = 0$ is not satisfied because the vertical discretization of Simmons and Burridge (1981) is used in Eqs. (C1)–(C3) and (C19)–(C21). However, the simultaneous equations (E5a) and (E6a) can be solved when $\mathcal{A}_1 \neq 0$. By vertically discretizing Eqs. (E5a) and (E6a), the following matrix equations are obtained:

$$[\mathbf{I} - \mathbf{B}\boldsymbol{\nabla}^2]\mathbf{D}^+ - \mathbf{C}\boldsymbol{\nabla}^2\hat{\hat{\boldsymbol{d}}}^+ = \mathbf{G}, \tag{E7}$$

$$\mathbf{Y}\hat{\hat{\boldsymbol{d}}}^+ - \mathbf{Z}\mathbf{D}^+ = \mathbf{H}, \tag{E8}$$

where $\mathbf{I}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{Y}$, and $\mathbf{Z}$ represent $K \times K$ square matrices, K is the number of vertical layers, $\mathbf{I}$ is the identity matrix, and $\mathbf{Y}$ is the tri-diagonal matrix. Substituting

$$\hat{\hat{\boldsymbol{d}}}^+ = \mathbf{Y}^{-1}\mathbf{Z}\mathbf{D}^+ + \mathbf{Y}^{-1}\mathbf{H} \tag{E8′}$$

obtained from Eq. (E8) into Eq. (E7) leads to

$$\mathbf{D}^+ - (\mathbf{B} + \mathbf{C}\mathbf{Y}^{-1}\mathbf{Z})\boldsymbol{\nabla}^2\mathbf{D}^+ = \mathbf{G} + \mathbf{C}\mathbf{Y}^{-1}\boldsymbol{\nabla}^2\mathbf{H}. \tag{E9}$$

The singular value decomposition (SVD) of the matrix $\mathbf{B} + \mathbf{C}\mathbf{Y}^{-1}\mathbf{Z}$ leads to

$$\mathbf{B} + \mathbf{C}\mathbf{Y}^{-1}\mathbf{Z} = \mathbf{U}^{-1}\mathbf{T}\mathbf{U}, \tag{E10}$$

where $\mathbf{T}$ is a diagonal matrix. Substituting Eq. (E10) into Eq, (E9) leads to

$$\mathbf{U}\mathbf{D}^+ - \mathbf{T}\nabla^2\mathbf{U}\mathbf{D}^+ = \mathbf{U}(\mathbf{G} + \mathbf{C}\mathbf{Y}^{-1}\nabla^2\mathbf{H}). \tag{E11}$$

Equation (E11) consists of independent Helmholtz equations for each vertical level. Solving Eq. (E11) gives $\mathbf{U}\mathbf{D}^+$, and multiplying $\mathbf{U}^{-1}$ gives $\mathbf{D}^+$. Substituting $\mathbf{D}^+$ into Eq. (E8’) gives $\hat{\hat{\boldsymbol{d}}}^+$. Substituting D^+ and $\hat{\hat{d}}^+$ into Eqs. (29), (30), and (31) gives $\ln T^+$, Q^+, and $\ln \pi_s^+$. In this way, the first approximate solutions $D^{+[0]}$, $\hat{\hat{d}}^{+[0]}$, $\ln T^{+[0]}$, $Q^{+[0]}$ and $\ln \pi_s^{+[0]}$ are obtained.

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In test cases without mountains, the first approximate solution obtained here can be used as the final solution without stability problems because the values of the non-constant coefficient linear terms $\boldsymbol{L}_{\mathrm{N},\nu}$ and $L_{\mathrm{N},\hat{d}}$, which contain $\nabla\Phi^{\#}$, are small. This method is referred to here as the no recalculation method.

E2. Single recalculation method

The simultaneous equations (29a), (30a), (31a), (32), (33), and (36a) can be written as

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{E21}$$

where $\mathbf{x}$ represents the variables D^{+}, $\hat{\hat{d}}^{+}$, $\ln T^{+}$, Q^{+}, and $\ln \pi_s^{+}$ in spectral space. $\mathbf{A}$ can be expressed as

$$\mathbf{A} = \mathbf{M} - \mathbf{L}_{\mathrm{N}}, \tag{E22}$$

where $-\mathbf{L}_{\mathrm{N}}$ represents the non-constant coefficient linear terms containing $\nabla\Phi^{\#}$ in the left side of Eqs. (32), (33), and (36a), and $\mathbf{M}$ represents the constant coefficient linear terms. By using the no recalculation method described in Appendix E1, the first approximate solution $\mathbf{x}^{[0]}$ is obtained by solving the simultaneous equations $\mathbf{M}\mathbf{x}^{[0]} = \mathbf{b} + \mathbf{L}_{\mathrm{N}}\mathbf{x}^{(+)}$ in spectral space as follows:

$$\mathbf{x}^{[0]} = \mathbf{M}^{-1}\left(\mathbf{b} + \mathbf{L}_{\mathrm{N}}\mathbf{x}^{(+)}\right), \tag{E23}$$

where $\boldsymbol{x}^{(+)} = 2\boldsymbol{x}^{0} - \boldsymbol{x}^{-}$ is the extrapolated future value. In the single recalculation method described here, a more accurate solution $\mathbf{x}^{\mathrm{r}}$ is calculated by recalculating the non-constant coefficient linear terms using the first approximate solution $\mathbf{x}^{[0]}$ as follows:

$$\mathbf{x}^{\mathrm{r}} = \mathbf{x}^{[0]} + \alpha^{\mathrm{r}}\mathbf{M}^{-1}\mathbf{L}_{\mathrm{N}}\left(\mathbf{x}^{[0]} - \mathbf{x}^{(+)}\right), \tag{E24}$$

where α^{r} is a parameter from 0.0 to 1.0. When $\alpha^{\mathrm{r}} = 1.0$, Eq. (E24) becomes $\mathbf{x}^{\mathrm{r}} = \mathbf{M}^{-1}\left(\mathbf{b} + \mathbf{L}_{\mathrm{N}}\mathbf{x}^{[0]}\right)$, which is the same as Eq. (E23) except that $\mathbf{x}^{(+)}$ is replaced by $\mathbf{x}^{[0]}$. The parameter α^{r} corresponds to $\alpha^{[0]}$ in (6) and (7) in Appendix E3 below, where $\alpha^{[0]}$ is calculated with higher computational cost using the GCR method to ensure that the error is minimized. Since $\alpha^{[0]}$ is usually <1.0 when using the GCR method in the test cases in Sect. 3, $\alpha^{\mathrm{r}} = 0.8$ is used instead of $\alpha^{\mathrm{r}} = 1.0$, which improves the numerical stability (Sect. 3.2). The term $\mathbf{M}^{-1}\mathbf{L}_{\mathrm{N}}\left(\mathbf{x}^{[0]} - \mathbf{x}^{(+)}\right)$ in Eq. (E24) is calculated as follows: $\mathbf{x}^{[0]}$ in spectral space is transformed to grid space, $\mathbf{L}_{\mathrm{N}}\left(\mathbf{x}^{[0]} - \mathbf{x}^{(+)}\right)$ is calculated in grid space and transformed to spectral space, and $\mathbf{M}^{-1}\mathbf{L}_{\mathrm{N}}\left(\mathbf{x}^{[0]} - \mathbf{x}^{(+)}\right)$ is calculated in spectral space using the method in Appendix E1.

When it is necessary to obtain more accurate solutions, the preconditioned GCR method described in Appendix E3 can be used. However, the single recalculation method using Eq. (E24) usually gives sufficiently accurate solutions, except for extremely steep mountain test cases (Sect. 3.3).

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E3. Preconditioned GCR method

Equation (E21) can be solved with the preconditioned GCR method (Eisenstat et al., 1983; Thomas et al., 2003), which is a fast-convergent iterative method. A unique feature of the GCR method in the spectral model is that the equation is solved in spectral space, except for the non-constant coefficient linear terms, which are calculated in grid space. The solver for the equations with constant coefficient linear terms in wave space ($\mathbf{M}^{-1}$; Appendices E1 and E2) is used as a preconditioner, which significantly improves the convergence of the iterations. Equation (E21) is solved as follows:

(1) Compute the first approximate solution $\mathbf{x}^{[0]}$ in Eq. (E23).

(2) Compute the residual $\mathbf{r}^{[0]} = \mathbf{b} - \mathbf{A}\mathbf{x}^{[0]} = \mathbf{b} - (\mathbf{M} - \mathbf{L}_\mathrm{N})\mathbf{x}^{[0]} = \mathbf{L}_\mathrm{N}\left(\mathbf{x}^{[0]} - \mathbf{x}^{(+)}\right)$.

(3) Compute the direction vector $\mathbf{p}^{[0]} = \mathbf{M}^{-1}\mathbf{r}^{[0]}$.

(4) Compute $\mathbf{A}\mathbf{p}^{[0]} = (\mathbf{M} - \mathbf{L}_\mathrm{N})\mathbf{p}^{[0]} = \mathbf{r}^{[0]} - \mathbf{L}_\mathrm{N}\mathbf{p}^{[0]}$.

(5) Loop: $n = 0, 1, 2, N_{\mathrm{max\_iter}} - 1$, where $N_{\mathrm{max\_iter}}$ is the maximum number of iterations.

(6) Compute $\alpha^{[n]} = \left(\mathbf{r}^{[n]}, \mathbf{A}\mathbf{p}^{[n]}\right) / \left(\mathbf{A}\mathbf{p}^{[n]}, \mathbf{A}\mathbf{p}^{[\mathrm{n}]}\right)$.

(7) Compute $\mathbf{x}^{[n+1]} = \mathbf{x}^{[n]} + \alpha^{[n]}\mathbf{p}^{[n]}$.

(8) Compute $\mathbf{r}^{[n+1]} = \mathbf{r}^{[n]} - \alpha^{[n]}\mathbf{A}\mathbf{p}^{[\mathrm{n}]}$.

(9) Exit Loop if $\left(\mathbf{r}^{[n+1]}, \mathbf{r}^{[n+1]}\right) < \varepsilon$ (error threshold) or $n = N_{\mathrm{max\_iter}} - 1$. Here, $\mathbf{x}^{[n+1]}$ is an obtained solution.

(10) Compute $\mathbf{q} = \mathbf{M}^{-1}\mathbf{r}^{[n+1]}$.

(11) Compute $\mathbf{A}\mathbf{q} = (\mathbf{M} - \mathbf{L}_\mathrm{N})\mathbf{q} = \mathbf{r}^{[n+1]} - \mathbf{L}_\mathrm{N}\mathbf{q}$.

(12) Compute $\beta^{[i,n]} = -\left(\mathbf{A}\mathbf{q}, \mathbf{A}\mathbf{p}^{[i]}\right) / \left(\mathbf{A}\mathbf{p}^{[i]}, \mathbf{A}\mathbf{p}^{[i]}\right)$, where $i = 0,1,2, \dots, n$.

(13) Compute $\mathbf{p}^{[n+1]} = \mathbf{q} + \sum_{i=0}^{n} \beta^{[i,n]}\, \mathbf{p}^{[i]}$.

(14) Compute $\mathbf{A}\mathbf{p}^{[n+1]} = \mathbf{A}\mathbf{q} + \sum_{i=0}^{n} \beta^{[i,n]}\, \mathbf{A}\mathbf{p}^{[i]}$.

(15) End Loop.

Here, $\mathbf{x}^{[n]}$, $\mathbf{r}^{[n]}$, $\mathbf{p}^{[n]}$, and $\mathbf{q}^{[n]}$ are the values in spectral space, $\mathbf{L}_\mathrm{N}$ represents the non-constant coefficient linear operators, and $(\mathbf{a}, \mathbf{b})$ is an inner product. $\mathbf{L}_\mathrm{N}\mathbf{x}$ is calculated as follows: $\mathbf{x}$ in spectral space is transformed to grid space, $\mathbf{L}_\mathrm{N}\mathbf{x}$ is calculated in grid space, and then transformed to spectral space. The inner product of the variables $A(x, z)$ and $B(x, z)$ is calculated in spectral space using the following equation:

$$\begin{aligned}\frac{1}{K}\sum_{k=1}^{K}\left\{\frac{1}{L}\int_0^L A(x, z_k)B(x, z_k)\mathrm{d}x\right\} &= \frac{1}{K}\sum_{k=1}^{K}\left\{A_0(z_k)B_0(z_k) + \sum_{n=1}^{N}\frac{1}{2}\left[A_n^\mathrm{c}(z_k)B_n^\mathrm{c}(z_k) + A_n^\mathrm{s}(z_k)B_n^\mathrm{s}(z_k)\right]\right\} \\ &= \frac{1}{K}\sum_{k=1}^{K}\left\{A_0(z_k)B_0(z_k) + \sum_{n=1}^{N} 2\left[\mathrm{Re}\left(A_n(z_k)\right)\mathrm{Re}\left(B_n(z_k)\right) + \mathrm{Im}\left(A_n(z_k)\right)\mathrm{Im}\left(B_n(z_k)\right)\right]\right\},\end{aligned} \tag{E1}$$

where Eq. (D2) is used to transform the equation.

Appendix F: Rayleigh damping

In the mountain wave test cases in Sects. 3.2–3.4, Rayleigh damping on w similar to that of Klemp et al. (2008) is used. The equation for the Rayleigh damping is as follows:

$$\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)_{\mathrm{R}} = -\nu \cdot w, \tag{F1}$$

and the vertical profile of ν is given as

$$\nu = \begin{cases} \nu_{\max} & \text{for } z_{\max} \le z, \\ \nu_{\max} \sin^2\left[\frac{\pi}{2} \cdot \frac{z - z_{\min}}{z_{\max} - z_{\min}}\right] & \text{for } z_{\min} \le z \le z_{\max}, \\ 0 & \text{for } z \le z_{\min}. \end{cases} \tag{F2}$$

In the mountain wave test cases, $\nu_{\max} = 25.0/\Delta x$ is used, except for case Schär2 where $\nu_{\max} = 50.0/\Delta x$ is used.

In the implicit Rayleigh damping of Klemp et al. (2008), Eq. (F1) is discretized as follows:

$$\frac{w^{(\mathrm{new})} - w^{(\mathrm{old})}}{\Delta t} = -\nu \cdot w^{(\mathrm{new})}, \tag{F3}$$

that is,

$$w^{(\mathrm{new})} = \frac{1}{1 + \nu \Delta t} w^{(\mathrm{old})}, \tag{F4}$$

where $w^{(\mathrm{old})}$ and $w^{(\mathrm{new})}$ are the values before and after applying Rayleigh damping, respectively. On the other hand, when solving Eq. (F1) analytically, the following equation is obtained:

$$w^{(\mathrm{new})} = w^{(\mathrm{old})} \exp(-\nu \Delta t). \tag{F5}$$

In cases Schär1 and Schär2, the results when using Eq. (F5) have less timestep dependency than those when using Eq. (F4). Therefore, Eq. (F5) is used here.

Author Contributions

HY developed the two-dimensional x–z nonhydrostatic spectral model using non-constant coefficient semi-implicit and vertically conservative semi-Lagrangian schemes, conducted the model experiments, analyzed the data, and wrote the paper.

Competing Interests

The author declares that there are no conflicts of interest.

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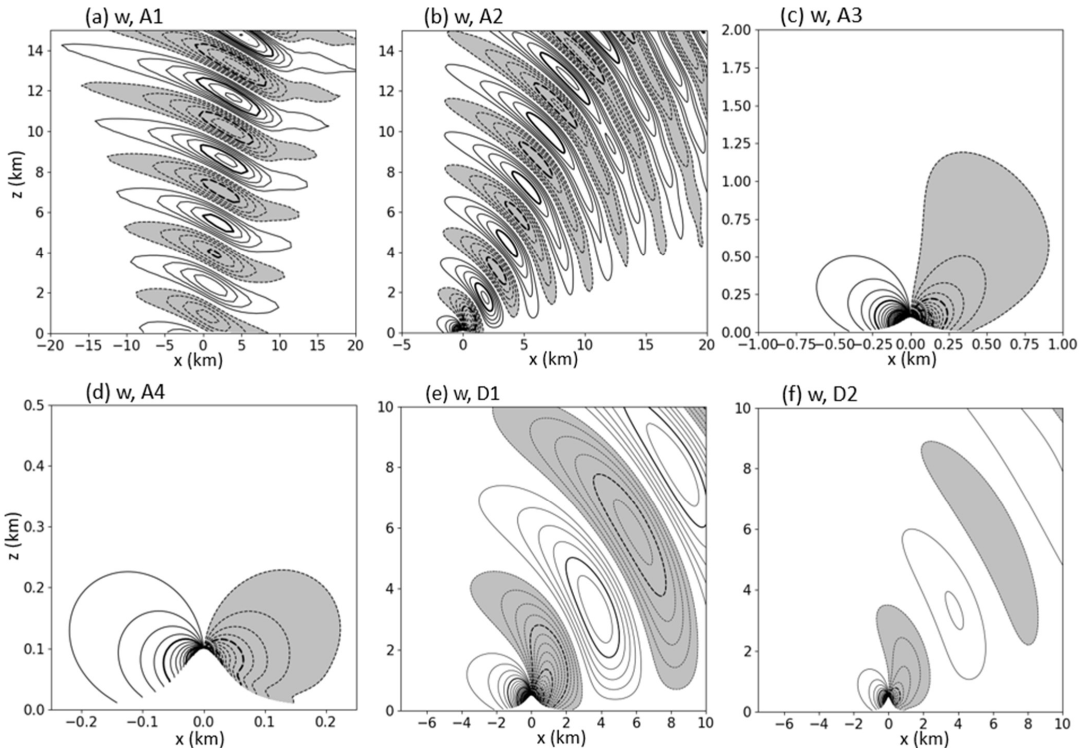

Figure 1. Simulated vertical velocity for cases (a) A1, (b) A2, (c) A3, (d) A4, (e) D1, and (f) D2. The contour intervals are 0.05, 0.1, 0.25, 1.0, 0.2, and 0.5 m s^{-1}, respectively.

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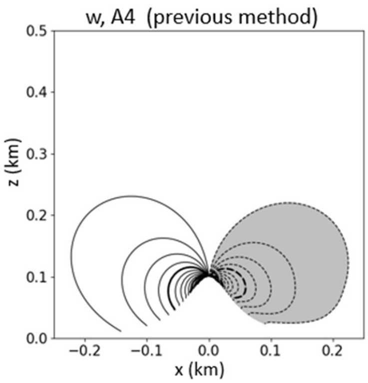
955 

Figure 2. Same as Fig. 1d, except for using $[1/m \cdot \partial(\pi\mathcal{P})/\partial\eta]_s = [1/m \cdot \partial(\pi\mathcal{P})/\partial\eta]_{3/2}$ for the surface boundary condition.

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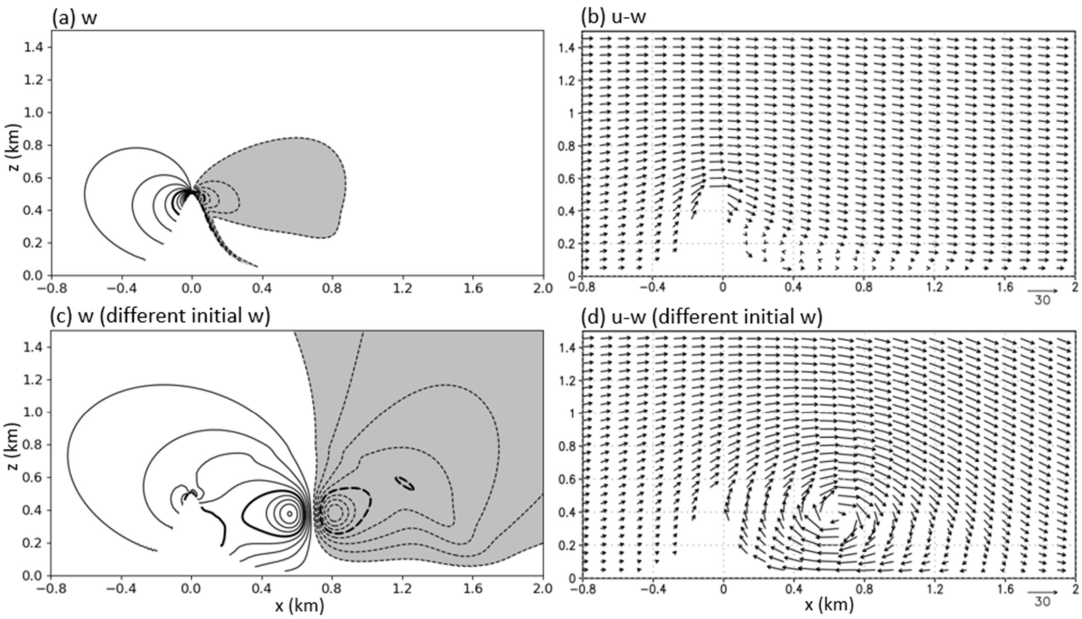

Figure 3. The results for case S1 with an extremely steep mountain. (a) Simulated vertical velocities (contour interval: 2.0 m s^{-1}) and (b) simulated wind vectors when $\Delta x = \Delta z = 25$ m, and $\Delta t = 0.75$ s. (c) and (d) are the same as (a) and (b), respectively, except that $\Delta x = \Delta z = 10$ m, and $\Delta t = 0.25$ s.

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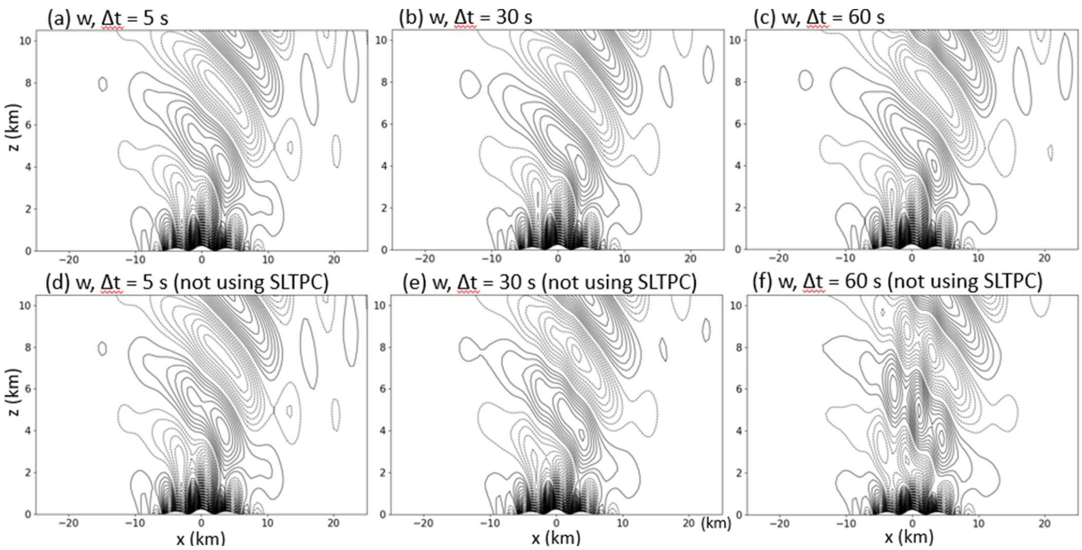

970 **Figure 4.** Simulated vertical velocity for case Schär1 using (a) $\Delta t = 5$ s, (b) $\Delta t = 30$ s, and (c) $\Delta t = 60$ s. (d)–(f) are the same as (a)–(c), respectively, except for not using the semi-Lagrangian treatment of the pressure change (SLTPC). The contour interval is 0.05 m s^{-1}.

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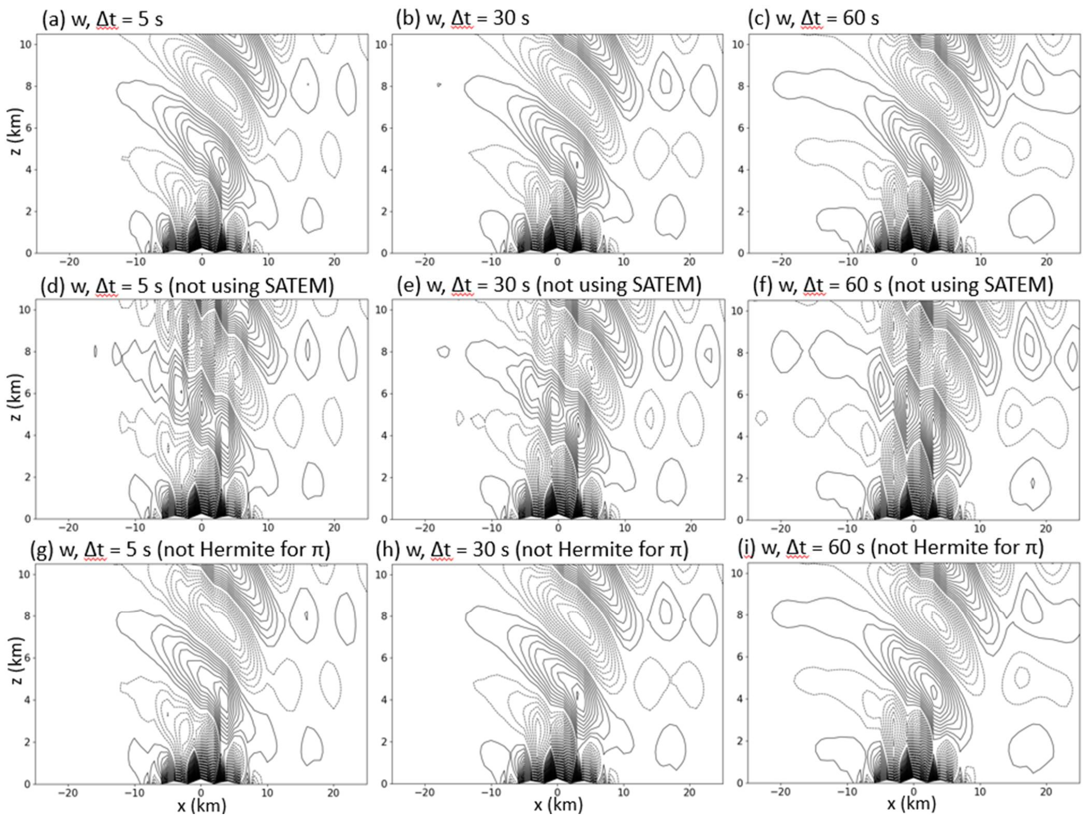

Figure 5. Simulated vertical velocity for case Schär2 using (a) $\Delta t = 5$ s, (b) $\Delta t = 30$ s, and (c) $\Delta t = 60$ s. (d)–(f) are the same as (a)–(c), respectively, except for not using the spatially averaged Eulerian treatment of the mountains (SAETM). (g)–(i) are
980 the same as (a)–(c), respectively, except for not using the cubic Hermite interpolation for the SLTPC. The contour interval is $0.09355\ \mathrm{m\ s^{-1}}$.

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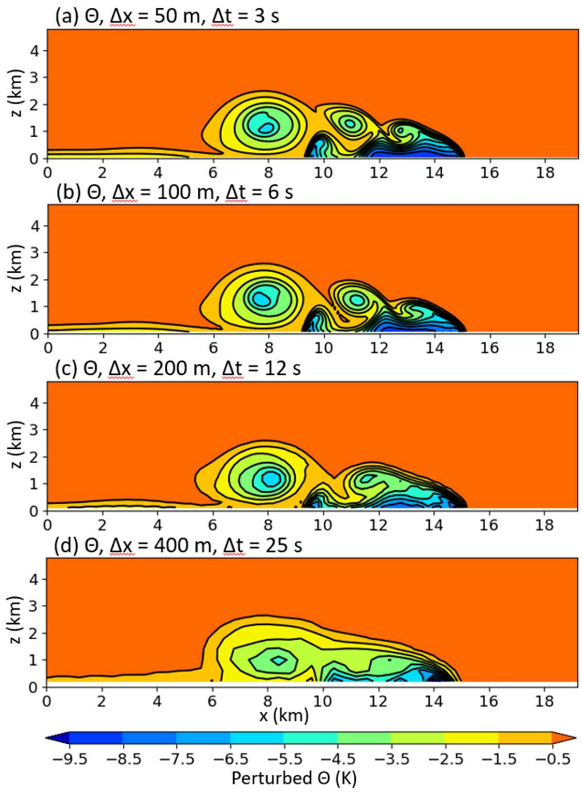

Figure 6. Simulated potential temperature perturbations (K) at 900 s in the density current test case of Straka et al. (1993). (a) $\Delta x = \Delta z = 50$ m and $\Delta t = 3$ s. (b) $\Delta x = \Delta z = 100$ m and $\Delta t = 6$ s. (c) $\Delta x = \Delta z = 200$ m and $\Delta t = 12$ s. (d) $\Delta x = \Delta z = 400$ m and $\Delta t = 25$ s.

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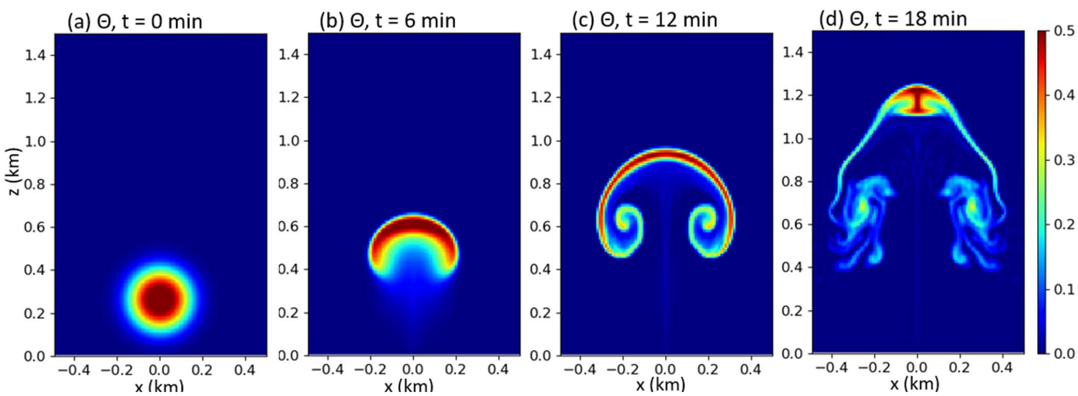

995 **Figure 7.** Simulated potential temperature perturbations (K) at (a) t = 0 min, (b) t = 6 min, (c) t = 12 min, and (d) t = 18 min in the warm bubble convection test case of Robert (1993).

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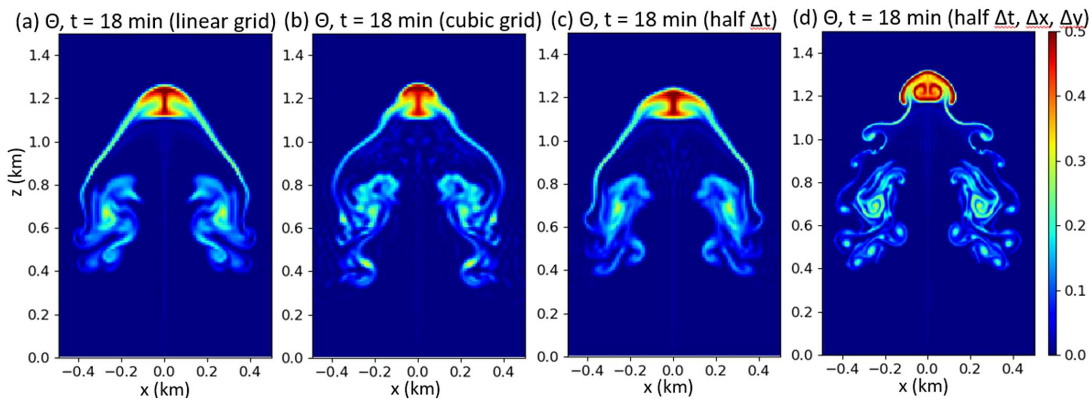

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Figure 8. (a) Same as Fig. 7d, except for using linear truncation instead of quadratic truncation. (b) Same as (a), except for using cubic truncation. (c) Same as Fig. 7d, except that $\Delta t = 2.5$ s instead of $\Delta t = 5$ s. (d) Same as (c), except that $\Delta x = \Delta z = 5$ m instead of $\Delta x = \Delta z = 10$ m.

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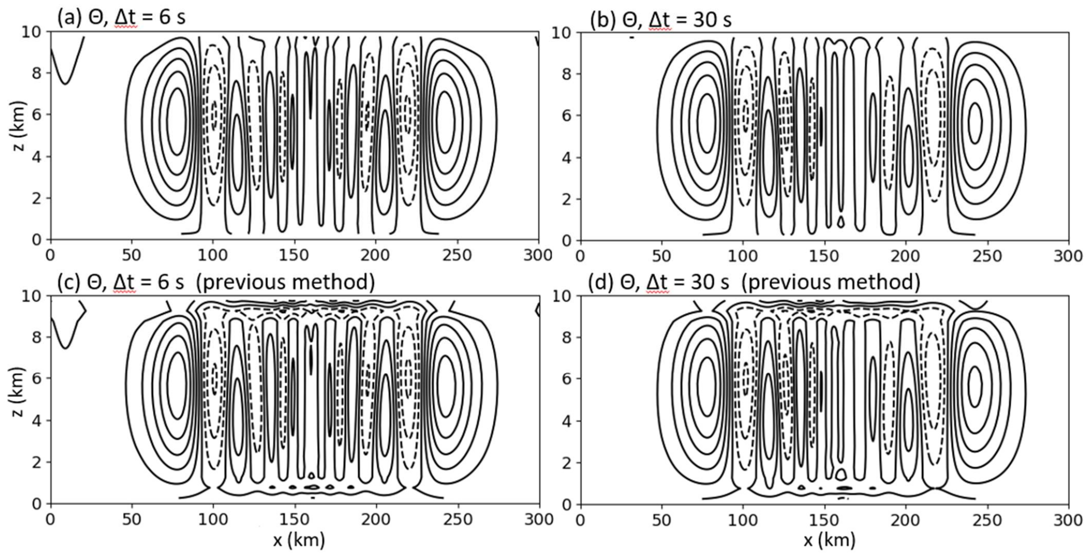

Figure 9. Simulated potential temperature perturbations at $t = 3000$ s using (a) $\Delta t = 6$ s and (b) $\Delta t = 30$ s in the gravity wave test case of Skamarock and Klemp (1994). (c) and (d) are the same as (a) and (b), respectively, except for using the previous method for vertical interpolation (Appendix B3). The contour interval is 0.0005 K.

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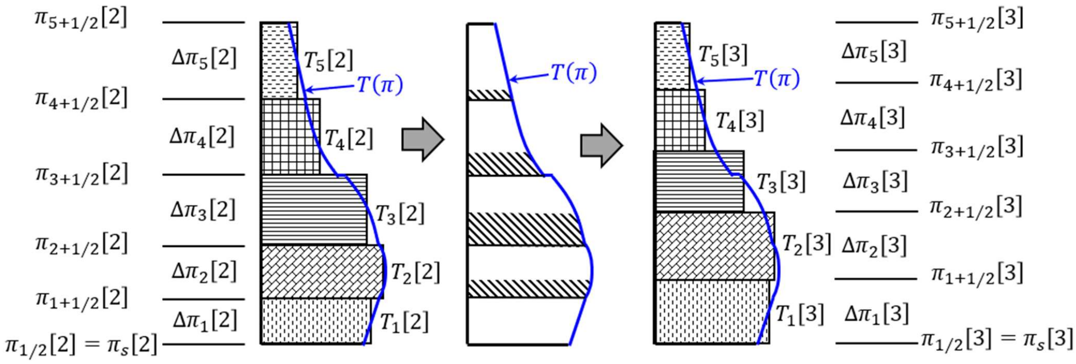

1020 **Figure B1.** Schematic diagram of the vertical flux calculation in the vertically conservative semi-Lagrangian scheme. $T(\pi)$ is an interpolation function obtained from the values of T_k [2]. The shaded areas of the middle subfigure are moved from the upper to lower levels.